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ABSTRACT. Let F be a field of characteristic 2. When K/F is a field extension of degree two or three, it is known to be excellent for quadratic as well as bilinear forms. In this paper, we discuss the excellence property when K/F is an inseparable quartic extension.

1. INTRODUCTION

Throughout this paper F denotes a field of characteristic 2. A field extension K/F is called *excellent* for quadratic forms if for any quadratic form φ over F, the anisotropic part $(\varphi_K)_{an}$ of the K-form φ_K is defined over F, i.e., there exists a quadratic form ψ over F such that $(\varphi_K)_{an}$ is isometric to ψ_K. The same definition can be formulated for bilinear forms. Moreover, since we are in characteristic 2, we need to consider a refinement of the above definition in the case of quadratic forms (introduced in [18]). More precisely, the extension K/F is called (r,s)-*excellent* if for any F-quadratic form φ of type (r,s), the K-form $(\varphi_K)_{\mathrm{an}}$ is defined over F (recall that a quadratic form is of type (r,s) if it is of dimension $2r+s$ and its quasilinear part has dimension s). In particular, an extension which is excellent is necessary (r,s)-excellent for any pair of positive integers r and s. Moreover, the excellence property holds for quadratic forms of type $(0,s)$ for any positive integer s, i.e., totally singular quadratic forms and any field extension of F [11, Proposition 8.1(iii)]. Other known examples of excellent extensions of F are: A purely transcendental extension, an algebraic extension of odd degree [8, Corollary 18.5], and multiquadratic purely inseparable extensions. This last result is due to Hoffmann for quadratic and bilinear forms [9], [10].

For extensions given by function fields of quadrics, the case of 2-dimensional quadratic forms is covered by quadratic extensions, which are excellent. For conics, one knows that any extension given by the function field of a nonsingular conic is excellent for both bilinear and quadratic forms [12, Cor. 5.7]. Recently in [18] the two authors proved that the extension given by the function field a singular conic is excellent for bilinear forms, and it is neither $(2,s+1)$-excellent nor $(3+m,s)$ for $m,s\geq 0$.

Our aim in this paper is to study the excellence of inseparable quartic extensions for quadratic forms. These extensions split in four types as given in Section 3. We prove in Corollary 7.2 that quartic simple purely inseparable extensions are not (r,s)-excellent when $r\geq 2$ and $s\geq 0$. To this end, we study the isotropy over such extensions using the work [5] of Aravire, Laghribi and O'Ryan. For quartic inseparable extensions which are simple but not purely inseparable, we use a transfer map that they induce in the setting of Kato-Milne cohomology (Corollary 5.6) to prove that these extensions are not (r,s)-excellent when $r\geq 2$ and $s\geq 1$ (Corollary 7.2). The other two types concern biquadratic extensions. The excellence of purely inseparable biquadratic extensions is due to Hoffmann [10] as we said before. We are yet to ascertain the excellence of mixed biquadratic extensions. For completeness we also give the proof that any separable biquadratic extension is not (r,s)-excellent when $r\geq 2$ and $s\geq 0$ (Example 6.1). This proof is an extension to characteristic 2 of that given by Sivatski in characteristic not 2, and is based on a result of Rowen about indecomposable central simple algebras of degree 8 and exponent 2.

MSC: 11E04, 11E81

Date: March 20, 2024.

Key words and phrases. Quadratic form, Isotropy, quartic extension, excellence.

2. Background and Preliminaries

2.1. **On quadratic forms.** Any quadratic form φ over F can be written up to isometry as follows:

$$\varphi \simeq [a_1, b_1] \perp [a_2, b_2] \perp \ldots \perp [a_r, b_r] \perp \langle c_1 \rangle \perp \ldots \perp \langle c_s \rangle, \tag{2.1}$$

where $\simeq$ and $\perp$ denotes the isometry and orthogonal sum of quadratic forms, and $[a, b]$ (*resp.* $\langle a \rangle$) denotes the quadratic form $ax^2+xy+by^2$ (*resp.* ax^2). The quadratic form $\langle c_1 \rangle \perp \ldots \perp \langle c_s \rangle$ is unique up to isometry, we call it the quasilinear part of φ, and denote it by $\mathrm{ql}(\varphi)$. As in equation (2.1), the form φ is called of type (r, s). We say that φ is:

- nonsingular (*resp.* totally singular) if $s = 0$ (*resp.* $r = 0$),
- singular if $s > 0$,

For $a_1, \ldots, a_n \in F$, let $\langle a_1, \ldots, a_n \rangle$ denote the totally singular quadratic form $\langle a_1 \rangle \perp \ldots \perp \langle a_n \rangle$.

The Arf invariant $\Delta(\varphi)$ of a nonsingular form $\varphi \simeq [a_1, b_1] \perp [a_2, b_2] \perp \ldots \perp [a_r, b_r]$ is defined to be $\sum_{i=1}^{r} a_i b_i + \wp(F)$ in $F/\wp(F)$, where $\wp(F) = \{\alpha^2 + \alpha \mid \alpha \in F\}$.

A quadratic form φ of underlying F-vector space V is called isotropic if there exists $v \in V \setminus \{0\}$ such that $\varphi(v) = 0$, otherwise φ is called anisotropic.

For an integer $n \geq 0$ and φ a quadratic form, let $n \times \varphi$ denote the quadratic form $\underbrace{\varphi \perp \ldots \perp \varphi}_{n \text{ times}}$.

Recall that any quadratic form φ over F uniquely decomposes as follows:

$$\varphi \simeq \varphi_{an} \perp i \times [0, 0] \perp j \times \langle 0 \rangle,$$

where φ_{an} is an anisotropic quadratic form. We call φ_{an} the anisotropic part of φ, and the integer i (*resp.* j) is called the Witt index (*resp.* the defect index) of φ. We denote i and j by $i_W(\varphi)$ and $i_d(\varphi)$, respectively.

Two quadratic forms φ_1 and φ_2 are called Witt-equivalent, denoted by $\varphi_1 \sim \varphi_2$, if there exists $m, n \in \mathbb{N}$ such that $\varphi_1 \perp m \times [0, 0] \simeq \varphi_2 \perp n \times [0, 0]$.

A quadratic form φ of underlying vector space V represents a scalar $\alpha \in F$ if there exists $v \in V$ such that $\varphi(v) = \alpha$. We denote by $D_F(\varphi)$ the set of scalars in $F^* := F \setminus \{0\}$ represented by φ.

We will need the following cancellation result:

Proposition 2.1. *(*[15, Proposition 1.2] *for (1),* [11, Lemma 2.6] *for (2)) Let φ_1, φ_2 be two quadratic forms (possibly singular). Suppose that one of the following conditions holds:*

(1) $\varphi_1 \perp \psi \simeq \varphi_2 \perp \psi$ for some nonsingular form ψ,
(2) $\varphi_1 \perp s \times \langle 0 \rangle \simeq \varphi_2 \perp s \times \langle 0 \rangle$ for some integer $s \geq 0$ such that $i_d(\varphi_1) = i_d(\varphi_2) = 0$.

Then, $\varphi_1 \simeq \varphi_2$.

We recall the definition of domination of quadratic forms that will often appear in the upcoming sections [11, Definition 3.4].

Definition 2.2. *Let φ and ψ be F-quadratic forms of underlying vector spaces V and W, respectively. We say that φ is dominated by ψ if there exists an injective isometry $(V, \varphi) \longrightarrow (W, \psi)$.*

For $\varphi \in F[x_1, x_2, \ldots, x_n]$ an irreducible polynomial, let $F(\varphi)$ be the field of fractions of the quotient ring $F[x_1, \ldots, x_n]/(\varphi)$. We call it the function field of φ.

For a field extension K/F and a quadratic form φ over F, let φ_K denote the quadratic form φ viewed as a quadratic form over K by scalar extension.

For $a_1, \ldots, a_n \in F^*$, let $\langle a_1, \ldots, a_n \rangle_b$ be the diagonal bilinear form defined by:

$$((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto \sum_{i=1}^{n} a_i x_i y_i.$$

An n-fold bilinear Pfister form is a bilinear form of type $\langle 1, a_1 \rangle_b \otimes \cdots \otimes \langle 1, a_n \rangle_b$ for some $a_i \in F^*$; we write $\langle\langle a_1, \cdots, a_n \rangle\rangle_b$ for short. An $(n+1)$-fold quadratic Pfister form is a nonsingular quadratic form of type $\langle\langle a_1, \cdots, a_n \rangle\rangle_b \otimes [1, b]$ for some $a_i \in F^*$, $b \in F$.

Any quadratic Pfister form π is round, i.e., $x \in F^*$ is represented by π if and only if $\pi \simeq x\pi$. The same fact is true for bilinear Pfister forms. The set of quadratc forms isometric (*resp.* similar) to n-fold quadratic Pfister forms will be denoted by $P_n F$ (*resp.* $GP_n F$).

Definition 2.3. *A quadratic form φ is said to be a Pfister neighbor if there exist a quadratic Pfister form π and $a \in F^*$ such that $a\varphi \prec \pi$ and $2 \dim \varphi > \dim \pi$.*

We summarize in the following proposition some properties of Pfister neighbors.

Proposition 2.4. *(*[16, Proposition 3.1]*)*

(1) If $\pi \in P_n F$ and φ are anisotropic. Then, φ is a Pfister neighbor of π if and only if $\dim \varphi > 2^{n-1}$ and $\pi_{F(\varphi)}$ is isotropic.

(2) If φ is a Pfister neighbor of π, then we have the following:

(a) φ is isotropic if and only if π is also isotropic.

(b) π is uniquely determined by φ up to isometry.

Let $W(F)$ (*resp.* $W_q(F)$) be the Witt ring of regular symmetric F-bilinear forms (*resp.* the Witt group of nonsingular F-quadratic forms). The group $W_q(F)$ is endowed with a $W(F)$-module structure as follows. For B (*resp.* φ) a regular F-bilinear form of underlying vector space V (*resp.* a nonsingular F-quadratic form of underlying vector space W), we associate a nonsingular quadratic form $B \otimes \varphi$ defined on $V \otimes_F W$ by:

$$B \otimes \varphi(v \otimes w) = \widetilde{B}(v)\varphi(w) \text{ for any } (v, w) \in V \times W \tag{2.2}$$

and whose polar form is $B \otimes B_\varphi$, where B_φ is the polar form of φ.

For any integer $n \geq 1$, let $I^n F$ be the n-th power of the fundamental ideal IF of $W(F)$. We take $I^0 F = WF$. For any integer $n \geq 0$, let $I_q^{n+1} F$ denote the $W(F)$-submodule of $W_q(F)$ given by $I^n F \otimes W_q(F)$. This submodule is additively generated by the $(n+1)$-fold quadratic Pfister forms $\langle\langle a_1, \cdots, a_n; b]]$, where $a_i \in F^*$, $b \in F$.

For any field extension L/F, we denote by $W_q(L/F)$ the kernel of the homomorphism $W_q(F) \to W_q(L)$ induced by scalar extension. Similarly, let $I_q^n(L/F)$ (*resp.* $\overline{I}_q^n(L/F)$) be the kernel of the homomorphism $I_q^n F \longrightarrow I_q^n L$ (*resp.* $\overline{I}_q^n F \longrightarrow \overline{I}_q^n L$).

We recall the following result known as the Hauptsatz of Arason and Pfister:

Theorem 2.5. *(*[6]*) Let φ be a nonzero anisotropic quadratic form that belongs to $I_q^n F$. Then, $\dim \varphi \geq 2^n$.*

2.2. **On differential forms.** For any integer $n \geq 1$, let $\Omega_F^n = \overset{n}{\wedge} \Omega_F^1$ where Ω_F^1 is the F-vector space of absolute 1-differential forms, i.e., the F-vector space generated by the symbols $\mathrm{d}da$ with $a \in F$ subject to the relations: $\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b$ and $\mathrm{d}(ab) = a\mathrm{d}b + b\mathrm{d}a$ for any $a, b \in F$. We set $\Omega_F^0 = F$. The differential operator $\mathrm{d} : \Omega_F^n \longrightarrow \Omega_F^{n+1}$ is given by: $a\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n \mapsto \mathrm{d}a \wedge \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n$. The Artin-Schreier operator $\wp : \Omega_F^n \longrightarrow \Omega_F^n / \mathrm{d}\Omega_F^{n-1}$ is defined by: $a\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n} \mapsto (a^2 + a)\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n} + \mathrm{d}\Omega_F^{n-1}$.

Let $H_p^{n+1}(F)$ denote the cokernel of $\wp$. Then, we have

$$H_p^{n+1}(F) = \Omega_F^n / (\wp(\Omega_F^n) + \mathrm{d}\Omega_F^{n-1}).$$

To translate some results from differential forms to quadratic forms and vice-versa, we will use the Kato's isomorphism [14]:

Theorem 2.6. *Let F be a field of characteristic 2. For every integer $n \geq 1$ there exists an isomorphism:*

$$\begin{array}{rcl} f_n : \overline{I}_q^n F & \to & H_2^{n+1}(F) \\ \overline{\langle\langle a_1, \ldots, a_n; b]]} & \mapsto & \overline{b\dfrac{\mathrm{d}a_1}{a_1} \wedge \ldots \wedge \dfrac{\mathrm{d}a_n}{a_n}}. \end{array} \tag{2.3}$$

2.3. **Preliminaries.** In this section we recall certain results to be useful in the upcoming sections.

Proposition 2.7. *(*[1, Cor. 2.8]*) Let φ be an anisotropic nonsingular F-quadratic form and $K = F(\sqrt{b})$ for some scalar $b \in F \setminus F^2$. Then φ_K is hyperbolic if and only if φ is divisible by $\langle 1, b\rangle_b$.*

Proposition 2.8. *Let L/F be an extension, and $z \in F$. Let $\langle\langle x, y]]$ be an anisotropic 2-fold Pfister form over F. We have:*
(1) The following two statements are equivalent:

- *(a) $x[1, y] \perp \langle 1, z\rangle$ is isotropic over L.*
- *(b) $z \in L^2$ or ($z \notin L^2$ and $\langle\langle x, y]]_{L(\sqrt{z})}$ is isotropic).*

(2) If L/F is a finite purely inseparable extension, then statements (a) and (b) are equivalent to the following:

- *(c) $z \in L^2$ or ($z \notin L^2$ and there exists $t \in F$ such that the Albert form $x[1, y] \perp z[1, t] \perp [1, y + t]$ is hyperbolic over L).*

Proof. (1) Let $\varphi = x[1, y] \perp \langle 1, z\rangle$.

(b) $\Longrightarrow$ (a) If $z \in L^2$, then $\langle 1, z\rangle_L$ is isotropic, and thus φ_L is isotropic. So suppose $z \notin L^2$ and $\langle\langle x, y]]_{L(\sqrt{z})}$ is isotropic. We have $\langle\langle x, y]]_{L(\sqrt{z})} \sim 0$.

(i) If $\langle\langle x, y]]_L$ is isotropic, then φ_L is isotropic because the Pfister neighbor $(x[1, y] \perp \langle 1\rangle)_L$ is isotropic and dominated by φ_L.

(ii) If $\langle\langle x, y]]_L$ is anisotropic, then we get by Proposition 2.7 and the roundness of Pfister forms that $\langle\langle x, y]]_L \simeq \langle\langle z, t]]$ for some $t \in L$. Adding the form $\langle 1, z\rangle_L$ in both sides of the previous isometry, we get $\varphi_L \perp [0, 0] \simeq 2 \times [0, 0] \perp \langle 1, z\rangle_L$. Canceling a hyperbolic plane (Proposition 2.1) implies that φ_L is isotropic.

(a) $\Longrightarrow$ (b) Suppose that φ_L is isotropic. If $i_d(\varphi_L) = 1$ then $\langle 1, z\rangle_L$ is isotropic and $z \in L^2$. If $i_d(\varphi_L) = 0$, then $\varphi_L \simeq [0, 0] \perp \langle 1, z\rangle_L$. Extending to $L(\sqrt{z})$ yields

$$(x[1, y] \perp \langle 1\rangle)_L \perp \langle 0\rangle \simeq [0, 0] \perp \langle 1\rangle_L \perp \langle 0\rangle .$$

Canceling the form $\langle 0\rangle$ (Proposition 2.1), we get $(x[1, y] \perp \langle 1\rangle)_L \simeq [0, 0] \perp \langle 1\rangle_L$. Hence, $\langle\langle x, y]]_{L(\sqrt{z})}$ is isotropic.

(2) Now we assume that L/F is purely inseparable of finite degree.

(b) $\Longrightarrow$ (c) Suppose that $z \notin L^2$ and $\langle\langle x, y]]_{L(\sqrt{z})}$ is isotropic. As we did in the proof of the implication (b) $\Longrightarrow$ (a), there exists $t \in L$ such that $\langle\langle x, y]]_L \simeq \langle\langle z, t]]$ (we take $t = 0$ when $\langle\langle x, y]]_L$ is isotropic). Since L/F is purely inseparable of finite degree, there exists an integer $m \geq 1$ such that $L^{2^m} \subset F$. Hence, we may suppose $t \in F$ because $[1, t] \simeq [1, t^{2^m}]$. Consequently, $(\langle\langle x, y]] \perp \langle\langle z, t]])_L$ is hyperbolic, in particular $(x[1, y] \perp z[1, t] \perp [1, y + t])_L$ is hyperbolic.

(c) $\Longrightarrow$ (b) If the Albert form $(x[1, y] \perp z[1, t] \perp [1, y + t])_L$ is hyperbolic for some $t \in F$, then passing to $L(\sqrt{z})$ we deduce that $\langle\langle x, y]]_{L(\sqrt{z})}$ is isotropic. □

We recall the following result which is well known:

Lemma 2.9. *Let L/F be a finite purely inseparable field extension and $x \in F$. If $x \in \wp(L)$, then $x \in \wp(F)$. In particular, if φ is an F-quadratic form such that $\Delta(\varphi_L) = 0 \in L/\wp(L)$, then $\Delta(\varphi) = 0 \in F/\wp(F)$.*

Proof. Let $n \geq 1$ be an integer such that $L^{2^n} \subset F$. Since $x \in \wp(L)$, we have $x^{2^n} \in \wp(F)$ and thus there exists $y \in F$ such that $x^{2^n} = y^2 + y$. Since $x^{2^n}, y^2 \in F^2$, we get $y \in F^2$. Hence, $x^{2^n}, y^2 \in F^4$, and thus $y \in F^4$. Repeating this argument n times, we get $y \in F^{2^n}$. Thus, $x = z^2 + z \in \wp(F)$ where $z^{2^n} = y$. □

Theorem 2.10. *(*[20, Theorem 2]*) Let φ be a nonsingular quadratic form over F.*

(1) $\varphi \in I_q^2F$ iff $\Delta(\varphi) = 0$.
(2) If $\varphi \in I_q^2F$, then $\varphi \in I_q^3F$ iff $C(\varphi) \sim 0$.

3. Isotropy of quadratic forms of dimension ≤ 4 over quartic extensions

Let us recall that an inseparable quartic field extension K of F is one of the following mutually exclusive types (we refer to [13, Section 3] for more details):

(Type 1) Simple purely inseparable: $K = F(\sqrt[4]{b})$ for some $b \in F \setminus F^2$.
(Type 2) Simple but not purely inseparable: $K = F(\alpha)$ such that α is a root of an irreducible polynomial $x^4 + ax^2 + b \in F[x]$ such that $\langle 1, a, b\rangle$ is anisotropic over F.
(Type 3) Purely inseparable biquadratic: $K = F(\sqrt{d_1}, \sqrt{d_2})$ for $d_1, d_2 \in F \setminus F^2$.
(Type 4) Mixed biquadratic: $K = F(\sqrt{b}, \alpha)$ such that $b \in F \setminus F^2$ and α is a root of an irreducible polynomial $x^2 + x + a \in F[x]$.

In this section we discuss the isotropy of quadratic forms of small dimension over quartic extensions of type 1. This will aid us in inspecting whether this field extension is excellent or not for quadratic forms. As we will see, we are mostly interested in the isotropy of nonsingular quadratic forms of dimension 4, but for completeness we consider the isotropy for all quadratic forms of dimension at most 4. Recall that quartic extensions of type 3 are excellent by a result of Hoffmann [10].

For the rest of this section, we fix $K = F(\sqrt[4]{b})$ such that $b \in F \setminus F^2$. We recall some results that will help us to classify F-quadratic forms of dimension ≤ 4 that become isotropic over K.

Lemma 3.1. *(*[16, Lemma 4.3]*) Let φ be a nonsingular quadratic form of dimension 4 and $a \in F$ such that $\triangle(\varphi) = a + \wp(F)$. Then, φ_L is isotropic if and only if φ is isotropic, where $L = F(\wp^{-1}(a))$.*

We deduce from the general result [5, Theorem 1.4] the following theorem:

Theorem 3.2. *Let K be the quartic extension as above, and $c \in F \setminus \wp(F)$. Then, we have:*

$$H_2^2(K(\wp^{-1}(c))/F) = H_2^2(K/F) + \overline{cG_K^1(1)}, \tag{3.1}$$

where $G_K^1(1)$ is the subgroup of Ω_F^1 additively generated by the differentials $d_{x,y} := \dfrac{x^2}{x^2+by^2}\dfrac{\mathrm{d}x}{x} + \dfrac{by^2}{x^2+by^2}\dfrac{\mathrm{d}y}{y}$ for $x, y \in F^*$.

This theorem can be rewritten as follows:

Corollary 3.3. *Let K be the quartic extension as above, and $c \in F \setminus \wp(F)$. Then, we have:*

$$H_2^2(K(\wp^{-1}(c))/F) = H_2^2(K/F) + \overline{c\nu_F(1)} + \overline{c\,\langle d_{x,1} \mid x \in F^*\rangle}. \tag{3.2}$$

Proof. It suffices to prove that $\nu_F(1) + \langle d_{x,1} \mid x \in F^* \rangle = G_K^1(1)$. In fact, the group $\nu_F(1)$ is contained in $G_K^1(1)$ because $d_{x,x} = \frac{\mathrm{d}x}{x}$ for all $x \in F^*$. Conversely, let $x, y \in F^*$. We have

$$\begin{aligned} d_{x,y} &= \frac{x^2}{x^2+by^2}\frac{\mathrm{d}x}{x} + \frac{by^2}{x^2+by^2}\frac{\mathrm{d}y}{y} \\ &= \frac{(xy^{-1})^2}{(xy^{-1})^2+b}\frac{\mathrm{d}(xy^{-1})}{xy^{-1}} + \frac{(xy^{-1})^2}{(xy^{-1})^2+b}\frac{\mathrm{d}(y^{-1})}{y^{-1}} + \frac{b}{(xy^{-1})^2+b}\frac{\mathrm{d}y}{y} \\ &= \frac{(xy^{-1})^2}{(xy^{-1})^2+b}\frac{\mathrm{d}(xy^{-1})}{xy^{-1}} + \frac{\mathrm{d}y}{y} \in \nu_F(1) + \langle d_{x,1} \mid x \in F^* \rangle . \quad \square \end{aligned}$$

Now applying the Kato's isomorphism (2.3) to equation (3.2), we obtain:

Corollary 3.4. *Let K be the quartic extension as above, $c \in F \setminus \wp(F)$. Then:*

$$\overline{I}_q^2(K(\wp^{-1}(c))/F) = \overline{I}_q^2(K/F) + \overline{IF \otimes [1,c]} + \sum_i \overline{\langle 1, x_i \rangle_b \otimes \left[1, \frac{cx_i^2}{x_i^2+b}\right]}, \tag{3.3}$$

where $x_i \in F^*$.

To complete the kernels given in Corollary 3.4, we recall the kernels $\overline{I}_q^2(K/F)$ and $I_q^2(K/F)$ studied by Sobiech:

Theorem 3.5. *Let L be the quartic extension as above. We have:*
(1) ([22, Corollary 5.2]*) The kernel $\overline{I}_q^2(K/F)$ is additively generated by the classes $\overline{\langle\langle u, bu^2]]}$ and $\overline{\langle\langle b, v]]}$ for $u, v \in F^*$.*
(2) ([22, Theorem 5.3]*) The kernel $I_q^2(K/F)$ coincides with the $W(F)$-module generated by the forms $\langle\langle u, bu^2]]$ and $\langle\langle b, v]]$ for $u, v \in F^*$.*

Now we treat the isotropy of F-quadratic forms of dimension ≤ 4 over K.

Theorem 3.6. *Let φ be an anisotropic F-quadratic form of dimension ≤ 4, and let K be as mentioned above. Then, φ_K is isotropic if and only if one of the following conditions holds:*
(1) φ is totally singular and $\alpha \langle 1, b \rangle \prec \varphi$ for some $\alpha \in F^*$.
(2) φ is a Pfister neighbor of a 2-fold Pfister form which is isotropic over K.
(3) $\varphi \simeq x[1,y] \perp \langle 1, z \rangle$ for some $x \neq 0, y, z \in F$ such that $z \in F^2(b)$ or ($z \notin F^2(b)$ and there exists $t \in F$ such that the Albert form $x[1,y] \perp z[1,t] \perp [1, y+t]$ is hyperbolic over K).
(4) φ is of type $(2,0)$ and there exist $\psi \in I_q^2(K/F)$ and $x_1, \ldots, x_s, y \in F$ for some integer $s \geq 1$ such that

$$\varphi \perp y[1,c] \perp \psi \perp \sum_{i=1}^{s} \langle 1, x_i \rangle_b \otimes \left[1, \frac{cx_i^2}{x_i^2+b}\right] \in I_q^3 F, \tag{3.4}$$

where $c \in F$ satisfies $\Delta(\varphi) = c + \wp(F)$.

Proof. When φ is of type $(2,0)$, we take $c \in F$ such that $\Delta(\varphi) = c + \wp(F)$, and consider the separable quadratic extension $F(\alpha)$ given by $\alpha^2 + \alpha = c$.

– If φ is totally singular, then φ_K is isotropic iff $\alpha \langle 1, b \rangle \prec \varphi$ for some $\alpha \in F^*$ [17, Lemma 2.10]. For the rest of the proof we suppose that φ is not totally singular.

– If φ is of type $(1,0)$ and φ_K is isotropic, then $\Delta(\varphi_K) = 0$. Lemma 2.9 implies $\Delta(\varphi) = 0$, i.e., φ is isotropic, which is excluded.

– If φ is of type $(1,1)$. Modulo a scalar we may suppose $\varphi \simeq x[1,y] \perp \langle 1 \rangle$ for some $x, y \in F^*$. Since φ is a Pfister neighbor of $\pi := \langle\langle x, y]]$, it follows that φ_K is isotropic iff π_K is isotropic (Proposition 2.4).

– If φ is of type $(1,2)$, then the assertion is a consequence of Proposition 2.8.

– Now suppose that φ is of type $(2,0)$ and φ_K isotropic. It follows that $\varphi_{K(\alpha)}$ is hyperbolic since $\Delta(\varphi_{F(\alpha)})=0$. In particular, we have $(\varphi \perp [1,c])_{K(\alpha)} \sim 0$, and thus $\varphi \perp [1,c] + I_q^3F \in \bar{I}_q^2(K(\alpha)/F)$. We now use Corollary 3.4 with Theorem 3.5 to get:

$$\varphi \perp [1,c] \perp \psi \perp \sum_{i=1}^{s} \langle 1, x_i\rangle_b \otimes \left[1, \frac{cx_i^2}{x_i^2+b}\right] \perp \rho \otimes [1,c] \in I_q^3F,$$

where $\psi \in I_q^2(K/F)$ and $\rho \in IF$. Let $y \in F^*$ be such that $\det \rho = yF^{*2}$ and use $\rho \perp \langle 1, y\rangle_b \in I^2F$ to get:

$$\varphi \perp y[1,c] \perp \psi \perp \sum_{i=1}^{s} \langle 1, x_i\rangle_b \otimes \left[1, \frac{cx_i^2}{x_i^2+b}\right] \in I_q^3F.$$

Conversely, if φ satisfies statement (4), then extending the equation (3.4) to $K(\alpha)$ yields $\varphi_{K(\alpha)} \sim 0$ by the Hauptsatz. Hence, Lemma 3.1 implies that φ_K is isotropic. □

Remark 3.7. *In the formulation of the statements (2), (3) and (4) of Theorem 3.6, we use some forms satisfying the hyperbolicity hypothesis over K. These forms are classified by Theorem 3.5.*

4. Nonexcellence of quartic extensions of Type 1

Our inspiration to build an example of a non-excellent quartic simple purely inseparable extension K/F is motived by statement (4) of Theorem 3.6 where some forms in $P_2(F)$ that become hyperbolic over K appear in equation (3.4). Starting with such a 2-fold Pfister form and modifying the Arf invariant, we get our example of nonexcellence as follows:

Example 4.1. *Let F_0 be a field of characteristic 2 and x, y, z be variables over F_0. Over the field $F := F_0(x,y,z)$, let us consider the F-quadratic form:*

$$\varphi = \left[1, z^2 + \frac{z^2x^2}{x^2+y}\right] \perp x\left[1, \frac{z^2x^2}{x^2+y}\right].$$

Then, for $K = F(\sqrt[4]{y})$ we have the following:
(1) φ is anisotropic over F.
(2) φ is isotropic over K.
(3) $(\varphi_K)_{\mathrm{an}}$ is not defined over F.
In particular, the extension K/F is not $(2,0)$-excellent.

Proof. (1) Let us take $u = \frac{z^2x^2}{x^2+y}$. Then, $\varphi = [1, z^2+u] \perp x[1,u]$ viewed as a quadratic form over $F_0(x,y,z) = F_0(x,u,z)$ (obviously, x,u,z remain variables over F_0). Since the forms $[1, z^2+u]$ and $[1,u]$ are anisotropic over $F_0(u,z)$, we deduce that φ is anisotropic over $F = F_0(x,u,z)$.

(2) Extending φ to K, we get

$$\begin{aligned}
\varphi_K &= \left[1, z^2 + \frac{z^2x^2}{x^2+y}\right]_K \perp x\left[1, \frac{z^2x^2}{x^2+y}\right]_K \\
&\simeq \left[1, z + \frac{zx}{x+\sqrt{y}}\right] \perp x\left[1, \frac{zx}{x+\sqrt{y}}\right] \\
&\sim [1,z]_K \perp \langle 1, x\rangle_b \otimes \left[1, \frac{zx}{x+\sqrt{y}}\right].
\end{aligned}$$

Using the fact that $\sqrt{y} \in K^2$ and [2, Lemma 2.1], we get

$$\begin{aligned}
\varphi_K &\sim [1,z]_K \perp \langle 1, x+\sqrt{y}\rangle_b \otimes [1,z] \\
&\sim ((x+\sqrt{y})[1,z])_K.
\end{aligned}$$

Moreover, the anisotropy of $[1,z]$ over F implies that $[1,z]_K$ is anisotropic since K/F is purely inseparable. Hence, φ_K is isotropic and $(\varphi_K)_{\mathrm{an}} \simeq ((x+\sqrt{y})[1,z])_K$.

(3) Suppose that $(\varphi_K)_{\mathrm{an}}$ is defined over F. Then, there exists $\alpha \in F^*$ such that

$$(\langle 1, x+\sqrt{y}\rangle_b \otimes [1,z])_K \simeq (\alpha[1,z])_K. \tag{4.1}$$

Consider the F-linear map $s : K \to F$ given by: $1 \mapsto 0$, $\sqrt[4]{y} \mapsto 0$, $(\sqrt[4]{y})^3 \mapsto 0$ and $\sqrt{y} \mapsto 1$. We compute $C_1 := s_*(\langle\alpha\rangle_b)$ and $C_2 := s_*(\langle x+\sqrt{y}\rangle_b)$ with respect to the ordered F-basis $\{1, \sqrt[4]{y}, \sqrt{y}, (\sqrt[4]{y})^3\}$ of K, we obtain

$$C_1 = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha y \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & x & 0 & 0 \\ x & y & 0 & 0 \\ 0 & 0 & x & y \\ 0 & 0 & y & xy \end{pmatrix}.$$

Applying the transfer map s_* to (4.1) and using the Frobenius reciprocity, we get $C_1 \otimes [1,z] \simeq C_2 \otimes [1,z]$. Since $C_1 \sim \begin{pmatrix} \alpha & 0 \\ 0 & \alpha y \end{pmatrix}$, hence $C_2 \otimes [1,z]$ is isotropic over F. But C_2 is anisotropic over F since its corresponding totally singular form $\widetilde{C_2} \simeq \langle 1, y, x, xy\rangle$ is anisotropic over F. Since C_2 is defined over $F_0(x,y)$, the form $C_2 \otimes [1,z]$ remains anisotropic over $F = F_0(x,y)(z)$, which is a contradiction. This finishes the proof of the example. □

5. Nonexcellence of quartic extensions of Type 2

In this section, we build an example of a quartic extension K/F of type 2 which is not excellent. Our example will not be inspired from the isotropy problem as we did for the quartic extensions of type 1, because we have no answer to the isotropy of nonsingular F-quadratic forms of dimension 4 over inseparable quartic extensions of type 2. For this reason our example will be given in dimension 5 using a transfer argument. More precisely, starting from the usual trace map of the separable quadratic extension of F contained in K, we first construct in the setting of Kato-Milne cohomology a transfer map from K to a quadratic inseparable extension of F. This construction is stated in a general setting in the subsection 5.1 below.

5.1. **A generalized transfer map.** Let $K = F(\alpha)$ be an extension of F which is inseparable but not purely inseparable. One knows that $\min(\alpha, F) = g(x^{2^l})$, where $l \geq 1$ and $g(x) \in F[x]$ is a separable polynomial of degree $n \geq 2$. Let us write $g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and let $L = F^2(a_0, \ldots, a_{n-1})$ with $[L : F^2] = 2^d$. Then, there exist $0 \leq i_1 < i_2 < \ldots < i_d \leq n-1$ such that $L = F^2(a_{i_1}, \ldots, a_{i_d})$, and thus $a_{i_1}, \ldots, a_{i_d}$ are 2-independent. Moreover, we suppose that n is even. Note that there exist an odd integer $i \in \{1, \ldots, n-1\}$ such that $a_i \neq 0$ (because $g(x)$ is separable). Let $\beta = \alpha^{2^l}$.

We thus have a sequence of fields $F \subset M := F(\beta) \subset K = F(\alpha) = M(\sqrt[2^l]{\beta})$, where M/F is separable and K/M is purely inseparable. Moreover, we choose a 2-basis $\mathcal{B}$ of F such that $\mathcal{B} = \{e_1 = a_{i_1}, \ldots, e_d = a_{i_d}, \cdots\} = \{e_i \mid i \in I\}$ for a suitable set I. Since M/F is separable, $\mathcal{B}$ remains a 2-basis of M. Since $a_{i_1}, \ldots, a_{i_d}$ are 2-dependent over K, we may choose $\mathcal{C} := (\mathcal{B} \setminus \{a_{i_1}\}) \cup \{\alpha\}$ as a 2-basis of K. We have $\Omega^m_M = \Omega^m_F \oplus \bigoplus_{1 \leq i \leq n-1} \beta^i \Omega^m_F$. In fact, since M/F is separable we can replace β by β^2, that is, $\Omega^m_M = \Omega^m_F \oplus \bigoplus_{1 \leq i \leq n-1} \beta^{2i} \Omega^m_F$.

Notation 5.1. *(1) Let $\mathcal{L}^m_F$ be the subgroup of Ω^m_F whose differentials are free from $\mathrm{d}a_{i_1}$.*

(2) Clearly, $(\Omega^m_M)_K = \{(w_0 + \sum_{i=1}^{n-1} \beta^{2i} w_i)_K \mid w_i \in \mathcal{L}^m_F\}$. We denote the group $(\Omega^m_M)_K$ by $\widehat{\Omega}^m_K$.

(3) Let S denote the field $F(\sqrt{a_{i_2}}, \cdots, \sqrt{a_{i_d}})$.

We then have the following result:

Lemma 5.2. *Let S and K be as mentioned above. Then, there exists a well-defined F-linear map $s : \widehat{\Omega}_K^m \to (\Omega_F^m)_S$ given by:*

$$\left(w_0 + \sum_{i=1}^{n-1} \beta^{2i} w_i\right)_K \mapsto \left(\sum_{i=1}^{n-1} \mathrm{Tr}(\beta^{2i}) w_i\right)_S$$

where $w_i \in \mathcal{L}_F^m$ for all $0 \leq i \leq n-1$.

Proof. Suppose $(w_0 + \sum_{i=1}^{n-1} \beta^{2i} w_i)_K = (w_0' + \sum_{i=1}^{n-1} \beta^{2i} w_i')_K$, where $w_i, w_i' \in \mathcal{L}_F^m$. Hence, $\sum_{i=0}^{n-1} \beta^{2i} w_i + \sum_{i=0}^{n-1} \beta^{2i} w_i' \in \mathrm{Ker}(\Omega_M^m \to \Omega_K^m)$. Using [4, Proposition 5.3], we get in Ω_M^m:

$$\sum_{i=0}^{n-1} \beta^{2i}(w_i + w_i') = \mathrm{d}\beta \wedge (\eta_0 + \mathrm{d}a_{i_1} \wedge \eta_1), \tag{5.1}$$

such that $\eta_0 \in M.\mathcal{L}_F^{m-1}$, $\eta_1 \in M.\mathcal{L}_F^{m-2}$. Since β is a root of $g(x)$ and there exists an odd integer $i \in \{1, \ldots, n-1\}$ such that $a_i \neq 0$, we pass to the derivative of $g(\beta) = 0$, we get $\mathrm{d}\beta \in \sum_{i=1}^{d} M.\mathrm{d}a_{i_j}$. Note that the left hand side of equation (5.1) is free from $\mathrm{d}a_{i_1}$, we thus get $\sum_{i=0}^{n-1} \beta^{2i}(w_i + w_i') \in \sum_{j=2}^{d} M.\mathrm{d}a_{i_j} \wedge \eta_0$. Hence, for all $0 \leq i \leq n-1$, we have $w_i + w_i' \in \sum_{j=2}^{d} \mathrm{d}a_{i_j} \wedge \Omega_F^{m-1}$, which implies that $(w_i + w_i')_S = 0$. Consequently, $(\sum_{i=1}^{n-1} \mathrm{Tr}(\beta^{2i}) w_i)_S = (\sum_{i=1}^{n-1} \mathrm{Tr}(\beta^{2i}) w_i')_S$. Hence, s is a well-defined map. □

Lemma 5.3. *We keep the same notations as mentioned above. Then, $\mathrm{d}\widehat{\Omega}_K^m \subset \widehat{\Omega}_K^{m+1}$ and $s(\mathrm{d}\widehat{\Omega}_K^m) \subset \mathrm{d}(\Omega_F^m)_S$.*

Proof. Let $w = (w_0 + \sum_{i=1}^{n-1} \beta^{2i} w_i)_K \in \widehat{\Omega}_K^m$, where $w_i \in \mathcal{L}_F^m$. Since we are over K, we may suppose that $w_i = \sum_\sigma c_\sigma \frac{\mathrm{d}e_\sigma}{e_\sigma}$ is free from $\mathrm{d}a_{i_1}$ and each $c_\sigma \in F$ does not contain an odd power of a_{i_1} in its expansion in $\mathcal{B}$. Hence, $\mathrm{d}w_i \in \mathcal{L}_F^{m+1}$ for all $0 \leq i \leq n-1$. Consequently, $\mathrm{d}w = (\mathrm{d}w_0 + \sum_{i=1}^{n-1} \beta^{2i} \mathrm{d}w_i)_K \in \widehat{\Omega}_K^{m+1}$. We have

$$\begin{aligned} s(\mathrm{d}w) &= \left(\sum_{i=1}^{n-1} \mathrm{Tr}(\beta^{2i}) \mathrm{d}w_i\right)_S \\ &= \left(\sum_{i=1}^{n-1} (\mathrm{Tr}(\beta^i))^2 \mathrm{d}w_i\right)_S \\ &= \mathrm{d}\left(\left(\sum_{i=1}^{n-1} (\mathrm{Tr}(\beta^i))^2 w_i\right)_S\right) \in \mathrm{d}(\Omega_F^m)_S. \quad \square \end{aligned}$$

Lemma 5.4. *We keep the same notations as mentioned above. Then, $s(\wp(\widehat{\Omega}_K^m)) \subset \wp((\Omega_F^m)_S) + \mathrm{d}(\Omega_F^m)_S$.*

Proof. Let $w = (w_0 + \sum_{i=1}^{n-1} \beta^{2i} w_i)_K \in \widehat{\Omega}_K^m$, where $w_i \in \mathcal{L}_F^m$. We have $\wp(w) = \wp(w_0)_K + \sum_{i=1}^{n-1} \wp(\beta^{2i} w_i)_K$, that is,

$$\wp(w) = \wp(w_0)_K + \sum_{i=1}^{n-1} ((\beta^{2i})^2 w_i^{[2]} + \beta^{2i} w_i)_K + \mathrm{d}\lambda \tag{5.2}$$

for some $\lambda \in \Omega_K^{m-1}$.

We claim that $\mathrm{d}\lambda \in \mathrm{d}\widehat{\Omega}_K^{m-1}$. In fact, we have $\lambda \in \Omega_K^{m-1} = K.\Omega_M^{m-1} \oplus K.\Omega_M^{m-2} \wedge \mathrm{d}\alpha$. Recall that $K.\Omega_M^{m-1} = \Omega_M^{m-1} \oplus \bigoplus_{i=1}^{2^l-1} \alpha^i \Omega_M^{m-1}$. Thus

$$\lambda \in \Omega_M^{m-1} \oplus \bigoplus_{i=1}^{2^l-1} \alpha^i \Omega_M^{m-1} \oplus K.\Omega_M^{m-2} \wedge \mathrm{d}\alpha.$$

Note that w_i are free from α and $\mathrm{d}\alpha$, so we get $\mathrm{d}\lambda = (\mathrm{d}\mu)_K$, where $\mu \in \Omega_M^{m-1}$. Now since equation (5.2) is considered over K and each w_i belongs to $\mathcal{L}_F^m$, we may take $\mu = \delta_0 + \sum_{i=1}^{n-1} \beta^{2i}\delta_i$ such that $\delta_i = \sum_\sigma c_{i,\sigma}\frac{\mathrm{d}e_\sigma}{e_\sigma} \in \mathcal{L}_F^{m-1}$ and each $c_{i,\sigma}$ has no odd power of a_{i_1} in its expansion in $\mathcal{B}$. Hence, $\mathrm{d}\lambda \in \mathrm{d}\widehat{\Omega}_K^{m-1}$. Lemma 5.3 implies that $s(\mathrm{d}\lambda) \in \mathrm{d}(\Omega_F^{m-1})_S$.

Now we apply the map s to equation (5.2), we get:

$$\begin{aligned} s(\wp(w)) &= \sum_{i=1}^{n-1}(\mathrm{Tr}((\beta^{2i})^2)w_i^{[2]} + \mathrm{Tr}(\beta^{2i})w_i) + s(\mathrm{d}\lambda) \\ &= \sum_{i=1}^{n-1}(\mathrm{Tr}(\beta^{2i})^2 w_i^{[2]} + \mathrm{Tr}(\beta^{2i})w_i) + s(\mathrm{d}\lambda) \\ &= \wp(\sum_{i=1}^{n-1}\mathrm{Tr}(\beta^{2i})w_i) + s(\mathrm{d}\lambda) \\ &= \wp(s(w)) + s(\mathrm{d}\lambda) \in \wp((\Omega_F^m)_S) + \mathrm{d}(\Omega_F^m)_S. \end{aligned}$$ □

Notation 5.5. *For any field extension E_2/E_1, let*

$$H_{2,E_1}^{m+1}(E_2) := (\Omega_{E_1}^m)_{E_2}/(\wp((\Omega_{E_1}^m)_{E_2}) + \mathrm{d}(\Omega_{E_1}^{m-1})_{E_2}).$$

A consequence of the last three lemmas is the following corollary:

Corollary 5.6. *Let K, M and S be as mentioned above. Then, the map s defined above extends to a group homomorphism $\widetilde{\mathrm{Tr}}_* : H_{2,M}^{m+1}(K) \to H_{2,F}^{m+1}(S)$ given by:*

$$\overline{w} \mapsto \overline{s(w)}.$$

5.2. **An application.** Our aim in this subsection is to apply the above construction to get a counterexample to the excellence of quartic inseparable extensions of type 2. So let $K = F(\alpha)$ be a quartic extension of F, where $\min(\alpha, F) = x^4 + bx^2 + d \in F[x]$ such that $\langle 1, b, d\rangle$ is anisotropic.

As in the previous subsection, we let: $L = F^2(b, d)$ and $M = F(\beta)$, where $\beta = \alpha^2$. Our 2-basis of F is $\mathcal{B} = \{b, d\} \cup \{e_i \mid i \in I\}$, and $\mathcal{C} = \{b, \alpha\} \cup \{e_i \mid i \in I\}$ is a 2-basis of K. We consider the sequence of fields: $F \subset M := F(\beta) \subset K = M(\sqrt{\beta})$. Finally, our field S would be $F(\sqrt{b})$.

Example 5.7. *We keep the same notations as above. Let $\varphi = [1, t_1^{-1}] \perp [b, t_2^{-1}] \perp \langle d\rangle$ be a quadratic form over $E := F((t_1))((t_2))$ the field of iterated Laurent series in the variables t_1, t_2. Then, we have:*

(1) φ is anisotropic over E.

(2) $\varphi_{E(\alpha)}$ is isotropic but $(\varphi_{E(\alpha)})_{an}$ is not defined over E.

Proof. (1) It is easy to verify that φ is anisotropic over E since the form $\langle 1, b, d\rangle$ is anisotropic over F.

(2) Extending φ to $E(\alpha)$, we get:

$$\begin{aligned} \varphi_{E(\alpha)} &\simeq [\alpha^4, \alpha^{-4}t_1^{-1}] \perp [b\alpha^2, \alpha^{-2}t_2^{-1}] \perp \langle d\rangle \\ &\simeq [\alpha^4 + b\alpha^2, \alpha^{-4}t_1^{-1}] \perp [b\alpha^2, \alpha^{-2}t_2^{-1} + \alpha^{-4}t_1^{-1}] \perp \langle d\rangle \\ &\simeq [\alpha^4 + b\alpha^2 + d, \alpha^{-4}t_1^{-1}] \perp [b\alpha^2, \alpha^{-2}t_2^{-1} + \alpha^{-4}t_1^{-1}] \perp \langle d\rangle \\ &\simeq \mathbb{H} \perp b[1, bt_2^{-1} + b\alpha^{-2}t_1^{-1}] \perp \langle d\rangle\,. \end{aligned}$$

Claim. The 2-fold Pfister form $\pi := \langle\langle bd, bt_2^{-1} + b\alpha^{-2}t_1^{-1}]]$ is not defined over E.

This claim implies that the form $\psi := b[1, bt_2^{-1} + b\alpha^{-2}t_1^{-1}] \perp \langle d\rangle$ is anisotropic and not defined over E. This proves that $(\varphi_{E(\alpha)})_{\mathrm{an}} \simeq \psi$ is not defined over E. So it remains to prove the claim.

Suppose that $\left\langle\left\langle bd, bt_2^{-1} + b\alpha^{-2}t_1^{-1}\right]\right]$ is defined over E. Then, there exists $\theta := x[1, y] \perp [1, z]$ an E-quadratic form such that $\pi \simeq \theta_{E(\alpha)}$. Comparing Arf invariants yields $\Delta(\theta_{E(\alpha)}) = 0$. Lemma 2.9 implies that $\Delta(\theta_{E(\beta)}) = 0$. Hence, $\Delta(\theta) \in \{\wp(E), b^{-2}d + \wp(E)\}$ because $M = F(\beta) = F(\wp^{-1}(b^{-2}d))$. If necessary, we may change z by $z + b^{-2}d$ and suppose that $\theta \in GP_2(E)$. By roundness, we take $\theta \simeq \langle\langle x, y]]$. Now we have this isometry of 2-fold Pfister forms:

$$\langle\langle x, y]]_{E(\alpha)} \simeq \left\langle\left\langle bd, bt_2^{-1} + b\beta^{-1}t_1^{-1}\right]\right]_{E(\alpha)}$$

(recall that $\beta = \alpha^2$). Consequently, we get in $H_2^2(E(\alpha))$

$$\begin{aligned}\left(\overline{y\frac{\mathrm{d}x}{x}}\right)_{E(\alpha)} &= \left(\overline{(bt_2^{-1} + bt_1^{-1}\beta^{-1})\frac{\mathrm{d}(bd)}{bd}}\right)_{E(\alpha)} \\ &\overset{(*)}{=} \left(\overline{(b^2t_2^{-2} + b^2t_1^{-2}\beta^{-2})\frac{\mathrm{d}(bd)}{bd}}\right)_{E(\alpha)}.\end{aligned}$$

$(*)$: We apply the Frobenius operator.

Hence, there exist $u \in \Omega^2_{E(\alpha)}$ and $v \in \Omega^1_{E(\alpha)}$ such that

$$\left(y\frac{\mathrm{d}x}{x}\right)_{E(\alpha)} + \left((b^2t_2^{-2} + b^2t_1^{-2}\beta^{-2})\frac{\mathrm{d}(bd)}{bd}\right)_{E(\alpha)} = \wp(u) + \mathrm{d}v. \tag{5.3}$$

The extension $E(\alpha)/E(\beta)$ is inseparable quadratic. Hence, using the same arguments as in the proof of [3, Proposition 2], we may suppose that $u \in \Omega^2_{E(\beta)}$ and $v \in \Omega^1_{E(\beta)}$. This means that we have

$$\left(\overline{y\frac{\mathrm{d}x}{x}}\right)_{E(\alpha)} = \left(\overline{(b^2t_2^{-2} + b^2t_1^{-2}\beta^{-2})\frac{\mathrm{d}(bd)}{bd}}\right)_{E(\alpha)} \text{ in } H^2_{2,E(\beta)}(E(\alpha)). \tag{5.4}$$

Notice that $\mathrm{Tr}(\beta^{-2}) = b^2d^{-2}$. Now applying $\widetilde{\mathrm{Tr}}_*$ to equation (5.4), we get:

$$0 = \left(\overline{b^4t_1^{-2}d^{-2}\frac{\mathrm{d}(bd)}{bd}}\right)_{E(\sqrt{b})} = \left(\overline{b^2t_1^{-1}d^{-1}\frac{\mathrm{d}(bd)}{bd}}\right)_{E(\sqrt{b})} \text{ in } H_{2,E}(E(\sqrt{b})).$$

Using Kato's isomorphism (2.3) with the Hauptsatz, we get $\left\langle\left\langle bd, b^2t_1^{-1}d^{-1}\right]\right]_{E(\sqrt{b})} \sim 0$, that is, $[1, b^2t_1^{-1}d^{-1}]_{E(\sqrt{b})} \simeq bd[1, b^2t_1^{-1}d^{-1}]_{E(\sqrt{b})}$. Now taking the first residue form with respect to the t_1-adic valuation of $E(\sqrt{b})$, we get:

$$\langle 1\rangle_{F((t_2))(\sqrt{b})} \simeq \langle bd\rangle_{F((t_2))(\sqrt{b})} \simeq \langle d\rangle_{F((t_2))(\sqrt{b})}.$$

In particular, the form $\langle 1, d\rangle_{F(\sqrt{b})}$ is isotropic, which implies that $\langle 1, b, d\rangle$ is isotropic, a contradiction. □

6. Nonexcellence of separable biquadratic extensions

For completeness we mention the case of biquadratic separable extensions. In general, such extensions are not excellent for quadratic forms. The counterexample concerns dimension 4 and is essentially due to Sivatski [21]. He assumed the characteristic to be different from 2 but the construction goes through basically without any change to characteristic 2 with the help of a result of Rowen.

By [19, Section 3], there exists a field F of characteristic 2, an indecomposable F-algebra D of degree 8 and exponent 2. Moreover, by [19, Theorem 2] we have $M_2(D) \simeq \otimes_{i=1}^4 [a_i, b_i)$ with $a_i, b_i \in F^*$ such that D is split by the triquadratic separable extension $M := F(\wp^{-1}(a_1), \wp^{-1}(a_2), \wp^{-1}(a_3))$. Then, we have the following example showing the nonexcellence of biquadratic separable extensions:

Example 6.1. *Let $K = F(\wp^{-1}(a_1), \wp^{-1}(a_2))$ and consider the F-quadratic form $\varphi = b_4[1, a_4] \perp [1, a_3 + a_4]$. Then, we have:*

(1) φ is anisotropic over F.

(2) φ_K is isotropic.

(3) $(\varphi_K)_{\mathrm{an}}$ is not defined over F.

In particular, K/F is not $(2, 0)$-excellent.

Proof. Let D, M be as mentioned above.

(1) Suppose that φ is isotropic over F. Then, there exist $x \in F^*$ such that $b_4[1, a_4] \simeq x[1, a_4]$ and $[1, a_3 + a_4] \simeq x[1, a_3 + a_4]$. Hence, $[a_4, b_4) \simeq [a_4, x)$ and $[a_3 + a_4, x) \sim 0$, that is, $[a_3, x) \simeq [a_4, x)$. Thus,

$$\begin{aligned} M_2(D) &\simeq [a_1, b_1) \otimes [a_2, b_2) \otimes [a_3, b_3) \otimes [a_4, b_4) \\ &\simeq [a_1, b_1) \otimes [a_2, b_2) \otimes [a_3, b_3) \otimes [a_4, x) \\ &\simeq [a_1, b_1) \otimes [a_2, b_2) \otimes [a_3, b_3) \otimes [a_3, x) \\ &\simeq [a_1, b_1) \otimes [a_2, b_2) \otimes M_2([a_3, b_3 x)) \\ &\simeq M_2([a_1, b_1) \otimes [a_2, b_2) \otimes [a_3, b_3 x)). \end{aligned}$$

This implies that $D \simeq [a_1, b_1) \otimes [a_2, b_2) \otimes [a_3, b_3 x)$, a contradiction to the hypothesis that D is indecomposable. Thus, φ is anisotropic over F.

(2) Recall that D is split by M, that is, $D_M \sim 0$. Thus $[a_4, b_4)_M = 0$, consequently $\varphi_M \sim ([1, a_4] \perp b_4[1, a_4])_M \sim 0$. It follows from Lemma 3.1 that φ_K is isotropic. Since $\triangle(\varphi_K) \neq 0$, we get $\dim(\varphi_K)_{\mathrm{an}} = 2$.

(3) Suppose that $(\varphi_K)_{\mathrm{an}}$ is defined over F. Then, there exist $\alpha \in F^*$ such that $(\varphi_K)_{\mathrm{an}} \simeq (\alpha[1, a_3])_K$. Hence, $(\varphi \perp \alpha[1, a_3])_K \sim 0$. It follows from a result of Baeza [7, Corollary 4.16] that

$$\varphi \perp \alpha[1, a_3] \sim \rho_1 \otimes [1, a_1] \perp \rho_2 \otimes [1, a_2], \tag{6.1}$$

for suitable bilinear forms ρ_1 and ρ_2. Taking the Clifford algebra in equation (6.1), we get

$$[a_4, b_4) \sim [a_3, \alpha) \otimes [a_1, r) \otimes [a_2, s),$$

where $r, s \in F^*$ satisfy $\det \rho_1 = rF^{*2}$ and $\det \rho_2 = sF^{*2}$. Substituting $[a_4, b_4)$ in $M_2(D)$ yields:

$$D \simeq [a_1, rb_1) \otimes [a_2, sb_2) \otimes [a_3, \alpha b_3),$$

and thus D is decomposable, a contradiction. □

7. Extending nonexcellence results to higher dimensional forms

The nonexcellence results that we established for quartic extensions concern quadratic forms of type $(2, 0)$ or $(2, 1)$. Based on a generic argument, we have that these extensions are also not (r, s)-excellent for larger r and s. In particular, Combining [18, Lemmas 4.5, 4.6], we have the following result:

Lemma 7.1. *Let L/F be a field extension which is not (r, s)-excellent. Let $x_1, \ldots, x_{2m+n}$ be variables over L, and let M (resp. N) be the field of iterated Laurent series $F((x_1)) \cdots ((x_{2m+n}))$ (resp. $L((x_1)) \cdots ((x_{2m+n}))$). Then, the extension N/M is not $(r + m, s + n)$-excellent.*

We summarize below what we know now for quartic extensions:

Corollary 7.2. *(1) Quartic extensions of type* 1 *are generally not* (r,s)*-excellent when* $r \geq 2$ *and* $s \geq 0$.
(2) Quartic extensions of type 2 *are generally not* (r,s)*-excellent when* $r \geq 2$ *and* $s \geq 1$.
(3) Quartic extensions of type 3 *are* (r,s)*-excellent for any* $r, s \geq 0$.
(4) Biquadratic separable extensions are generally not (r,s)*-excellent when* $r \geq 2$ *and* $s \geq 0$.

Proof. Statement (3) is given in [10]. For statements (1), (2) and (4) we combine Lemma 7.1 with Examples 4.1, 5.7 and 6.1, respectively. □

Now, we are left to address the excellence for type 4 extensions and the following question to completely classify inseparable quartic extensions:

Question 7.3. *Is it true that any quartic extension of type* 2 *is* $(2,0)$*-excellent?*

In contrast of what we did for quartic inseparable extensions of type 1, and in order to answer this question, it seems natural to study the isotropy in dimension 4 over quartic inseparable extensions of type 2.

The generic argument proposed above extends nonexcellence for same type of field extensions to higher dimensional quadratic forms. Naturally, one can pose the question if such an extension is also possible for higher degree field extensions. To be precise, we ask the following:

Question 7.4. *Given a tower of fields* $F \subseteq L \subseteq K$ *and a nonexcellent field extension* L/F*. Is it true in general that* K/F *is also not excellent?*

While the answer is trivially positive when K/L is purely transcendental or odd-degree algebraic extension, it is not apparent for non-trivial cases. In fact, we have the following example that gives a negative answer to the question posed above:

Example 7.5. *Let* t_1, t_2, t_3, t_4 *be independent variables over* $\mathbb{F}_2$ *(the finite field with two elements). Let* $\gamma = [t_1, t_2] \perp [t_3, t_4] \perp [1, t_1t_2 + t_3t_4]$ *be an Albert quadratic form over the rational function field* $F := \mathbb{F}_2(t_1, t_2, t_3, t_4)$*. Let* Q *be the totally singular quadratic form* $\langle 1, t_1, t_3\rangle$*. Then,* $F(Q)/F$ *is not* $(3,0)$*-excellent but* $F(Q)(\sqrt{t_1})/F$ *is excellent.*

Proof. Consider the tower of field extensions $F \subseteq F(Q) \subseteq F(Q)(\sqrt{t_1})$. The fact that the extension $F(Q)/F$ is not $(3,0)$-excellent was proved in [18, Proposition 4.3]. Since $Q_{F(\sqrt{t_1})} \simeq \langle 0, 1, t_3\rangle$, we get the following equalities:

$$F(Q)(\sqrt{t_1}) = F(\sqrt{t_1})(Q) = F(\sqrt{t_1})(\langle 0, 1, t_3\rangle) = F(x, y)(\langle\sqrt{t_1}, \sqrt{t_3}\rangle),$$

for some variables x, y over F. By the result of Hoffmann [10], we get $F(x, y)(\langle\sqrt{t_1}, \sqrt{t_3}\rangle)/F$ is excellent. Hence, $F(Q)(\sqrt{t_1})/F$ is excellent. □

The above example shows that it is possible to have an excellent field extension even if it contains a subextension that is not excellent. This example reiterates why the question of excellence is obstinate to study in general.

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