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On some questions about composition operators on weighted Hardy spaces

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Dedicated to Gilles Godefroy for his 70th birthday

May 18, 2024

Abstract. We first consider some questions raised by N. Zorboska in her thesis. In particular she asked for which sequences β every symbol $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ with $\varphi \in H^2(\beta)$ induces a bounded composition operator C_φ on the weighted Hardy space $H^2(\beta)$. We give partial answers and investigate when $H^2(\beta)$ is an algebra. We answer negatively another question in showing that there are a sequence β and $\varphi \in H^2(\beta)$ such that $\|\varphi\|_\infty < 1$ and the composition operator C_φ is not bounded on $H^2(\beta)$.

In a second part, we show that for $p \neq 2$, no automorphism of \mathbb{D} , except those that fix 0, induces a bounded composition operator on the Beurling-Sobolev space ℓ_A^p , and even on any weighted version of this space.

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1 Introduction

Let $\beta = (\beta_n)_{n \geq 0}$ be a sequence of positive numbers such that

$$(1.1) \quad \liminf_{n \rightarrow \infty} \beta_n^{1/n} \geq 1.$$

The associated weighted Hardy space $H^2(\beta)$ is the Hilbert space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that

$$(1.2) \quad \|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty,$$

if $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We shall denote $a_n = \widehat{f}(n)$. Note that condition (1.1) ensures that those functions are indeed analytic on \mathbb{D} , and conversely.

More generally, for $1 \leq p \leq \infty$, the weighted space $h^p(\beta)$ (called Beurling-Sobolev space in [3] and [10]) is the space of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk \mathbb{D} such that $\|f\|_{\beta}$ is finite, where

$$\|f\|_{\beta}^p := \sum_{n=0}^{\infty} |a_n|^p \beta_n.$$

for $1 \leq p < \infty$, and $\|f\|_{\infty} = \sup_{n \geq 0} |a_n| \beta_n$. For $\beta \equiv 1$, we write simply h^p or ℓ_A^p .

Observe that h^2 is none other than the usual Hardy space H^2 , and that $h^2(\beta) = H^2(\beta)$. However, for $p \neq 2$, h^p has nothing to do with the usual Hardy space H^p , even if, for example, $H^p \subseteq h^{\infty}$. But in the spirit of our recent work [4], the case of h^p seems a natural one to consider.

Again, (1.1) is equivalent to the inclusion $h^p(\beta) \subseteq \mathcal{H}(\mathbb{D})$ and allows to treat the elements of $h^p(\beta)$ as analytic functions on \mathbb{D} .

Though most of our results hold for all $p \geq 1$, we first restrict ourselves to $p = 2$, and only consider the other exponents p in Section 3.

If $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ is a non-constant analytic self-map of \mathbb{D} , we say that φ is a *symbol*. To each symbol, we associate the *composition operator* C_{φ} defined as $C_{\varphi}(f) = f \circ \varphi$ for every analytic function $f: \mathbb{D} \rightarrow \mathbb{C}$.

In her thesis, N. Zorboska [11] raised the following problems.

- 1) Determine the spaces $H^2(\beta)$ for which every symbol φ , with $\varphi \in H^2(\beta)$, induces a bounded composition operator on $H^2(\beta)$ (Problem 2).
- 2) If $\varphi \in H^2(\beta)$ and $\|\varphi\|_{\infty} < 1$, is the composition operator C_{φ} necessarily bounded on $H^2(\beta)$? (Problem 3).

In Section 2, we give partial answers to the first problem (Theorem 2.2 and Theorem 2.7), and a negative one to the second problem (Theorem 2.11).

In Section, 3, we show that for $p \neq 2$, no automorphism T_a of \mathbb{D} (to be defined right after), with $a \neq 0$, induces a bounded composition operator on the weighted Hardy space $h^p(\beta)$, whatever β .

After the first version of this paper was completed, we learned the existence of [7] and [3]. This version takes these papers into account. We thank O. El-Fallah for sending us a copy of the english version of [3]. We thank also N. Nikolski who learned us that [8, §3.2] deals with the question of weighted algebras, and that this question goes back to [5] (without proof), [6], and [7] (without proof).

2 Answers to Zorboska's questions

2.1 A necessary condition

In [4], we characterized the weights β for which all composition operators are bounded on $H^2(\beta)$: this happens if and only if β is *essentially decreasing*

and *slowly oscillating*.

Definition 2.1. A sequence of positive numbers $\beta = (\beta_n)_{n \geq 0}$ is said to be

1) essentially decreasing if, for some constant $C \geq 1$, we have

$$(2.1) \quad \beta_m \leq C \beta_n \quad \text{for all } m \geq n \geq 0;$$

2) slowly oscillating if there are positive constants $c < 1 < C$ such that

$$(2.2) \quad c \beta_n \leq \beta_m \leq C \beta_n \quad \text{when } n/2 \leq m \leq 2n.$$

Note that if β is slowly oscillating, then $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$.

First, for the first Zorboska's question, we have the following necessary condition.

Theorem 2.2. Assume that $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$.

If every symbol $\varphi \in H^2(\beta)$ induces a bounded composition operator on $H^2(\beta)$, then β is slowly oscillating.

Proof. For all $a \in \mathbb{D}$, we consider the automorphism T_a defined by

$$(2.3) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}.$$

Let us point out that for every integer $n \geq 1$, we have

$$\widehat{T_a}(n) = (-1)^{n-1} \bar{a}^{n-1} (1 - |a|^2).$$

Therefore, all the symbols T_a are in $H^2(\beta)$ if and only if for every $a \in \mathbb{D}$, we have $\sum \beta_n |a|^{2n} < \infty$, which is equivalent to the fact that the Taylor series $\sum \beta_n z^n$ has a radius of convergence at least 1 i.e. $\limsup \beta_n^{1/n} \leq 1$.

Since $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$, by hypothesis, the preceding remark implies that all the symbols T_a define bounded composition operators on $H^2(\beta)$, by the hypothesis. By [4, Theorem 4.9], it follows that β is slowly oscillating. \square

2.2 $H^2(\beta)$ as an algebra

In order to motivate the content of this section, we point out the following easy fact.

Proposition 2.3. Assume that every symbol $\varphi \in H^2(\beta)$ induces a bounded composition operator on $H^2(\beta)$. Then $H^2(\beta) \cap H^\infty$ is an algebra.

Proof. It suffices to show that $f^2 \in H^2(\beta) \cap H^\infty$ for every $f \in H^2(\beta) \cap H^\infty$. Let $f \in H^2(\beta) \cap H^\infty$, not constant. If $M > \|f\|_\infty$, then $\varphi = f/M$ is a symbol. With $e_n(z) = z^n$, we have, by hypothesis, $\varphi^n = C_\varphi(e_n) \in H^2(\beta)$. Hence $f^n \in H^2(\beta)$. Since it is clear that $f^n \in H^\infty$, we are done. \square

Remark. The conclusion of Proposition 2.3 can be obtained with a hypothesis of a different nature.

Proposition 2.4. *If $(\beta_n/n^2)_{n \geq 1}$ is equivalent to a sequence of moments, then $H^2(\beta) \cap H^\infty$ is an algebra.*

Proof. By hypothesis, $\beta_n \approx n^2 \gamma_n$, where $\gamma_n = \int_0^1 t^{n-1} d\lambda(t)$ and where λ is a non negative finite measure on $[0, 1]$. Let μ be the image of λ by the square-root function $r \mapsto r^{1/2}$ on $[0, 1]$ and ν the non negative measure on \mathbb{D} defined by

$$\int_{\mathbb{D}} h(z) d\nu(z) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 h(r e^{i\theta}) d\theta d\mu(r).$$

We have, for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\beta)$:

$$\begin{aligned} I(f) &:= \int_{\mathbb{D}} |f'(z)|^2 d\nu(z) = \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 r^{2n-2} d\mu(r) \\ &= \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 t^{n-1} d\lambda(t) \approx \sum_{n=1}^{\infty} |a_n|^2 \beta_n. \end{aligned}$$

Now, we observe that $\|f\|_{H^2(\beta)}^2 \approx |f(0)|^2 + I(f)$ and that

$$I(f^2) = \int_{\mathbb{D}} 4 |f(z)|^2 |f'(z)|^2 d\nu(z) \leq 4 \|f\|_{\infty}^2 I(f),$$

which clearly gives the result. \square

A specific example (the Dirichlet space case corresponds to $d\lambda(t) = \omega(t) dt = dt$ and $\gamma_n = 1/n$, or to $\alpha = 0$) is:

$$\beta_0 = \beta_1 = 1 \quad \text{and} \quad \beta_n = n (\log n)^\alpha, \quad 0 \leq \alpha \leq 1, \quad \text{for } n \geq 2,$$

for which $\sum_{n=0}^{\infty} \frac{1}{\beta_n} = +\infty$. Indeed, if $d\lambda(t) = \omega(t) dt$ with

$$\omega(t) = \left(\log^+ \left(\frac{1}{\log(1/t)} \right) \right)^\alpha, \quad 0 \leq t \leq 1,$$

we have, making the changes of variables $t = e^{-x}$ and $x = y/n$:

$$\gamma_n = \int_0^1 t^{n-1} \omega(t) dt = \frac{1}{n} \int_0^n e^{-y} \left(\log \frac{n}{y} \right)^\alpha dy \sim \frac{(\log n)^\alpha}{n}.$$

When β is a bounded sequence, it is easy to see that $H^\infty \subseteq H^2 \subseteq H^2(\beta)$ (see [12, Proposition 2], for instance).

Proposition 2.5. *Assume that β is a non-increasing sequence. Then $H^2(\beta)$ is an H^∞ -module.*

Proof. First of all, since β is then bounded, we have $H^\infty \subseteq H^2(\beta)$. Moreover, the shift $S: H^2(\beta) \rightarrow H^2(\beta)$, defined as $(Sf)(z) = zf(z)$, is a contraction.

By von Neumann's inequality, we have $\|P(S)\| \leq \|P\|_\infty$ for all polynomials P . Since $P(S)f = Pf$, it follows, by approximation, that

$$\|gf\|_{H^2(\beta)} \leq \|g\|_\infty \|f\|_{H^2(\beta)}$$

for all $g \in H^\infty$ and all $f \in H^2(\beta)$, and that proves that $H^2(\beta)$ is an H^∞ -module. \square

We will now study when $H^2(\beta)$ is an algebra, and we begin with an elementary remark.

Proposition 2.6. *Assume that $H^2(\beta)$ is an algebra. Then*

(i) $\sup_{n \geq 0} \beta_{n+1}/\beta_n < +\infty$.

(ii) the sequence $(\beta_n^{1/n})_{n \geq 1}$ has a limit.

Proof. The shift, which associates to $f \in H^2(\beta)$ the function $z \mapsto zf(z)$, is bounded from $H^2(\beta)$ into $H^2(\beta)$. That means that $\sup_{n \geq 0} \beta_{n+1}/\beta_n < +\infty$ and (i) is proved.

Since $H^2(\beta)$ is an algebra, there exists a positive constant C such that $\|fg\| \leq C\|f\|\|g\|$ for all $f, g \in H^2(\beta)$. Applying that to $f(z) = e_m(z) = z^m$ and $g(z) = e_n(z) = z^n$, we get that, for all $m, n \geq 0$:

$$(2.4) \quad \beta_{m+n} = \|e_{m+n}\| = \|e_m e_n\| \leq C \|e_m\| \|e_n\| = C \beta_m \beta_n.$$

Setting $\alpha_n = C \beta_n$, that means that $\alpha_{m+n} \leq \alpha_m \alpha_n$ for all $m, n \geq 0$, i.e. the sequence $(\alpha_n)_{n \geq 0}$ is submultiplicative.

Now (ii) follows from the multiplicative version of Fekete's lemma (see the proof of the spectral radius formula). \square

Remark. Given $q > 1$ and writing $\tilde{\beta}_n = q^n \beta_n$, we have:

$$(2.5) \quad H^2(\beta) \text{ is an algebra if and only if } H^2(\tilde{\beta}) \text{ is an algebra.}$$

Indeed, it is clear that $f \in H^2(\tilde{\beta})$ if and only if $g \in H^2(\beta)$ where $g(z) = f(\sqrt{q}z)$.

As a consequence of this remark, we have the following partial answer of the first question raised in the Introduction.

Theorem 2.7. *Assume that $H^2(\beta)$ is an algebra. Let φ be a symbol which is in $H^2(\beta)$. Then φ induces a bounded composition operator on $H^2(\beta)$ in the following cases*

- 1) when $\liminf_{n \rightarrow \infty} \beta_n^{1/n} > 1$;
- 2) when $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$ and $\varphi(0) = 0$;
- 3) when β is slowly oscillating.

Recall that the slow oscillation of β implies that $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$.

Remark. We will see in Corollary 2.11 that we can have $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$ without the boundedness of all the composition operators on $H^2(\beta)$ with symbol in $H^2(\beta)$.

Proof. 1) Since $\liminf_{n \rightarrow \infty} \beta_n^{1/n} > 1$, the functions in $H^2(\beta)$ are actually analytic in a disk $D(0, R)$ containing the closed unit disk $\overline{\mathbb{D}}$. Since $H^2(\beta)$ is an algebra, we have $H^2(\beta) = \mathcal{M}(H^2(\beta))$ (see (2.6)). By [9, Proposition 20 and Corollary 1 of Proposition 31], the spectrum of each $\varphi \in H^2(\beta)$, as an element of the algebra $H^2(\beta) = \mathcal{M}(H^2(\beta))$ is $\overline{\varphi(\mathbb{D})}$. If φ is moreover a symbol, we have $\overline{\varphi(\mathbb{D})} \subseteq \overline{\mathbb{D}}$. By the analytic functional calculus, we have $f(\varphi) \in H^2(\beta)$ for every function f analytic in an open neighborhood of $\overline{\mathbb{D}}$. In particular, $f \circ \varphi = f(\varphi) \in H^2(\beta)$ for every $f \in H^2(\beta)$.

2) Let $f \in H^2(\beta)$. Let $q > 1$ and set $\tilde{\beta}_n = q^n \beta_n$; then $\liminf_{n \rightarrow \infty} \tilde{\beta}_n^{1/n} > 1$. We set $\tilde{\varphi}(z) = \sqrt{q} \varphi(z/\sqrt{q})$; we have $\tilde{\varphi} \in H^2(\tilde{\beta})$ and, since $\varphi(0) = 0$, $\tilde{\varphi}$ maps \mathbb{D} into \mathbb{D} , by the Schwarz lemma, so $\tilde{\varphi}$ is a symbol. We set now $g(z) = f(z/\sqrt{q})$; then $g \in H^2(\tilde{\beta})$ and, by 1), we have $g \circ \tilde{\varphi} \in H^2(\tilde{\beta})$. But

$$g[\tilde{\varphi}(z)] = f[\varphi(z/\sqrt{q})],$$

and saying that $g \circ \tilde{\varphi} \in H^2(\tilde{\beta})$ is equivalent to saying that $f \circ \varphi \in H^2(\beta)$. Hence C_φ is bounded on $H^2(\beta)$.

3) By the parts 1) and 2), all the symbols $\varphi \in H^2(\beta)$ with $\varphi(0) = 0$ induce a bounded composition operator on $H^2(\beta)$, whatever β . Now, if β is slowly oscillating, by [4, Theorem 4.6], all the automorphisms T_a induce bounded composition operators on $H^2(\beta)$. It follows, as in 1), because T_a is analytic in an open neighborhood of \mathbb{D} , that then all symbols $\varphi \in H^2(\beta)$ induce bounded composition operators (if $a = -\varphi(0)$, then $\psi = T_a \circ \varphi = \sum_{n=0}^{\infty} \widehat{T}_a(n) \varphi^n$ is in $H^2(\beta)$ and $\psi(0) = 0$; the result follows since $\varphi = T_{-a} \circ \psi$ and $C_\varphi = C_\psi \circ C_{T_{-a}}$). \square

The space $\mathcal{M}(H^2(\beta))$ of multipliers of $H^2(\beta)$ is, by definition, the set of all analytic functions h on \mathbb{D} such that $hf \in H^2(\beta)$ for all $f \in H^2(\beta)$. It is easy to see (see [4, beginning of Section 6]) that $\mathcal{M}(H^2(\beta)) \subseteq H^\infty$. Actually, we have

$$\mathcal{M}(H^2(\beta)) \subseteq H^2(\beta) \cap H^\infty.$$

Clearly $H^2(\beta) = \mathcal{M}(H^2(\beta))$ when $H^2(\beta)$ is an algebra; hence

$$(2.6) \quad \text{If } H^2(\beta) \text{ is an algebra, then } H^2(\beta) \subseteq H^\infty,$$

Proposition 2.8. *We have $H^2(\beta) \subseteq H^\infty$ if and only if $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$.*

Note that this last condition implies that $H^2(\beta) \subseteq W^+(\mathbb{D})$, where $W^+(\mathbb{D})$ is the Wiener algebra of all analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that $\sum_{n=0}^{\infty} |a_n| < +\infty$ if $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Proof. Assume that $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$. Then, for $f \in H^2(\beta)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have, by the Cauchy-Schwarz inequality:

$$\sum_{n=0}^{\infty} |a_n| \leq \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n \right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{1}{\beta_n} \right)^{1/2} < +\infty.$$

Hence $f \in W^+(\mathbb{D})$. Therefore $H^2(\beta) \subseteq W^+(\mathbb{D}) \subseteq A(\mathbb{D}) \subseteq H^\infty$.

Conversely, assume that $H^2(\beta) \subseteq H^\infty$. Let $(a_n)_{n \geq 0}$ be a sequence such that $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$. Then, setting $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have $f \in H^2(\beta)$. Now, if $g(z) = \sum_{n=0}^{\infty} |a_n| z^n$, we have $g \in H^2(\beta)$ and $\|g\| = \|f\|$. By hypothesis, we hence have $g \in H^\infty$. Then

$$\sup_{|z| < 1} \left| \sum_{n=0}^{\infty} |a_n| z^n \right| = \|g\|_\infty$$

implies that $\sum_{n=0}^{\infty} |a_n| \leq \|g\|_\infty$. We have proved that $\sum_{n=0}^{\infty} |a_n| < +\infty$ whenever $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$. That means that $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$. \square

Corollary 2.9. *If $H^2(\beta)$ is an algebra, then $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$.*

This corollary appeared already in [3, Theorem 3.2.7]. Moreover, [3, Theorem 3.2.7] states that the converse holds when β is “regular”, namely if β is log-concave, $\beta_n \geq 1$, $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$, and, either β_n/n^c eventually decreases for some $c > 0$, or either β_n/n^4 eventually increases and is log-concave. However, the converse does not hold in general, as the following result shows.

Theorem 2.10. *There exists a sequence β such that $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$, and for which:*

- 1) $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$; so $H^2(\beta) \subseteq H^\infty$;
- 2) $H^2(\beta)$ is not an algebra.

Proof. Take $m_k = 3^k$ and

$$\begin{cases} \beta_{m_k} &= m_k^2 \\ \beta_{2m_k} &= m_k^5, \end{cases}$$

and

$$\beta_{n+1} = \begin{cases} m_k^{3/m_k} \beta_n & \text{for } m_k \leq n \leq 2m_k - 1; \\ 9^{1/m_k} m_k^{-3/m_k} \beta_n & \text{for } 2m_k \leq n \leq m_{k+1} - 1. \end{cases}$$

Since $\beta_n \gtrsim n^2$, we have $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$. Since $m_k^{1/m_k} \xrightarrow[k \rightarrow \infty]{} 1$, we have $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$. Moreover, setting $e_n(z) = z^n$, we have

$$\frac{\|e_{m_k}^2\|}{\|e_{m_k}\|^2} = \frac{\|e_{2m_k}\|}{\|e_{m_k}\|^2} = \frac{\beta_{2m_k}^{1/2}}{\beta_{m_k}} = m_k^{1/2} = 3^{k/2} \xrightarrow[k \rightarrow \infty]{} +\infty;$$

hence $H^2(\beta)$ is not an algebra, since otherwise, we would have a positive constant C for which $\|fg\| \leq C\|f\|\|g\|$ for all $f, g \in H^2(\beta)$. \square

Theorem 2.10 allows to answer negatively Zorboska's question 2).

Theorem 2.11. *There exist a sequence β such that $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ and $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$, and a symbol φ such that $\varphi \in H^2(\beta)$ and $\|\varphi\|_{\infty} < 1$, but C_{φ} is not bounded on $H^2(\beta)$.*

Proof. We use the sequence obtained in Theorem 2.10. There exists $\varphi \in H^2(\beta) \subset H^{\infty}$ such that $\varphi^2 \notin H^2(\beta)$; we can assume that $\|\varphi\|_{\infty} < 1$. Clearly C_{φ} is not bounded on $H^2(\beta)$ since $z^2 \in H^2(\beta)$ and $C_{\varphi}(z^2) = \varphi^2$. \square

2.3 Slowly oscillating sequences

In Corollary 2.9, we saw that $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < \infty$ is necessary for $H^2(\beta)$ to be an algebra, and in [3, Theorem 3.2.7], it is shown that the converse holds when β satisfies some conditions. Here, we see that the converse holds with another regularity condition on β , namely when β is slowly oscillating (recall that we saw in Theorem 2.2, assuming that $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$, that β slowly oscillating is necessary in order that every symbol $\varphi \in H^2(\beta)$ induces a bounded composition operator on $H^2(\beta)$).

Theorem 2.12. *If $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ and β is slowly oscillating, then $H^2(\beta)$ is an algebra.*

Using Theorem 2.7, we get the following corollary.

Theorem 2.13. *If $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ and β is slowly oscillating, then all symbols $\varphi \in H^2(\beta)$ induce bounded composition operators on $H^2(\beta)$.*

Theorem 2.12 can be deduced from the condition (3.6.1) of [3, Lemma 3.6.3], but we give a direct proof for sake of completeness. We introduce the following notation:

$$(2.7) \quad B_n = \sum_{k=0}^n \frac{1}{\beta_k \beta_{n-k}}.$$

Note that $B_n \geq \frac{1}{\beta_0 \beta_n}$ so $\inf_{n \geq 0} B_n \beta_n > 0$.

The proof will follow from the next two lemmas.

Lemma 2.14. *If $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ and β is slowly oscillating, then $\beta_n B_n = O(1)$.*

Proof. Set $M = \sum_{n=0}^{\infty} \frac{1}{\beta_n}$. We have

$$B_n \leq 2 \sum_{0 \leq k \leq n/2} \frac{1}{\beta_k \beta_{n-k}}.$$

But $\frac{1}{\beta_{n-k}} \leq C \frac{1}{\beta_n}$, since $n/2 \leq n-k \leq n$ for $0 \leq k \leq n/2$, and β is slowly oscillating. Hence

$$B_n \leq 2C \frac{1}{\beta_n} \sum_{0 \leq k \leq n/2} \frac{1}{\beta_k} \leq 2CM \frac{1}{\beta_n}. \quad \square$$

The second lemma is nothing else than the condition (3.2.2) of [3] (see also [7, Lemma, page 207]).

Lemma 2.15. *If $\beta_n B_n = O(1)$, then $H^2(\beta)$ is an algebra.*

Proof. It suffices to show that $f^2 \in H^2(\beta)$ for all $f \in H^2(\beta)$. Let $f \in H^2(\beta)$ and write $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We have

$$[f(z)]^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k a_{n-k} \right) z^n.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_k a_{n-k} \right|^2 \beta_n &\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^n |a_k a_{n-k}|^2 \beta_k \beta_{n-k} \right) \left(\sum_{k=0}^n \frac{1}{\beta_k \beta_{n-k}} \right) \beta_n \\ &\lesssim \sum_{n=0}^{\infty} \left(\sum_{k=0}^n |a_k a_{n-k}|^2 \beta_k \beta_{n-k} \right) \\ &= \left(\sum_{j=0}^{\infty} |a_j|^2 \beta_j \right)^2 = \|f\|^4 < +\infty, \end{aligned}$$

which says that $f^2 \in H^2(\beta)$. \square

As said, when β is “regular”, $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ is also necessary in order that $H^2(\beta)$ be an algebra ([3, Theorem 3.2.7]). In general, we have no necessary and sufficient condition. However, we can state the following necessary conditions.

Proposition 2.16. *If $H^2(\beta)$ is an algebra, then*

- 1) $\beta_n B_n = O(n)$;
- 2) $\sum_{n=0}^{\infty} \beta_n B_n^2 < +\infty$.

Proof. 1) By (2.4), we have $\frac{1}{\beta_k \beta_l} \leq C \frac{1}{\beta_{k+l}}$; so

$$B_n = \sum_{k+l=n} \frac{1}{\beta_k \beta_l} \leq Cn \frac{1}{\beta_n}.$$

2) Let $f(z) = \sum_{n=0}^{\infty} \frac{1}{\beta_n} z^n$; we have $\|f\|^2 = \sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$, so $f \in H^2(\beta)$. Since $H^2(\beta)$ is an algebra, we have $f^2 \in H^2(\beta)$. But $[f(z)]^2 = \sum_{n=0}^{\infty} B_n z^n$, so we get $\sum_{n=0}^{\infty} B_n^2 \beta_n < +\infty$. \square

The necessary conditions of Corollary 2.9 and of Proposition 2.16 are not sufficient.

Proposition 2.17. *There exists a sequence β such that $\sum_{n=0}^{\infty} \beta_n B_n^2 < +\infty$, hence $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$, but for which $H^2(\beta)$ is not an algebra.*

Proof. We take $\beta_0 = 1$, and, for $n \geq 1$, $\beta_n = n^\gamma$ for n even and $\beta_n = n^{\gamma'}$ for n odd, where $1 < \gamma' < \gamma$ and $2\gamma' > \gamma + 1$ (for example, $\gamma' = 3$ and $\gamma = 4$). Since β_{n+1}/β_n is not bounded, $H^2(\beta)$ is not an algebra. It is clear that $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$.

Now

$$\begin{aligned} B_{2n} &= \sum_{k=0}^{2n} \frac{1}{\beta_k \beta_{2n-k}} \approx \sum_{k \text{ even}} \frac{1}{k^\gamma (2n-k)^\gamma} + \sum_{k \text{ odd}} \frac{1}{k^{\gamma'} (2n-k)^{\gamma'}} \\ &\lesssim \frac{1}{n^\gamma} + \frac{1}{n^{\gamma'}} \lesssim \frac{1}{n^{\gamma'}}, \end{aligned}$$

so $(2n)^\gamma B_{2n}^2 \lesssim (2n)^{\gamma-2\gamma'}$.

Similarly $(2n+1)^{\gamma'} B_{2n+1}^2 \lesssim (2n+1)^{-\gamma'}$.

Since $\gamma - 2\gamma' < -1$ and $\gamma' > 1$, we obtain the convergence of the series $\sum_{n \geq 0} \beta_n B_n^2$. \square

Nevertheless, β slowly oscillating is not necessary for $H^2(\beta)$ to be an algebra.

Example 2.18. *If $\beta_n = e^{\sqrt{n}}$ for all $n \geq 0$, then $H^2(\beta)$ is an algebra, though β is not slowly oscillating.*

That β is not slowly oscillating is clear. That $H^2(\beta)$ is an algebra, follows from [3, Lemma 3.5.5].

As a consequence, using Theorem 2.2, we get the following result.

Theorem 2.19. *There exist a sequence β such that $H^2(\beta)$ is an algebra, and a symbol $\varphi \in H^2(\beta)$ for which C_φ is not bounded on $H^2(\beta)$.*

2.4 Hankel matrices

Actually the problem to know whether $H^2(\beta)$ is an algebra or not can be formulated in terms of Schur multipliers acting on a family of Hankel matrices.

Theorem 2.20. *$H^2(\beta)$ is an algebra if and only if the map*

$$\begin{aligned} \Psi: \ell^2 &\longrightarrow B(\ell^2) \\ u &\longmapsto \left(u_{k+l} \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}} \right)_{k,l} \end{aligned}$$

is bounded.

Let us point out that Lemma 2.15 also follows from this theorem. Actually, the condition $\beta_n B_n = O(1)$ is equivalent to the fact that Ψ is bounded from ℓ^2 to $HS(\ell^2) \subseteq B(\ell^2)$, where $HS(\ell^2)$ stands for the space of Hilbert-Schmidt operators on ℓ^2 .

Indeed, for every $u \in \ell^2$, we have $\|\Psi(u)\|_{HS}^2 = \sum_n |u_n|^2 \beta_n B_n$.

This remark gives a hint on the gap between the sufficient condition $\beta_n B_n = O(1)$ and a potential characterization.

Proof of Theorem 2.20. The vector space $H^2(\beta)$ is an algebra if and only if $fg \in H^2(\beta)$ for all $f, g \in H^2(\beta)$. In other words, $H^2(\beta)$ is an algebra if and only if

$$\sum_{n=0}^{\infty} \left| \sum_{p=0}^n a_p b_{n-p} \right|^2 \beta_n < +\infty$$

whenever $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$ and $\sum_{n=0}^{\infty} |b_n|^2 \beta_n < +\infty$. Equivalently,

$$\sup_{\substack{a, b \in \ell^2 \\ \|a\| = \|b\| = 1}} \sum_{n=0}^{\infty} \left| \sum_{k+l=n} \frac{a_k}{\sqrt{\beta_k}} \frac{b_l}{\sqrt{\beta_l}} \sqrt{\beta_{k+l}} \right|^2 < +\infty$$

or, in other words,

$$\sup_{\substack{a, b, u \in \ell^2 \\ \|a\| = \|b\| = \|u\| = 1}} \left| \sum_{n=0}^{\infty} \sum_{k+l=n} a_k b_l u_n \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}} \right| < +\infty.$$

Let us point out that

$$\sum_{n=0}^{\infty} \sum_{k+l=n} a_k b_l u_n \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}} = \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} M_{k,l} b_l = \sum_{k=0}^{\infty} a_k \left(\Psi(u)(b) \right)_k$$

where $M_{k,l} = u_{k+l} \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}}$.

Therefore, $H^2(\beta)$ is an algebra if and only if

$$\sup_{\substack{b, u \in \ell^2 \\ \|b\| = \|u\| = 1}} \left\| \Psi(u)(b) \right\| < +\infty.$$

and the result follows. \square

As an application of Theorem 2.20, we see that the equivalence between “ $H^2(\beta)$ is an algebra” and $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ does not follow from the submultiplicativity of β .

Theorem 2.21. *There exists a submultiplicative sequence β such that $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$ and $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$, and for which $H^2(\beta)$ is not an algebra.*

Proof. Let $m_k = 3^k$ and $\sigma(n)$ defined by $\sigma(0) = 1$ and

$$\sigma(n) = k \quad \text{for } m_{k-1} \leq n < m_k \text{ and } k \geq 1.$$

We define

$$\beta_n = \exp(n/\sigma(n)).$$

- The submultiplicativity of β is obvious since σ is nondecreasing:

$$\frac{m+n}{\sigma(m+n)} = \frac{m}{\sigma(m+n)} + \frac{n}{\sigma(m+n)} \leq \frac{m}{\sigma(m)} + \frac{n}{\sigma(n)}.$$

- That $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$ is clear.
- That $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ is easy: since $m_k - m_{k-1} = 2m_k/3$, we get using geometric series:

$$\begin{aligned} \sum_{n=0}^{\infty} \exp[-n/\sigma(n)] &= \sum_{k=1}^{\infty} \left(\sum_{m_{k-1} \leq n < m_k} \exp(-n/k) \right) \lesssim \sum_{k=1}^{\infty} \frac{\exp(-2m_k/3)}{1 - e^{-1/k}} \\ &\lesssim \sum_{k=1}^{\infty} k \exp(-2m_k/3) < +\infty. \end{aligned}$$

- We give two proofs that $H^2(\beta)$ is not an algebra.

– *First proof.* Let A_n be the $(n \times n)$ self-adjoint matrix with all entries equal to 1, which satisfies $A_n^2 = nA_n$. Hence $\|A_n\|^2 = \|A_n^2\| = n\|A_n\|$ and so $\|A_n\| = n$.

Let us be more specific. Consider the interval of integers $I_k = [m_k/3, m_k/2)$ and let $u = \mathbf{1}_{2I_k} = \mathbf{1}_{[2m_k/3, m_k)} \in \ell^2$. We have $\|u\|_{\ell^2} \approx \sqrt{m_k}$.

Now, let $\Psi(u)$ be the matrix $\left(u_{i+j} \sqrt{\frac{\beta_{i+j}}{\beta_i \beta_j}} \right)_{i,j}$. We recall that, with obvious notations:

$$|a_{i,j}| \leq b_{i,j} \quad \implies \quad \|A\| \leq \|B\|.$$

Therefore, since $i, j \in I_k \Rightarrow i+j \in 2I_k$:

$$\begin{aligned} \|\Psi(u)\| &\geq \left\| \left(u_{i+j} \sqrt{\frac{\beta_{i+j}}{\beta_i \beta_j}} \right)_{i,j \in I_k} \right\| = \left\| \left(\sqrt{\frac{\beta_{i+j}}{\beta_i \beta_j}} \right)_{i,j \in I_k} \right\| \\ &= \|(1)_{i,j \in I_k}\| = |I_k| \approx m_k. \end{aligned}$$

Indeed, for $i, j \in I_k$, we have $i, j, i+j \in [m_k/3, m_k) = [m_{k-1}, m_k)$, so that

$$\beta_{i+j} = \exp[(i+j)/k] = \exp(i/k) \exp(j/k) = \beta_i \beta_j,$$

and we use the norm of the matrix $A_{|I_k|}$.

Hence the quotient $\frac{\|\Psi(u)\|_{B(\ell^2)}}{\|u\|_{\ell^2}}$ is unbounded as $k \rightarrow \infty$.

– *Second proof.* Actually, this proof was obtained thanks to the first one. Let $f = f_k$ be the function defined by

$$f(z) = \sum_{m_k/3 \leq n < m_k/2} z^n = \sum_{n \in I_k} z^n.$$

We are going to prove that

$$(2.8) \quad \frac{\|f^2\|}{\|f\|^2} \xrightarrow{k \rightarrow \infty} +\infty.$$

For that, we first prove that

$$(2.9) \quad \|f\|^2 \lesssim k \exp\left(\frac{m_k}{2k}\right).$$

Indeed, setting again $I_k = [m_k/3, m_k/2)$ and using $1/3 + 1/6 = 1/2$:

$$\|f\|^2 = \sum_{n \in I_k} \exp\left(\frac{n}{k}\right) = e^{m_k/3k} \sum_{0 \leq j \leq m_k/6} e^{j/k} \lesssim \frac{\exp\left(\frac{m_k}{2k}\right)}{\exp\left(\frac{1}{k}\right) - 1} \lesssim k \exp\left(\frac{m_k}{2k}\right).$$

It remains to show that

$$(2.10) \quad \|f^2\| \gtrsim k^{3/2} \exp\left(\frac{m_k}{2k}\right).$$

For that, setting $J_k = [2m_k/3, m_k) = 2I_k$, we have $f^2 = \sum_{n \in J_k} c_n z^n$, and

$$\|f^2\|^2 = \sum_{n \in J_k} c_n^2 \exp\left(\frac{n}{k}\right),$$

where c_n is the number of indices $i \in I_k$ such that $n - i \in I_k$, that is

$$\max(m_k/3, n - m_k/2) \leq i \leq \min(m_k/2, n - m_k/3).$$

When $(5/6)m_k \leq n < m_k$, this amounts to $n - m_k/2 \leq i < m_k/2$, implying $c_n \geq m_k - n$.

Setting $q = e^{1/k}$ and observing that $m_k^2 q^{-m_k/6} \xrightarrow{k \rightarrow \infty} 0$, we get

$$\|f^2\|^2 \geq \sum_{5/6 m_k \leq n < m_k} (m_k - n)^2 q^n = q^{m_k} \sum_{0 \leq j \leq m_k/6} j^2 q^{-j} \approx q^{m_k} (1 - q^{-1})^{-3},$$

that is $\|f^2\|^2 \gtrsim k^3 \exp(m_k/k)$, since $(1 - q^{-1})^{-1} \sim k$, and that proves (2.10). \square

Remark. Let us observe that, in the above proof, $\sigma(n)$ behaves, roughly speaking, as $\log n$, so that β_n behaves as $\gamma_n = \exp(n/\log n)$, although $H^2(\gamma)$ is an algebra, as said in the next proposition. The difference between $\beta_n = \exp[n/\sigma(n)]$, and γ_n , can be heuristically explained as follows. Set $\tau(x) = \frac{x}{\log x}$ for $x > 1$ and $\tilde{\tau}_n = n/\sigma(n)$. In the discrete case, we have

$$j, n \in [m_k/3, m_k/2] \implies \tilde{\tau}(n+j) - \tilde{\tau}(n) - \tilde{\tau}(j) = 0,$$

meaning that $\tilde{\tau}'' = 0$, while in the continuous case we have

$$\tau''(x) \sim \frac{-1}{x(\log x)^2} < 0.$$

Proposition 2.22 follows from [3, Theorem 3.2.7], but, for sake of completeness, we prefer to give a short elementary proof of this particular case.

Proposition 2.22. *For $\gamma(n) = e^{n/\log n}$, $n \geq 2$ (and $\gamma_0 = \gamma_1 = 1$, for instance), the space $H^2(\gamma)$ is an algebra.*

Proof. Let

$$\alpha_{n,k} = \frac{n}{\log n} - \frac{k}{\log k} - \frac{n-k}{\log(n-k)}.$$

We have to prove, using Lemma 2.15, that

$$\sup_n \left(\sum_{k=2}^{n/2} \exp(\alpha_{n,k}) \right) < +\infty.$$

Observe that, for $2 \leq k \leq \sqrt{n}$, we have

$$\alpha_{n,k} \leq \frac{n}{\log n} - \frac{k}{\log k} - \frac{n-k}{\log n} = k \left(\frac{1}{\log n} - \frac{1}{\log k} \right) \leq -\frac{k}{2 \log k};$$

hence

$$\sum_{k=2}^{\sqrt{n}} \exp(\alpha_{n,k}) \leq \sum_{k=2}^{\infty} \exp \left(-\frac{k}{2 \log k} \right) := S < +\infty.$$

For $\sqrt{n} \leq k \leq n/2$, we have, in the same way:

$$\begin{aligned} \alpha_{n,k} &\leq k \left(\frac{1}{\log n} - \frac{1}{\log k} \right) \leq \sqrt{n} \left(\frac{1}{\log n} - \frac{1}{\log n - \log 2} \right) \\ &= -\frac{\sqrt{n} \log 2}{(\log n)(\log n - \log 2)}; \end{aligned}$$

hence

$$\sum_{\sqrt{n} \leq k \leq n/2} \exp(\alpha_{n,k}) \leq \frac{n}{2} \exp \left(-\frac{\sqrt{n} \log 2}{(\log n)(\log n - \log 2)} \right) \xrightarrow{n \rightarrow \infty} 0. \quad \square$$

3 Composition operators induced by the automorphisms on $h^p(\beta)$

For $p = 2$, it is easy to check that all the composition operators C_{T_a} , for $a \in \mathbb{D}$, generated by the automorphisms T_a , defined in (2.3), (and actually all composition operators, by Littlewood's subordination principle) are bounded

on H^2 . We recently studied [4] the weighted version $H^2(\beta)$ of this space and gave a complete characterization of those weights $\beta = (\beta_n)$ for which either C_φ is bounded on $H^2(\beta)$ for all automorphisms, or for all symbols. In this section, we show that this is never the case for $p \neq 2$. We mention in passing that this boundedness issue was previously considered by Blyudze and Shimorin [1] in the case when the initial space is h^1 and h^p is the target space. In that case, the authors show that $C_{T_a} : h^1 \rightarrow h^p$ is bounded if and only if $p \geq 2$. This has been made much more precise in [10].

We will prove the following result.

Theorem 3.1. *Let $p \in [1, +\infty]$, with $p \neq 2$. Then C_{T_a} is unbounded on h^p for all $a \in \mathbb{D} \setminus \{0\}$.*

Moreover, C_{T_a} is never bounded on $h^p(\beta)$, whatever the choice of β satisfying (1.1).

S. Charpentier, N. Espouiller and R. Zarouf informed us that they proved, independently, Theorem 3.1 for $\beta \equiv 1$, using [10] (see [2]).

Proof. Set

$$I = \left[\frac{1}{2}, \frac{2}{3} \right] \quad \text{and} \quad J = [\alpha^{-1}, \alpha],$$

($\alpha = 5/4$ for instance).

As in [4, Proposition 4.2], the rotation invariance of $h^p(\beta)$ allows us to claim that, if C_{T_a} is bounded on $h^p(\beta)$ for some $a \in \mathbb{D}$, $a \neq 0$, then there exists a constant K such that

$$(3.1) \quad \|C_{T_a}\| \leq K, \quad \forall a \in I.$$

Now, everything will rely on the matrix $A = (a_{m,n})_{m,n}$, where

$$a_{m,n} = \widehat{(T_a)^n}(m) \left(\frac{\beta_m}{\beta_n} \right)^{1/p},$$

which represents C_{T_a} on the canonical (Schauder) basis of $h^p(\beta)$. If this matrix defines a bounded operator, its columns and rows (the columns of the transposed operator) are respectively uniformly bounded on ℓ^p and on the dual space ℓ^q (q the conjugate exponent of p), that is (with some R independent of $a \in I$):

$$(3.2) \quad C_n = \sum_{m=0}^{\infty} |a_{m,n}|^p \frac{\beta_m}{\beta_n} \quad \text{satisfies} \quad \sup_n C_n \leq R < +\infty,$$

and

$$(3.3) \quad L_m = \sum_{n=0}^{\infty} |a_{m,n}|^q \left(\frac{\beta_m}{\beta_n} \right)^{q/p} \quad \text{satisfies} \quad \sup_m L_m \leq R < +\infty.$$

We will show that, for $p \neq 2$, one of the necessary conditions (3.2) or (3.3) fails. We will hence separate two cases.

We need an auxiliary result. Recall first the following elementary lemma (see [4, Lemma 4.16]).

Lemma 3.2. Let $f: [A, B] \rightarrow \mathbb{R}$ be a \mathcal{C}^2 -function such that $|f'| \geq \delta$ and $|f''| \leq M$. Then

$$\left| \int_A^B e^{if(x)} dx \right| \leq \frac{2}{\delta} + \frac{M(B-A)}{\delta^2}.$$

This lemma implies the following extended version of [4, Proposition 4.12].

Proposition 3.3. If $r \in J = [\alpha^{-1}, \alpha]$ and $s \geq 1$, it holds, when $r = m/n$:

$$(3.4) \quad \int_I |\widehat{T}_a^n(m)|^s da \gtrsim \left(\int_I |\widehat{T}_a^n(m)| da \right)^s \geq \delta n^{-s/2}.$$

Proof. The first inequality in (3.4) is just Hölder's inequality. For the second one, we proved in [4, Proposition 4.14] that

$$(3.5) \quad |\widehat{T}_a^n(m)| \geq \delta n^{-1/2} |\cos(n\psi_r(a) + \pi/4)| + O(n^{-3/5}),$$

where $\psi_r = f$ satisfies the assumptions of Lemma 3.2, and the constant controlling the error term $O(n^{-3/5})$ does not depend on a . Moreover, we have the classical Fourier expansion

$$|\cos x| = c + \sum_{l=1}^{\infty} \delta_l \cos lx,$$

with $c > 0$ and $\delta_l = O(l^{-2})$. Hence (actually $\sum_{l=1}^{\infty} |\delta_l| < +\infty$ would suffice):

$$(3.6) \quad |\widehat{T}_a^n(m)| \geq \delta n^{-1/2} \left(c + \sum_{l=1}^{\infty} \delta_l \cos(l(n\psi_r(a) + \pi/4)) \right) + O(n^{-3/5}).$$

We now apply Lemma 3.2 with $f(a) = l(n\psi_r(a) + \pi/4)$. Here, for given l , we have $|f'| \geq \delta nl$ and $|f''| \leq Mnl$ on I , so that

$$\left| \int_I \cos(l(n\psi_r(a) + \pi/4)) da \right| \lesssim \left(\frac{1}{nl} + \frac{nl}{n^2 l^2} \right) \approx \frac{1}{nl}.$$

It now follows from (3.5) that

$$\begin{aligned} \int_I |\widehat{T}_a^n(m)| da &\geq \delta n^{-1/2} \left(c + O\left(\sum_{l=1}^{\infty} \frac{|\delta_l|}{nl} \right) \right) + O(n^{-3/5}) \\ &\geq \delta c n^{-1/2} + O(n^{-3/5}) \geq \delta' n^{-1/2}, \end{aligned}$$

and this ends the proof. \square

We now come back to the proof of Theorem 3.1. We will reason by contradiction, and separate two cases.

We set $J_l = [l/\sqrt{\alpha}, \sqrt{\alpha} l]$.

- Case $p < 2$.

It follows from (3.2) that, for $n \in J_l$, we have

$$\sum_{m \in J_l} |\widehat{T}_a^n(m)|^p \frac{\beta_m}{\beta_n} \leq R.$$

Integrating on I and using Proposition 3.3 give, since $m, n \in J_l$ (hence $m/n \in J$) have the same size as l :

$$l^{-p/2} \sum_{m \in J_l} \frac{\beta_m}{\beta_n} \lesssim R.$$

Now, summing up over $n \in J_l$ further gives

$$l^{-p/2} \left(\sum_{k \in J_l} \beta_k \right) \left(\sum_{k \in J_l} \beta_k^{-1} \right) \lesssim Rl.$$

But, by the Cauchy-Schwarz inequality:

$$l^2 \lesssim |J_l|^2 \leq \left(\sum_{k \in J_l} \beta_k \right) \left(\sum_{k \in J_l} \beta_k^{-1} \right),$$

and we get $l^{2-p/2} \lesssim Rl$ or $R \gtrsim l^{1-p/2}$.

Since $1 - p/2 > 0$, we have a contradiction for large l .

- *Case $p > 2$.*

The proof is nearly the same. It now follows from (3.3) that, for $m \in J_l$:

$$\sum_{n \in J_l} |\widehat{T}_a^n(m)|^q \left(\frac{\beta_m}{\beta_n} \right)^{q/p} \leq R.$$

Set $\gamma_k = \beta_k^{q/p}$ and proceed as in the case $p < 2$ to get

$$l^{-q/2} \left(\sum_{k \in J_l} \gamma_k \right) \left(\sum_{k \in J_l} \gamma_k^{-1} \right) \lesssim Rl,$$

or, again by the Cauchy-Schwarz inequality, $l^{2-q/2} \lesssim Rl$ or $R \gtrsim l^{1-q/2}$.

Since this time $1 - q/2 > 0$, we have again a contradiction for large l . \square

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