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On some questions about composition operators on weighted Hardy spaces

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Dedicated to Gilles Godefroy for its 70th birthday

October 31, 2023

Abstract. We first consider some questions raised by N. Zorboska in her thesis. In particular she asked for which sequences β every symbol $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ with $\varphi \in H^2(\beta)$ induces a bounded composition operator C_φ on the weighted Hardy space $H^2(\beta)$. We give partial answers and investigate when $H^2(\beta)$ is an algebra. We answer negatively to another question in showing that there are a sequence β and $\varphi \in H^2(\beta)$ such that $\|\varphi\|_\infty < 1$ and the composition operator C_φ is not bounded on $H^2(\beta)$.

In a second part, we show that for $p \neq 2$, no automorphism of \mathbb{D} , except those that fix 0, induces a bounded composition operator on the Beurling-Sobolev space ℓ_A^p , and even on the weighted versions of this space.

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1 Introduction

Let $\beta = (\beta_n)_{n \geq 0}$ be a sequence of positive numbers such that

$$(1.1) \quad \liminf_{n \rightarrow \infty} \beta_n^{1/n} \geq 1.$$

The associated weighted Hardy space is the Hilbert space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that

$$(1.2) \quad \|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty,$$

if $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We shall denote $a_n = \widehat{f}(n)$. Note that condition (1.1) ensures that those functions are indeed analytic on \mathbb{D} .

If $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ is a non-constant analytic self-map of \mathbb{D} , we say that φ is a *symbol*. To each symbol, we associate the *composition operator* C_φ defined as $C_\varphi(f) = f \circ \varphi$ for every analytic function $f: \mathbb{D} \rightarrow \mathbb{C}$.

In [3], we characterized the weights β for which all composition operators are bounded on $H^2(\beta)$: this happens if and only if β is *essentially decreasing* and *slowly oscillating*.

Definition 1.1. *A sequence of positive numbers $\beta = (\beta_n)_{n \geq 0}$ is said to be*

1) *essentially decreasing if, for some constant $C \geq 1$, we have*

$$(1.3) \quad \beta_m \leq C \beta_n \quad \text{for all } m \geq n \geq 0;$$

2) *slowly oscillating if there are positive constants $c < 1 < C$ such that*

$$(1.4) \quad c \leq \frac{\beta_m}{\beta_n} \leq C \quad \text{when } n/2 \leq m \leq 2n.$$

In her thesis, N. Zorboska [6] raised the following problems.

- 1) Determine the spaces $H^2(\beta)$ for which every symbol φ , with $\varphi \in H^2(\beta)$, induces a bounded composition operator on $H^2(\beta)$ (Problem 2).
- 2) If $\varphi \in H^2(\beta)$ and $\|\varphi\|_\infty < 1$, is the composition operator C_φ necessarily bounded on $H^2(\beta)$? (Problem 3).

In the first part of this paper, we give a partial answer to the first problem (Theorem 2.5), and a negative one to the second problem (Corollary 2.15).

First, for the first problem, we have the following necessary condition.

Theorem 1.2. *Assume that $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$.*

If every symbol $\varphi \in H^2(\beta)$ induces a bounded composition operator on $H^2(\beta)$, then β is slowly oscillating.

Proof. For all $a \in \mathbb{D}$, we consider the automorphism T_a defined by

$$(1.5) \quad T_a(z) = \frac{a + z}{1 + \bar{a}z}.$$

Let us point out that for every integer $n \geq 1$, we have

$$\widehat{T_a}(n) = (-1)^{n-1} \bar{a}^{n-1} (1 - |a|^2).$$

Therefore, all the symbols T_a are in $H^2(\beta)$ if and only for every $a \in \mathbb{D}$, we have $\sum \beta_n |a|^{2n} < \infty$, which is equivalent to the fact that the Taylor series $\sum \beta_n z^n$ has a radius of convergence at least 1 i.e. $\limsup \beta_n^{1/n} \leq 1$.

Since $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$, the preceding remark implies that all the symbols T_a define bounded composition operators on $H^2(\beta)$, by the hypothesis. By [3, Theorem 4.9], it follows that β is slowly oscillating. \square

In the second part of this paper, we show that for $p \neq 2$, no automorphism T_a of \mathbb{D} , with $a \neq 0$, induces a bounded composition operator on the weighted Hardy space $h^p(\beta)$ (defined in Section 3), whatever β .

2 $H^2(\beta)$ as an algebra

In order to motivate the content of this section, we point out the following easy fact.

Proposition 2.1. *Assume that every symbol $\varphi \in H^2(\beta)$ induces a bounded composition operator on $H^2(\beta)$. Then $H^2(\beta) \cap H^\infty$ is an algebra.*

Proof. It suffices to show that $f^2 \in H^2(\beta) \cap H^\infty$ for every $f \in H^2(\beta) \cap H^\infty$. Let $f \in H^2(\beta) \cap H^\infty$, not constant. If $M > \|f\|_\infty$, then $\varphi = f/M$ is a symbol. With $e_n(z) = z^n$, we have, by hypothesis, $\varphi^n = C_\varphi(e_n) \in H^2(\beta)$. Hence $f^n \in H^2(\beta)$. Since it is clear that $f^n \in H^\infty$, we are done. \square

Remark. The conclusion of Proposition 2.1 can be obtained with a hypothesis of a different nature.

Proposition 2.2. *If $(\beta_n/n^2)_{n \geq 1}$ is a sequence of moments, then $H^2(\beta) \cap H^\infty$ is an algebra.*

Proof. By hypothesis, $\beta_n \approx n^2 \gamma_n$, where $\gamma_n = \int_0^1 t^{n-1} \omega(t) dt$ with ω a positive measurable function on $[0, 1]$. We have, for $f(z) = \sum_{n=0}^\infty a_n z^n$ (A is the normalized area measure on \mathbb{D}):

$$\begin{aligned} I(f) &:= \int_{\mathbb{D}} |f'(z)|^2 \omega(|z|^2) dA(z) = \sum_{n=1}^\infty n^2 |a_n|^2 \int_0^1 r^{2n-2} \omega(r^2) 2r dr \\ &= \sum_{n=1}^\infty n^2 |a_n|^2 \int_0^1 t^{n-1} \omega(t) dt \approx \sum_{n=1}^\infty |a_n|^2 \beta_n. \end{aligned}$$

Now, we observe that $\|f\|_{H^2(\beta)}^2 \approx |f(0)|^2 + I(f)$ and that

$$I(f^2) = \int_{\mathbb{D}} 4 |f(z)|^2 |f'(z)|^2 \omega(|z|^2) dA(z) \leq 4 \|f\|_\infty^2 I(f),$$

which clearly gives the result. \square

A specific example (different from the Dirichlet space case (!), which corresponds to $\omega(t) = 1$ and $\gamma_n = 1/n$) is:

$$\beta_0 = \beta_1 = 1 \quad \text{and} \quad \beta_n = n \log n, \quad \text{for } n \geq 2,$$

for which $\sum_{n=0}^\infty \frac{1}{\beta_n} = +\infty$. Indeed, if

$$\omega(t) = \log^+ \left(\frac{1}{\log(1/t)} \right), \quad 0 \leq t \leq 1,$$

we have, making the changes of variables $t = e^{-x}$ and $x = y/n$:

$$\gamma_n = \int_0^1 t^{n-1} \omega(t) dt = \frac{1}{n} \int_0^n e^{-y} \log \frac{n}{y} dy \sim \frac{\log n}{n}.$$

When β is a bounded sequence, it is easy to see that $H^\infty \subseteq H^2 \subseteq H^2(\beta)$ (see [7, Proposition 2], for instance).

Proposition 2.3. *Assume that β is a non-increasing sequence. Then $H^2(\beta)$ is an H^∞ -module.*

Proof. First of all, since β is then bounded, we have $H^\infty \subseteq H^2(\beta)$. Moreover, the shift $S: H^2(\beta) \rightarrow H^2(\beta)$, defined as $(Sf)(z) = zf(z)$, is a contraction. By von Neumann's inequality, we have $\|P(S)\| \leq \|P\|_\infty$ for all polynomials P . Since $P(S)f = Pf$, it follows, by approximation, that

$$\|gf\|_{H^2(\beta)} \leq \|g\|_\infty \|f\|_{H^2(\beta)}$$

for all $g \in H^\infty$ and all $f \in H^2(\beta)$, and that proves that $H^2(\beta)$ is an H^∞ -module. \square

We will now study when $H^2(\beta)$ is an algebra, and we begin with an elementary remark.

Proposition 2.4. *Assume that $H^2(\beta)$ is an algebra. Then*

(i) $\sup_{n \geq 0} \beta_{n+1}/\beta_n < +\infty$.

(ii) *the sequence $(\beta_n^{1/n})_{n \geq 1}$ has a limit.*

Proof. The shift, which associates to $f \in H^2(\beta)$ the function $z \mapsto zf(z)$, is bounded from $H^2(\beta)$ into $H^2(\beta)$. That means that $\sup_{n \geq 0} \beta_{n+1}/\beta_n < +\infty$ and (i) is proved.

Since $H^2(\beta)$ is an algebra, there exists a positive constant C such that $\|fg\| \leq C \|f\| \|g\|$ for all $f, g \in H^2(\beta)$. Applying that to $f(z) = e_m(z) = z^m$ and $g(z) = e_n(z) = z^n$, we get that, for all $m, n \geq 0$:

$$\beta_{m+n} = \|e_{m+n}\| = \|e_m e_n\| \leq C \|e_m\| \|e_n\| = C \beta_m \beta_n.$$

Setting $\alpha_n = C \beta_n$, that means that $\alpha_{m+n} \leq \alpha_m \alpha_n$ for all $m, n \geq 0$, i.e. the sequence $(\alpha_n)_{n \geq 0}$ is submultiplicative.

Now (ii) follows from Fekete's lemma. \square

Remark. Given $q > 1$ and writing $\tilde{\beta}_n = q^n \beta_n$, we have:

(2.1) $H^2(\beta)$ is an algebra if and only if $H^2(\tilde{\beta})$ is an algebra.

Indeed, it is clear that $f \in H^2(\tilde{\beta})$ if and only if $g \in H^2(\beta)$ where $g(z) = f(\sqrt[q]{q}z)$.

As a consequence of this remark, we have the following partial answer of the first question raised in the Introduction.

Theorem 2.5. *Assume that $H^2(\beta)$ is an algebra. Let φ be a symbol which is in $H^2(\beta)$. Then φ induces a bounded composition operator on $H^2(\beta)$ in the following cases*

- 1) *when $\liminf_{n \rightarrow \infty} \beta_n^{1/n} > 1$;*
- 2) *when $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$ and $\varphi(0) = 0$;*
- 3) *when β is slowly oscillating.*

Remark. We will see in Corollary 2.15 that we can have $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$ without the boundedness of all the composition operators on $H^2(\beta)$ with symbol in $H^2(\beta)$.

Proof. 1) Since $\liminf_{n \rightarrow \infty} \beta_n^{1/n} > 1$, the functions in $H^2(\mathbb{D})$ are actually analytic in a disk $D(0, R)$ containing the closed unit disk $\overline{\mathbb{D}}$. Since $H^2(\beta)$ is an algebra, we have $H^2(\beta) = \mathcal{M}(H^2(\beta))$ (see (2.2)). By [4, Proposition 20 and Corollary 1 of Proposition 31], the spectrum of each $\varphi \in H^2(\beta)$, as an element of the algebra $H^2(\beta) = \mathcal{M}(H^2(\beta))$ is $\overline{\varphi(\mathbb{D})}$. If φ is moreover a symbol, we have $\overline{\varphi(\mathbb{D})} \subseteq \overline{\mathbb{D}}$. By the analytic functional calculus, we have $f(\varphi) \in H^2(\beta)$ for every function f analytic in an open neighborhood of $\overline{\mathbb{D}}$. In particular, $f \circ \varphi = f(\varphi) \in H^2(\beta)$ for every $f \in H^2(\beta)$.

2) Let $f \in H^2(\beta)$. Let $q > 1$ and set $\tilde{\beta}_n = q^n \beta_n$; then $\liminf_{n \rightarrow \infty} \tilde{\beta}_n^{1/n} > 1$. We set $\tilde{\varphi}(z) = \sqrt{q} \varphi(z/\sqrt{q})$; we have $\tilde{\varphi} \in H^2(\tilde{\beta})$ and, since $\varphi(0) = 0$, $\tilde{\varphi}$ maps \mathbb{D} into \mathbb{D} , by the Schwarz lemma, so $\tilde{\varphi}$ is a symbol. We set now $g(z) = f(z/\sqrt{q})$; then $g \in H^2(\tilde{\beta})$ and, by 1), we have $g \circ \tilde{\varphi} \in H^2(\tilde{\beta})$. But

$$g[\tilde{\varphi}(z)] = f[\varphi(z/\sqrt{q})],$$

and saying that $g \circ \tilde{\varphi} \in H^2(\tilde{\beta})$ is equivalent to saying that $f \circ \varphi \in H^2(\beta)$. Hence C_φ is bounded on $H^2(\beta)$.

3) By the parts 1) and 2), all the symbols $\varphi \in H^2(\beta)$ with $\varphi(0) = 0$ induce a bounded composition operator on $H^2(\beta)$, whatever β . Now, if β is slowly oscillating, by [3, Theorem 4.6], all the automorphisms T_a induce bounded composition operators on $H^2(\beta)$. It follows classically that then all symbols $\varphi \in H^2(\beta)$ induce bounded composition operators (if $a = -\varphi(0)$, then $\psi = T_a \circ \varphi = \sum_{n=0}^{\infty} \widehat{T}_a(n) \varphi^n$ is in $H^2(\beta)$ and $\psi(0) = 0$; the result follows since $\varphi = T_{-a} \circ \psi$ and $C_\varphi = C_\psi \circ C_{T_{-a}}$). \square

The space $\mathcal{M}(H^2(\beta))$ of multipliers of $H^2(\beta)$ is, by definition, the set of all analytic functions h on \mathbb{D} such that $hf \in H^2(\beta)$ for all $f \in H^2(\beta)$. It is easy to see (see [3, beginning of Section 6]) that $\mathcal{M}(H^2(\beta)) \subseteq H^\infty$. Actually, we have

$$\mathcal{M}(H^2(\beta)) \subseteq H^2(\beta) \cap H^\infty.$$

Clearly $H^2(\beta) = \mathcal{M}(H^2(\beta))$ when $H^2(\beta)$ is an algebra; hence

$$(2.2) \quad \text{If } H^2(\beta) \text{ is an algebra, then } H^2(\beta) \subseteq H^\infty,$$

Theorem 2.6. *We have $H^2(\beta) \subseteq H^\infty$ if and only if $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$.*

Note that this last condition implies that $H^2(\beta) \subseteq W^+(\mathbb{D})$, where $W^+(\mathbb{D})$ is the Wiener algebra of all analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that $\sum_{n=0}^{\infty} |a_n| < +\infty$ if $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Corollary 2.7. *If $H^2(\beta)$ is an algebra, then $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$.*

Proof of Theorem 2.6. Assume that $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$. Then, for $f \in H^2(\beta)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have, by the Cauchy-Schwarz inequality:

$$\sum_{n=0}^{\infty} |a_n| \leq \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n \right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{1}{\beta_n} \right)^{1/2} < +\infty.$$

Hence $f \in W^+(\mathbb{D})$. Therefore $H^2(\beta) \subseteq W^+(\mathbb{D}) \subseteq A(\mathbb{D}) \subseteq H^\infty$.

Conversely, assume that $H^2(\beta) \subseteq H^\infty$. Let $(a_n)_{n \geq 0}$ be a sequence such that $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$. Then, setting $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have $f \in H^2(\beta)$. Now, if $g(z) = \sum_{n=0}^{\infty} |a_n| z^n$, we have $g \in H^2(\beta)$ and $\|g\| = \|f\|$. By hypothesis, we hence have $g \in H^\infty$. Then

$$\sup_{|z| < 1} \left| \sum_{n=0}^{\infty} |a_n| z^n \right| = \|g\|_\infty$$

implies that $\sum_{n=0}^{\infty} |a_n| \leq \|g\|_\infty$. We have proved that $\sum_{n=0}^{\infty} |a_n| < +\infty$ whenever $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$. That means that $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$. \square

Theorem 2.8. *If $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ and β is slowly oscillating, then $H^2(\beta)$ is an algebra.*

Using Theorem 2.5, we get the following corollary.

Corollary 2.9. *If $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ and β is slowly oscillating, then all symbols $\varphi \in H^2(\beta)$ induce bounded composition operators on $H^2(\beta)$.*

In order to prove this theorem, we introduce the following notation:

$$(2.3) \quad B_n = \sum_{k=0}^n \frac{1}{\beta_k \beta_{n-k}}.$$

Note that $B_n \geq \frac{1}{\beta_0 \beta_n}$ so $\inf_{n \geq 0} B_n \beta_n > 0$.

The proof will follow from the next two lemmas.

Lemma 2.10. *If $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ and β is slowly oscillating, then $B_n \lesssim 1/\beta_n$.*

Proof. Set $M = \sum_{n=0}^{\infty} \frac{1}{\beta_n}$. We have

$$B_n \leq 2 \sum_{0 \leq k \leq n/2} \frac{1}{\beta_k \beta_{n-k}}.$$

But $\frac{1}{\beta_{n-k}} \leq C \frac{1}{\beta_n}$, since $n/2 \leq n-k \leq n$ for $0 \leq k \leq n/2$, and β is slowly oscillating. Hence

$$B_n \leq 2C \frac{1}{\beta_n} \sum_{0 \leq k \leq n/2} \frac{1}{\beta_k} \leq 2CM \frac{1}{\beta_n}. \quad \square$$

Lemma 2.11. *If $B_n \lesssim 1/\beta_n$, then $H^2(\beta)$ is an algebra.*

Proof. It suffices to show that $f^2 \in H^2(\beta)$ for all $f \in H^2(\beta)$. Let $f \in H^2(\beta)$ and write $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We have

$$[f(z)]^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k a_{n-k} \right) z^n.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_k a_{n-k} \right|^2 \beta_n &\leq \sum_{n=0}^{\infty} \left(\sum_{k=0}^n |a_k a_{n-k}|^2 \beta_k \beta_{n-k} \right) \left(\sum_{k=0}^n \frac{1}{\beta_k \beta_{n-k}} \right) \beta_n \\ &\lesssim \sum_{n=0}^{\infty} \left(\sum_{k=0}^n |a_k a_{n-k}|^2 \beta_k \beta_{n-k} \right) \\ &= \left(\sum_{j=0}^{\infty} |a_j|^2 \beta_j \right)^2 = \|f\|^4 < +\infty, \end{aligned}$$

which says that $f^2 \in H^2(\beta)$. \square

Proposition 2.12. *If $H^2(\beta)$ is an algebra, then $\sum_{n=0}^{\infty} \beta_n B_n^2 < +\infty$.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} \frac{1}{\beta_n} z^n$; we have $\|f\|^2 = \sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$, so $f \in H^2(\beta)$. Since $H^2(\beta)$ is an algebra, we have $f^2 \in H^2(\beta)$. But $[f(z)]^2 = \sum_{n=0}^{\infty} B_n z^n$, so we get $\sum_{n=0}^{\infty} B_n^2 \beta_n < +\infty$. \square

The necessary conditions of Corollary 2.7 and of Proposition 2.12 are not sufficient.

Proposition 2.13. *There exists a sequence β such that $\sum_{n=0}^{\infty} \beta_n B_n^2 < +\infty$, hence $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$, but for which $H^2(\beta)$ is not an algebra.*

Proof. We take $\beta_n = n^\gamma$ for n even and $\beta_n = n^{\gamma'}$ for n odd, where $1 < \gamma' < \gamma$ and $2\gamma' > \gamma + 1$ (for example, $\gamma' = 3$ and $\gamma = 4$). Since β_{n+1}/β_n is not bounded, $H^2(\beta)$ is not an algebra. It is clear that $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$.

Now

$$\begin{aligned} B_{2n} &= \sum_{k=0}^{2n} \frac{1}{\beta_k \beta_{2n-k}} \\ &= \sum_{k \text{ even}} \frac{1}{k^\gamma (2n-k)^\gamma} + \sum_{k \text{ odd}} \frac{1}{k^{\gamma'} (2n-k)^{\gamma'}} \lesssim \frac{1}{n^\gamma} + \frac{1}{n^{\gamma'}} \lesssim \frac{1}{n^{\gamma'}}, \end{aligned}$$

so $(2n)^\gamma B_{2n}^2 \lesssim (2n)^{\gamma-2\gamma'}$.

Similarly $(2n+1)^{\gamma'} B_{2n+1}^2 \lesssim (2n+1)^{-\gamma'}$.

Since $\gamma - 2\gamma' < -1$ and $\gamma' > 1$, we obtain the convergence of the series $\sum_{n \geq 0} \beta_n B_n^2$. \square

Actually, we can improve Proposition 2.13, with the additional requirement that $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$.

Theorem 2.14. *There exists a sequence β such that $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$, and for which:*

- 1) $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$; so $H^2(\beta) \subseteq H^\infty$;
- 2) $H^2(\beta)$ is not an algebra.

Corollary 2.15. *There exist a sequence β such that $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$, and a symbol φ such that $\varphi \in H^2(\beta)$ and $\|\varphi\|_\infty < 1$, but C_φ is not bounded on $H^2(\beta)$.*

Proof of Corollary 2.15. We use the sequence obtained in Theorem 2.14. There exists $\varphi \in H^2(\beta) \subset H^\infty$ such that $\varphi^2 \notin H^2(\beta)$; we can assume that $\|\varphi\|_\infty < 1$. Clearly C_φ is not bounded on $H^2(\beta)$ since $z^2 \in H^2(\beta)$ and $C_\varphi(z^2) = \varphi^2$. \square

Proof of Theorem 2.14. Take $m_k = 3^k$ and

$$\begin{cases} \beta_{m_k} &= m_k^2 \\ \beta_{2m_k} &= m_k^5, \end{cases}$$

and

$$\beta_{n+1} = \begin{cases} m_k^{3/m_k} \beta_n & \text{for } m_k \leq n \leq 2m_k - 1; \\ (9/m_k)^{3/m_k} \beta_n & \text{for } 2m_k \leq n \leq m_{k+1} - 1. \end{cases}$$

Since $\beta_n \geq n^2$, we have $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$. Since $m_k^{1/m_k} \xrightarrow[k \rightarrow \infty]{} 1$, we have $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$. Moreover, setting $e_n(z) = z^n$, we have

$$\frac{\|e_{m_k}^2\|^2}{\|e_{m_k}\|^2} = \frac{\|e_{2m_k}\|^2}{\|e_{m_k}\|^2} = \frac{\beta_{2m_k}}{\beta_{m_k}} = m_k^3 = 3^{3k} \xrightarrow[k \rightarrow \infty]{} +\infty;$$

hence $H^2(\beta)$ is not an algebra, since otherwise, we would have a positive constant C for which $\|fg\| \leq C\|f\|\|g\|$ for all $f, g \in H^2(\beta)$. \square

We now see that β slowly oscillating is not necessary for $H^2(\beta)$ to be an algebra.

Example 2.16. *Let $\beta_n = e^{\sqrt{n}}$ for all $n \geq 0$. Then $H^2(\beta)$ is an algebra, though β is not slowly oscillating.*

That β is not slowly oscillating is clear. To see that $H^2(\beta)$ is an algebra, we will prove a more general result: Theorem 2.19 below. Let us give right now the following consequence, using Theorem 1.2.

Corollary 2.17. *There exist a sequence β such that $H^2(\beta)$ is an algebra, and a symbol $\varphi \in H^2(\beta)$ for which C_φ is not bounded on $H^2(\beta)$.*

The following notion will be used.

Definition 2.18. A function $\tau: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said good if $\tau(0) = 0$ and

1. τ is \mathcal{C}^1 on $(0, +\infty)$ with $\tau' \geq 0$ and τ' decreasing;
2. $\liminf_{x \rightarrow +\infty} \frac{\tau(x)}{\log x} > 1$;
3. $\sup_{x \geq 1} \frac{\tau'(tx)}{\tau'(x)} \xrightarrow{t \rightarrow +\infty} 0$.

Examples.

- ▷ $\tau(x) = a \log(x + 1)$, with $a > 1$;
- ▷ $\tau(x) = x^a$, with $0 < a < 1$;

We now have the following result.

Theorem 2.19. Let τ be a good function and $\beta_n = e^{\tau(n)}$. Then, $H^2(\beta)$ is an algebra.

For $\tau(x) = a \log(x + 1)$ with $a > 1$, we have the weighted Dirichlet space \mathcal{D}_a .

Note that if, moreover, we have $\lim_{x \rightarrow +\infty} \frac{\tau(x)}{x} = 0$, then $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$.

Proof. By Lemma 2.11, it suffices to show that the sequence of numbers

$$I_n = \sum_{1 \leq k \leq n/2} \exp[-(\tau(k) + \tau(n - k) - \tau(n))]$$

is bounded.

We set, for each $n \geq 2$ and $x \in [1, n/2]$:

$$\kappa(x) = \tau(x) + \tau(n - x) - \tau(n).$$

Let $\rho > 1$ such that $\liminf_{x \rightarrow +\infty} \frac{\tau(x)}{\log x} > \rho$. There exists some $n_0 \geq 1$ such that $\tau(x) \geq \rho \log x$ for every $x \geq n_0$.

Fix $\varepsilon > 0$ so that $(1 - \varepsilon)\rho > 1$. By the goodness condition 3, we can choose an integer $A = A_\varepsilon > 1$ such that for every $x \geq 1$ and every $t \geq A$, we have $\tau'(tx) \leq \varepsilon \tau'(x)$.

Take $n \geq (A + 1)n_0$. Then, with $\lambda = 1/(A + 1)$, we have, for $1 \leq x \leq \lambda n$:

$$\frac{n - x}{x} = \frac{n}{x} - 1 \geq \frac{1}{\lambda} - 1 = A.$$

Therefore

$$1 \leq x \leq \lambda n \implies \tau'(n - x) = \tau'\left(\frac{n - x}{x} \cdot x\right) \leq \varepsilon \tau'(x).$$

Hence

$$\kappa'(x) = \tau'(x) - \tau'(n - x) \geq (1 - \varepsilon)\tau'(x) \quad \text{for } 1 \leq x \leq \lambda n.$$

Moreover, $\kappa'(x) = \tau'(x) - \tau'(n-x)$ is positive on $[0, n/2]$, since τ' is decreasing. Hence κ is increasing on $[0, n/2]$. Since $\kappa(0) = 0$, we get also that κ is positive on $[0, n/2]$.

Now, we consider N the integer part of λn , and we have, for $1 \leq k \leq N \leq \lambda n$:

$$(2.4) \quad \kappa(k) \geq \int_1^k \kappa'(t) dt \geq (1-\varepsilon) \int_1^k \tau'(t) dt = (1-\varepsilon) (\tau(k) - \tau(1)).$$

Summing up, and using the condition 2, we get

$$(2.5) \quad \sum_{k=1}^N e^{-\kappa(k)} \lesssim \sum_{k=1}^N \exp[-(1-\varepsilon)\tau(k)] \leq n_0 + \sum_{k=n_0}^N \exp[-(1-\varepsilon)\rho \log k]$$

which is bounded since $(1-\varepsilon)\rho > 1$.

On the other hand, we already pointed out that κ is increasing on $[0, n/2]$, and we get, using (2.4):

$$(2.6) \quad \sum_{k=N+1}^{n/2} e^{-\kappa(k)} \leq n e^{-\kappa(\lambda n)} \lesssim n \exp[-(1-\varepsilon)\tau(\lambda n)] \lesssim n^{1-\rho(1-\varepsilon)} = o(1),$$

using the condition 2 and the fact that $n \geq (A+1)n_0$ (i.e. $\lambda n \geq n_0$) and

$$\tau(\lambda n) \geq \rho \log(\lambda n) = \rho \log n - \rho \log(\lambda),$$

and once more that $\rho(1-\varepsilon) > 1$.

Putting (2.5) and (2.6) together, that proves the theorem. \square

The function τ defined by $\tau(x) = x/\log x$ for $x > 1$ is not a good function (because the condition 3 is not fulfilled). Nevertheless, it induces an algebra.

Proposition 2.20. *For $\beta(n) = e^{n/\log n}$, $n \geq 2$ (and $\beta_0 = \beta_1 = 1$, for instance), the space $H^2(\beta)$ is an algebra.*

Proof. The proof follows the same lines as that of Theorem 2.19.

We set $\tau(0) = \tau(1) = 0$ and $\tau(x) = \frac{x}{\log x}$ for $x > 1$. We wish to control

$$I_n = \sum_{1 \leq k \leq n/2} \exp[-(\tau(k) + \tau(n-k) - \tau(n))].$$

We have, for $x > 1$, $\tau'(x) = \frac{1}{\log x} - \frac{1}{\log^2 x}$; in particular $\tau'(x) > 0$ for $x > e$.

Since $\frac{1}{X} - \frac{1}{X^2} \geq \frac{1}{2X}$ if $X \geq 2$, we have $\tau'(x) \geq \frac{1}{2 \log x}$ if $x \geq 8 > e^2$. Hence, for $y \geq x^4$, we have

$$\tau'(x) \geq \frac{1}{2 \log x} \geq \frac{2}{\log y} \geq 2\tau'(y).$$

Let $n \geq 2^{13}$ be an integer. For every $x \geq 8$ such that $2x^4 \leq n$, we have $n-x \geq x^4$ hence $\tau'(x) - \tau'(n-x) \geq \tau'(x)/2$.

Now we consider the function, defined for $x \in [1, n/2]$, as

$$\kappa_n(x) = \tau(x) + \tau(n-x) - \tau(n).$$

We first point out that $\lim_{x \rightarrow +\infty} \tau'(x) = 0$, hence, for each $x \geq 1$, the sequence $(\kappa_n(x))_n$ is bounded and there exists $M > 0$, not depending on n , such that $\kappa_n(k) \geq -M$ for every $1 \leq k \leq 8$. In particular, we have

$$\sum_{1 \leq k \leq 7} e^{-\kappa_n(k)} \leq 7e^M.$$

Let k_n be the integer part of $(n/2)^{1/4}$, we have $\kappa'_n(x) \geq \tau'(x)/2$ for $8 \leq x \leq k_n$ hence, for every $8 \leq k \leq k_n$, we get

$$\kappa_n(k) \geq \kappa_n(8) + \int_8^k \kappa'_n(t) dt \geq \kappa_n(8) + \frac{1}{2} \int_8^k \tau'(t) dt \geq -M + \frac{1}{2} (\tau(k) - \tau(8)).$$

Therefore,

$$\sum_{k=8}^{k_n} e^{-\kappa_n(k)} \leq \exp(M + \tau(8)/2) \sum_{k=8}^{\infty} e^{-\frac{k}{2 \log k}} < +\infty.$$

Now, τ' is decreasing on $[8, +\infty)$, so $\kappa'_n(x) \geq 0$ when $8 \leq x \leq n/2$, and $\kappa_n(k) \geq \kappa_n(k_n) \geq c + \frac{1}{2} \tau(k_n)$ for $k_n \leq k \leq n/2$, where $c \in \mathbb{R}$. We get

$$\sum_{k_n < k \leq n/2} e^{-\kappa_n(k)} \lesssim n e^{-\frac{k_n}{2 \log k_n}} = O(1).$$

That finishes the proof. \square

Actually the problem to know whether $H^2(\beta)$ is an algebra or not can be formulated in terms of Schur multipliers acting on a family of Hankel matrices.

Theorem 2.21. $H^2(\beta)$ is an algebra if and only if the map

$$\begin{aligned} \Psi: \ell^2 &\longrightarrow B(\ell^2) \\ u &\longmapsto \left(u_{k+l} \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}} \right)_{k,l} \end{aligned}$$

is bounded.

Let us point out that Lemma 2.11 also follows from the previous theorem. Actually condition $B_n \lesssim 1/\beta_n$ is equivalent to the fact that Ψ is bounded from ℓ^2 to $HS(\ell^2) \subseteq B(\ell^2)$ (where $HS(\ell^2)$ stands for the space of Hilbert-Schmidt operators on ℓ^2).

Indeed, for every $u \in \ell^2$, we have $\|\Psi(u)\|_{HS}^2 = \sum_{k,l} |u_{k+l}|^2 \beta_n B_n$.

This remark gives a hint on the gap between the sufficient condition $B_n \lesssim 1/\beta_n$ and a potential characterization.

Proof of Theorem 2.21. The vector space $H^2(\beta)$ is an algebra if and only if $fg \in H^2(\beta)$ for all $f, g \in H^2(\beta)$. In other words, $H^2(\beta)$ is an algebra if and only if

$$\sum_{n=0}^{\infty} \left| \sum_{p=0}^n a_p b_{n-p} \right|^2 \beta_n < +\infty$$

whenever $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$ and $\sum_{n=0}^{\infty} |b_n|^2 \beta_n < +\infty$. Equivalently,

$$\sup_{\substack{a, b \in \ell^2 \\ \|a\| = \|b\| = 1}} \sum_{n=0}^{\infty} \left| \sum_{k+l=n} \frac{a_k}{\sqrt{\beta_k}} \frac{b_l}{\sqrt{\beta_l}} \sqrt{\beta_{k+l}} \right|^2 < +\infty$$

or, in other words,

$$\sup_{\substack{a, b, u \in \ell^2 \\ \|a\| = \|b\| = \|u\| = 1}} \left| \sum_{n=0}^{\infty} \sum_{k+l=n} a_k b_l u_n \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}} \right| < +\infty.$$

Let us point out that

$$\sum_{n=0}^{\infty} \sum_{k+l=n} a_k b_l u_n \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}} = \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} M_{k,l} b_l = \sum_{k=0}^{\infty} a_k \left(\Psi(u)(b) \right)_k$$

where $M_{k,l} = u_{k+l} \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}}$.

Therefore, $H^2(\beta)$ is an algebra if and only if

$$\sup_{\substack{b, u \in \ell^2 \\ \|b\| = \|u\| = 1}} \left\| \Psi(u)(b) \right\| < +\infty.$$

and the result follows. \square

3 Composition operators induced by the automorphisms on $h^p(\beta)$

For $1 \leq p \leq \infty$, the Banach space $\ell_A^p = h^p$ (a copy of ℓ^p , called the Beurling-Sobolev space in [5]) is the space of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk \mathbb{D} such that

$$\|f\|_p^p := \sum_{n=0}^{\infty} |a_n|^p < +\infty.$$

If $\beta = (\beta_n)_{n \geq 0}$ is a sequence of positive numbers satisfying (1.1), the weighted space $h^p(\beta)$ is defined by the finiteness of

$$\|f\|_{\beta}^p := \sum_{n=0}^{\infty} |a_n|^p \beta_n.$$

Observe that h^2 is none other than the usual Hardy space H^2 , and that $h^2(\beta) = H^2(\beta)$. But note also that h^p , for $p \neq 2$, has nothing to do with the usual Hardy space H^p , even if, for example, $H^p \subseteq h^\infty$. But in the spirit of our recent work [3], the case of h^p seems a natural one to consider.

Remark that (1.1) is equivalent to the inclusion $h^p(\beta) \subseteq \mathcal{H}(\mathbb{D})$ and allows to treat the elements of $h^p(\beta)$ as analytic functions on \mathbb{D} .

For $p = 2$, it is easy to check that all the composition operators C_{T_a} , for $a \in \mathbb{D}$, generated by the automorphisms T_a , defined in (1.5), (and actually all composition operators, by Littlewood's subordination principle) are bounded on H^2 . We recently studied [3] the weighted version $H^2(\beta)$ of this space and gave a complete characterization of those weights $\beta = (\beta_n)$ for which either C_φ is bounded on $H^2(\beta)$ for all automorphisms, or for all symbols. In this section, we show that this is never the case for $p \neq 2$. We mention in passing that this boundedness issue was previously considered by Blyudze and Shimorin [1] in the case when the initial space is h^1 and h^p is the target space. In that case, the authors show that $C_{T_a} : h^1 \rightarrow h^p$ is bounded if and only if $p \geq 2$. This has been made much more precise in [5].

We will prove the following result.

Theorem 3.1. *Let $p \in [1, +\infty]$, with $p \neq 2$. Then C_{T_a} is unbounded on h^p for all $a \in \mathbb{D} \setminus \{0\}$.*

Moreover, C_{T_a} is never bounded on $h^p(\beta)$, whatever the choice of β satisfying (1.1).

S. Charpentier, N. Espouiller and R. Zarouf informed us that they proved, independently, Theorem 3.1 for $\beta \equiv 1$, using [5] (see [2]).

Nevertheless, we present below a self-contained argument, more elementary, directly relying on our paper [3], and which works for all weights β .

Proof. Set

$$I = [1/2, 2/3] \quad \text{and} \quad J = [\alpha^{-1}, \alpha],$$

($\alpha = 5/4$ for instance).

As in [3, Proposition 4.2], the rotation invariance of $h^p(\beta)$ allows us to claim that, if C_{T_a} is bounded on $h^p(\beta)$ for some $a \in \mathbb{D}$, $a \neq 0$, then there exists a constant K such that

$$(3.1) \quad \|C_{T_a}\| \leq K, \quad \forall a \in I.$$

Now, everything will rely on the matrix $A = (a_{m,n})_{m,n}$, where

$$a_{m,n} = \widehat{(T_a)^n}(m) \left(\frac{\beta_m}{\beta_n} \right)^{1/p},$$

which represents C_{T_a} on the canonical (Schauder) basis of $h^p(\beta)$. If this matrix defines a bounded operator, its columns and rows (the columns of the transposed

operator) are respectively uniformly bounded on ℓ^p and on the dual space ℓ^q (q the conjugate exponent of p), that is (with some R independent of $a \in I$):

$$(3.2) \quad C_n = \sum_{m=0}^{\infty} |a_{m,n}|^p \frac{\beta_m}{\beta_n} \quad \text{satisfies} \quad \sup_n C_n \leq R < +\infty,$$

and

$$(3.3) \quad L_m = \sum_{n=0}^{\infty} |a_{m,n}|^q \left(\frac{\beta_m}{\beta_n} \right)^{q/p} \quad \text{satisfies} \quad \sup_m L_m \leq R < +\infty.$$

We will show that, for $p \neq 2$, one of the necessary conditions (3.2) or (3.3) fails. We will hence separate two cases.

We need an auxiliary result. Recall first the following elementary lemma (see [3, Lemma 4.16]).

Lemma 3.2. *Let $f: [A, B] \rightarrow \mathbb{R}$ be a C^2 -function such that $|f'| \geq \delta$ and $|f''| \leq M$. Then*

$$\left| \int_A^B e^{if(x)} dx \right| \leq \frac{2}{\delta} + \frac{M(B-A)}{\delta^2}.$$

This lemma implies the following extended version of [3, Proposition 4.12].

Proposition 3.3. *If $r \in J = [\alpha^{-1}, \alpha]$ and $s \geq 1$, it holds, when $r = m/n$:*

$$(3.4) \quad \int_I |\widehat{T}_a^n(m)|^s da \gtrsim \left(\int_I |\widehat{T}_a^n(m)| da \right)^s \geq \delta n^{-s/2}.$$

Proof. The first inequality in (3.4) is just Hölder's inequality. For the second one, we proved in [3, Proposition 4.14] that

$$(3.5) \quad |\widehat{T}_a^n(m)| \geq \delta n^{-1/2} |\cos(n\psi_r(a) + \pi/4)| + O(n^{-3/5}),$$

where $\psi_r = f$ satisfies the assumptions of Lemma 3.2. Moreover, we have the classical Fourier expansion

$$|\cos x| = c + \sum_{l=1}^{\infty} \delta_l \cos lx,$$

with $c > 0$ and $\delta_l = O(l^{-2})$. Hence (actually $\sum_{l=1}^{\infty} |\delta_l| < +\infty$ would suffice):

$$(3.6) \quad |\widehat{T}_a^n(m)| \geq \delta n^{-1/2} \left(c + \sum_{l=1}^{\infty} \delta_l \cos(l(n\psi_r(a) + \pi/4)) \right) + O(n^{-3/5}).$$

We now apply Lemma 3.2 with $f(a) = l(n\psi_r(a) + \pi/4)$. Here, for given l , we have $|f'| \geq \delta nl$ and $|f''| \leq Mnl$ on I , so that

$$\left| \int_I \cos(l(n\psi_r(a) + \pi/4)) da \right| \leq C \left(\frac{1}{nl} + \frac{nl}{n^2 l^2} \right) = \frac{C}{nl}.$$

It now follows from (3.2) that

$$\begin{aligned} \int_I |\widehat{T}_a^n(m)| da &\geq \delta n^{-1/2} \left(c + O \left(\sum_{l=1}^{\infty} \frac{|\delta_l|}{nl} \right) \right) + O(n^{-3/5}) \\ &\geq \delta c n^{-1/2} + O(n^{-3/5}) \geq \delta' n^{-1/2}, \end{aligned}$$

and this ends the proof. \square

We now come back to the proof of Theorem 3.1. We will reason by contradiction, and separate two cases.

We set $J_l = [l/\sqrt{\alpha}, \sqrt{\alpha} l]$.

- *Case $p < 2$.*

It follows from (3.2) that, for $n \in J_l$, we have

$$\sum_{m \in J_l} |\widehat{T}_a^n(m)|^p \frac{\beta_m}{\beta_n} \leq R.$$

Integrating on I and using Proposition 3.3 give, since $m, n \in J_l$ (hence $m/n \in J$) have the same size as l :

$$l^{-p/2} \sum_{m \in J_l} \frac{\beta_m}{\beta_n} \lesssim R.$$

Now, summing up over $n \in J_l$ further gives

$$l^{-p/2} \left(\sum_{k \in J_l} \beta_k \right) \left(\sum_{k \in J_l} \beta_k^{-1} \right) \lesssim Rl.$$

But, by the Cauchy-Schwarz inequality:

$$l^2 \lesssim |J_l|^2 \leq \left(\sum_{k \in J_l} \beta_k \right) \left(\sum_{k \in J_l} \beta_k^{-1} \right),$$

and we get $l^{2-p/2} \lesssim Rl$ or $R \gtrsim l^{1-p/2}$.

Since $1 - p/2 > 0$, we have a contradiction for large l .

- *Case $p > 2$.*

The proof is nearly the same. It now follows from (3.3) that, for $m \in J_l$:

$$\sum_{n \in J_l} |\widehat{T}_a^n(m)|^q \left(\frac{\beta_m}{\beta_n} \right)^{q/p} \leq R.$$

Set $\gamma_k = \beta_k^{q/p}$ and proceed as in the case $p < 2$ to get

$$l^{-q/2} \left(\sum_{k \in J_l} \gamma_k \right) \left(\sum_{k \in J_l} \gamma_k^{-1} \right) \lesssim Rl,$$

or, again by the Cauchy-Schwarz inequality, $l^{2-q/2} \lesssim Rl$ or $R \gtrsim l^{1-q/2}$.

Since this time $1 - q/2 > 0$, we have again a contradiction for large l . \square

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References

- [1] M. Blyudze, S. Shimorin, Estimates of the norms of powers of functions in certain Banach spaces, *J. Math. Sci.*, 80, no. 4 (1996), 1880–1891.
- [2] S. Charpentier, N. Espoullier, R. Zarouf, Composition operators on Beurling-Sobolev spaces, *preprint*.
- [3] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Characterization of weighted Hardy spaces on which all composition operators are bounded, *preprint*, available on HAL: <https://hal.science/hal-04108498v2>.
- [4] A. L. Shields, Weighted shift operators and analytic function theory, *Topics in operator theory*, pp. 49–128, *Math. Surveys*, No. 13, Amer. Math. Soc., Providence, R.I. (1974).
- [5] O. Szehr, R. Zarouf, l_p -norms of Fourier coefficients of powers of a Blaschke factor, *J. Anal. Math.* 140 (2020), 1–30.
- [6] N. Zorboska, Composition operators on weighted Hardy spaces. Thesis (Ph. D.), University of Toronto (Canada) (1988).
- [7] N. Zorboska, Composition operators induced by functions with supremum strictly smaller than 1, *Proc. Amer. Math. Soc.* 106, No 3 (1989), 679–684.

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