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# On some questions about composition operators on weighted Hardy spaces

*Pascal Lefèvre, Daniel Li,  
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Dedicated to Gilles Godefroy for his 70<sup>th</sup> birthday

May 18, 2024

**Abstract.** We first consider some questions raised by N. Zorboska in her thesis. In particular she asked for which sequences  $\beta$  every symbol  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  with  $\varphi \in H^2(\beta)$  induces a bounded composition operator  $C_\varphi$  on the weighted Hardy space  $H^2(\beta)$ . We give partial answers and investigate when  $H^2(\beta)$  is an algebra. We answer negatively another question in showing that there are a sequence  $\beta$  and  $\varphi \in H^2(\beta)$  such that  $\|\varphi\|_\infty < 1$  and the composition operator  $C_\varphi$  is not bounded on  $H^2(\beta)$ .

In a second part, we show that for  $p \neq 2$ , no automorphism of  $\mathbb{D}$ , except those that fix 0, induces a bounded composition operator on the Beurling-Sobolev space  $\ell_A^p$ , and even on any weighted version of this space.

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**Key-words.** composition operator; oscillatory integrals; weighted Hardy space.

## 1 Introduction

Let  $\beta = (\beta_n)_{n \geq 0}$  be a sequence of positive numbers such that

$$(1.1) \quad \liminf_{n \rightarrow \infty} \beta_n^{1/n} \geq 1.$$

The associated weighted Hardy space  $H^2(\beta)$  is the Hilbert space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that

$$(1.2) \quad \|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty,$$

if  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ . We shall denote  $a_n = \widehat{f}(n)$ . Note that condition (1.1) ensures that those functions are indeed analytic on  $\mathbb{D}$ , and conversely.

More generally, for  $1 \leq p \leq \infty$ , the weighted space  $h^p(\beta)$  (called Beurling-Sobolev space in [3] and [10]) is the space of all analytic functions  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  on the unit disk  $\mathbb{D}$  such that  $\|f\|_{\beta}$  is finite, where

$$\|f\|_{\beta}^p := \sum_{n=0}^{\infty} |a_n|^p \beta_n.$$

for  $1 \leq p < \infty$ , and  $\|f\|_{\infty} = \sup_{n \geq 0} |a_n| \beta_n$ . For  $\beta \equiv 1$ , we write simply  $h^p$  or  $\ell_A^p$ .

Observe that  $h^2$  is none other than the usual Hardy space  $H^2$ , and that  $h^2(\beta) = H^2(\beta)$ . However, for  $p \neq 2$ ,  $h^p$  has nothing to do with the usual Hardy space  $H^p$ , even if, for example,  $H^p \subseteq h^{\infty}$ . But in the spirit of our recent work [4], the case of  $h^p$  seems a natural one to consider.

Again, (1.1) is equivalent to the inclusion  $h^p(\beta) \subseteq \mathcal{H}(\mathbb{D})$  and allows to treat the elements of  $h^p(\beta)$  as analytic functions on  $\mathbb{D}$ .

Though most of our results hold for all  $p \geq 1$ , we first restrict ourselves to  $p = 2$ , and only consider the other exponents  $p$  in Section 3.

If  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  is a non-constant analytic self-map of  $\mathbb{D}$ , we say that  $\varphi$  is a *symbol*. To each symbol, we associate the *composition operator*  $C_{\varphi}$  defined as  $C_{\varphi}(f) = f \circ \varphi$  for every analytic function  $f: \mathbb{D} \rightarrow \mathbb{C}$ .

In her thesis, N. Zorboska [11] raised the following problems.

- 1) Determine the spaces  $H^2(\beta)$  for which every symbol  $\varphi$ , with  $\varphi \in H^2(\beta)$ , induces a bounded composition operator on  $H^2(\beta)$  (Problem 2).
- 2) If  $\varphi \in H^2(\beta)$  and  $\|\varphi\|_{\infty} < 1$ , is the composition operator  $C_{\varphi}$  necessarily bounded on  $H^2(\beta)$ ? (Problem 3).

In Section 2, we give partial answers to the first problem (Theorem 2.2 and Theorem 2.7), and a negative one to the second problem (Theorem 2.11).

In Section, 3, we show that for  $p \neq 2$ , no automorphism  $T_a$  of  $\mathbb{D}$  (to be defined right after), with  $a \neq 0$ , induces a bounded composition operator on the weighted Hardy space  $h^p(\beta)$ , whatever  $\beta$ .

After the first version of this paper was completed, we learned the existence of [7] and [3]. This version takes these papers into account. We thank O. El-Fallah for sending us a copy of the english version of [3]. We thank also N. Nikolski who learned us that [8, §3.2] deals with the question of weighted algebras, and that this question goes back to [5] (without proof), [6], and [7] (without proof).

## 2 Answers to Zorboska's questions

### 2.1 A necessary condition

In [4], we characterized the weights  $\beta$  for which all composition operators are bounded on  $H^2(\beta)$ : this happens if and only if  $\beta$  is *essentially decreasing*

and *slowly oscillating*.

**Definition 2.1.** A sequence of positive numbers  $\beta = (\beta_n)_{n \geq 0}$  is said to be

1) essentially decreasing if, for some constant  $C \geq 1$ , we have

$$(2.1) \quad \beta_m \leq C \beta_n \quad \text{for all } m \geq n \geq 0;$$

2) slowly oscillating if there are positive constants  $c < 1 < C$  such that

$$(2.2) \quad c \beta_n \leq \beta_m \leq C \beta_n \quad \text{when } n/2 \leq m \leq 2n.$$

Note that if  $\beta$  is slowly oscillating, then  $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$ .

First, for the first Zorboska's question, we have the following necessary condition.

**Theorem 2.2.** Assume that  $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$ .

If every symbol  $\varphi \in H^2(\beta)$  induces a bounded composition operator on  $H^2(\beta)$ , then  $\beta$  is slowly oscillating.

*Proof.* For all  $a \in \mathbb{D}$ , we consider the automorphism  $T_a$  defined by

$$(2.3) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}.$$

Let us point out that for every integer  $n \geq 1$ , we have

$$\widehat{T_a}(n) = (-1)^{n-1} \bar{a}^{n-1} (1 - |a|^2).$$

Therefore, all the symbols  $T_a$  are in  $H^2(\beta)$  if and only if for every  $a \in \mathbb{D}$ , we have  $\sum \beta_n |a|^{2n} < \infty$ , which is equivalent to the fact that the Taylor series  $\sum \beta_n z^n$  has a radius of convergence at least 1 i.e.  $\limsup \beta_n^{1/n} \leq 1$ .

Since  $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$ , by hypothesis, the preceding remark implies that all the symbols  $T_a$  define bounded composition operators on  $H^2(\beta)$ , by the hypothesis. By [4, Theorem 4.9], it follows that  $\beta$  is slowly oscillating.  $\square$

## 2.2 $H^2(\beta)$ as an algebra

In order to motivate the content of this section, we point out the following easy fact.

**Proposition 2.3.** Assume that every symbol  $\varphi \in H^2(\beta)$  induces a bounded composition operator on  $H^2(\beta)$ . Then  $H^2(\beta) \cap H^\infty$  is an algebra.

*Proof.* It suffices to show that  $f^2 \in H^2(\beta) \cap H^\infty$  for every  $f \in H^2(\beta) \cap H^\infty$ . Let  $f \in H^2(\beta) \cap H^\infty$ , not constant. If  $M > \|f\|_\infty$ , then  $\varphi = f/M$  is a symbol. With  $e_n(z) = z^n$ , we have, by hypothesis,  $\varphi^n = C_\varphi(e_n) \in H^2(\beta)$ . Hence  $f^n \in H^2(\beta)$ . Since it is clear that  $f^n \in H^\infty$ , we are done.  $\square$

**Remark.** The conclusion of Proposition 2.3 can be obtained with a hypothesis of a different nature.

**Proposition 2.4.** *If  $(\beta_n/n^2)_{n \geq 1}$  is equivalent to a sequence of moments, then  $H^2(\beta) \cap H^\infty$  is an algebra.*

*Proof.* By hypothesis,  $\beta_n \approx n^2 \gamma_n$ , where  $\gamma_n = \int_0^1 t^{n-1} d\lambda(t)$  and where  $\lambda$  is a non negative finite measure on  $[0, 1]$ . Let  $\mu$  be the image of  $\lambda$  by the square-root function  $r \mapsto r^{1/2}$  on  $[0, 1]$  and  $\nu$  the non negative measure on  $\mathbb{D}$  defined by

$$\int_{\mathbb{D}} h(z) d\nu(z) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 h(r e^{i\theta}) d\theta d\mu(r).$$

We have, for  $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\beta)$ :

$$\begin{aligned} I(f) &:= \int_{\mathbb{D}} |f'(z)|^2 d\nu(z) = \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 r^{2n-2} d\mu(r) \\ &= \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 t^{n-1} d\lambda(t) \approx \sum_{n=1}^{\infty} |a_n|^2 \beta_n. \end{aligned}$$

Now, we observe that  $\|f\|_{H^2(\beta)}^2 \approx |f(0)|^2 + I(f)$  and that

$$I(f^2) = \int_{\mathbb{D}} 4 |f(z)|^2 |f'(z)|^2 d\nu(z) \leq 4 \|f\|_{\infty}^2 I(f),$$

which clearly gives the result.  $\square$

A specific example (the Dirichlet space case corresponds to  $d\lambda(t) = \omega(t) dt = dt$  and  $\gamma_n = 1/n$ , or to  $\alpha = 0$ ) is:

$$\beta_0 = \beta_1 = 1 \quad \text{and} \quad \beta_n = n (\log n)^\alpha, \quad 0 \leq \alpha \leq 1, \quad \text{for } n \geq 2,$$

for which  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} = +\infty$ . Indeed, if  $d\lambda(t) = \omega(t) dt$  with

$$\omega(t) = \left( \log^+ \left( \frac{1}{\log(1/t)} \right) \right)^\alpha, \quad 0 \leq t \leq 1,$$

we have, making the changes of variables  $t = e^{-x}$  and  $x = y/n$ :

$$\gamma_n = \int_0^1 t^{n-1} \omega(t) dt = \frac{1}{n} \int_0^n e^{-y} \left( \log \frac{n}{y} \right)^\alpha dy \sim \frac{(\log n)^\alpha}{n}.$$

When  $\beta$  is a bounded sequence, it is easy to see that  $H^\infty \subseteq H^2 \subseteq H^2(\beta)$  (see [12, Proposition 2], for instance).

**Proposition 2.5.** *Assume that  $\beta$  is a non-increasing sequence. Then  $H^2(\beta)$  is an  $H^\infty$ -module.*

*Proof.* First of all, since  $\beta$  is then bounded, we have  $H^\infty \subseteq H^2(\beta)$ . Moreover, the shift  $S: H^2(\beta) \rightarrow H^2(\beta)$ , defined as  $(Sf)(z) = zf(z)$ , is a contraction.

By von Neumann's inequality, we have  $\|P(S)\| \leq \|P\|_\infty$  for all polynomials  $P$ . Since  $P(S)f = Pf$ , it follows, by approximation, that

$$\|gf\|_{H^2(\beta)} \leq \|g\|_\infty \|f\|_{H^2(\beta)}$$

for all  $g \in H^\infty$  and all  $f \in H^2(\beta)$ , and that proves that  $H^2(\beta)$  is an  $H^\infty$ -module.  $\square$

We will now study when  $H^2(\beta)$  is an algebra, and we begin with an elementary remark.

**Proposition 2.6.** *Assume that  $H^2(\beta)$  is an algebra. Then*

(i)  $\sup_{n \geq 0} \beta_{n+1}/\beta_n < +\infty$ .

(ii) *the sequence  $(\beta_n^{1/n})_{n \geq 1}$  has a limit.*

*Proof.* The shift, which associates to  $f \in H^2(\beta)$  the function  $z \mapsto zf(z)$ , is bounded from  $H^2(\beta)$  into  $H^2(\beta)$ . That means that  $\sup_{n \geq 0} \beta_{n+1}/\beta_n < +\infty$  and (i) is proved.

Since  $H^2(\beta)$  is an algebra, there exists a positive constant  $C$  such that  $\|fg\| \leq C\|f\|\|g\|$  for all  $f, g \in H^2(\beta)$ . Applying that to  $f(z) = e_m(z) = z^m$  and  $g(z) = e_n(z) = z^n$ , we get that, for all  $m, n \geq 0$ :

$$(2.4) \quad \beta_{m+n} = \|e_{m+n}\| = \|e_m e_n\| \leq C \|e_m\| \|e_n\| = C \beta_m \beta_n.$$

Setting  $\alpha_n = C \beta_n$ , that means that  $\alpha_{m+n} \leq \alpha_m \alpha_n$  for all  $m, n \geq 0$ , i.e. the sequence  $(\alpha_n)_{n \geq 0}$  is submultiplicative.

Now (ii) follows from the multiplicative version of Fekete's lemma (see the proof of the spectral radius formula).  $\square$

**Remark.** Given  $q > 1$  and writing  $\tilde{\beta}_n = q^n \beta_n$ , we have:

$$(2.5) \quad H^2(\beta) \text{ is an algebra if and only if } H^2(\tilde{\beta}) \text{ is an algebra.}$$

Indeed, it is clear that  $f \in H^2(\tilde{\beta})$  if and only if  $g \in H^2(\beta)$  where  $g(z) = f(\sqrt[q]{q}z)$ .

As a consequence of this remark, we have the following partial answer of the first question raised in the Introduction.

**Theorem 2.7.** *Assume that  $H^2(\beta)$  is an algebra. Let  $\varphi$  be a symbol which is in  $H^2(\beta)$ . Then  $\varphi$  induces a bounded composition operator on  $H^2(\beta)$  in the following cases*

- 1) *when  $\liminf_{n \rightarrow \infty} \beta_n^{1/n} > 1$ ;*
- 2) *when  $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$  and  $\varphi(0) = 0$ ;*
- 3) *when  $\beta$  is slowly oscillating.*

Recall that the slow oscillation of  $\beta$  implies that  $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$ .

**Remark.** We will see in Corollary 2.11 that we can have  $\liminf_{n \rightarrow \infty} \beta_n^{1/n} = 1$  without the boundedness of all the composition operators on  $H^2(\beta)$  with symbol in  $H^2(\beta)$ .

*Proof.* 1) Since  $\liminf_{n \rightarrow \infty} \beta_n^{1/n} > 1$ , the functions in  $H^2(\beta)$  are actually analytic in a disk  $D(0, R)$  containing the closed unit disk  $\overline{\mathbb{D}}$ . Since  $H^2(\beta)$  is an algebra, we have  $H^2(\beta) = \mathcal{M}(H^2(\beta))$  (see (2.6)). By [9, Proposition 20 and Corollary 1 of Proposition 31], the spectrum of each  $\varphi \in H^2(\beta)$ , as an element of the algebra  $H^2(\beta) = \mathcal{M}(H^2(\beta))$  is  $\overline{\varphi(\mathbb{D})}$ . If  $\varphi$  is moreover a symbol, we have  $\overline{\varphi(\mathbb{D})} \subseteq \overline{\mathbb{D}}$ . By the analytic functional calculus, we have  $f(\varphi) \in H^2(\beta)$  for every function  $f$  analytic in an open neighborhood of  $\overline{\mathbb{D}}$ . In particular,  $f \circ \varphi = f(\varphi) \in H^2(\beta)$  for every  $f \in H^2(\beta)$ .

2) Let  $f \in H^2(\beta)$ . Let  $q > 1$  and set  $\tilde{\beta}_n = q^n \beta_n$ ; then  $\liminf_{n \rightarrow \infty} \tilde{\beta}_n^{1/n} > 1$ . We set  $\tilde{\varphi}(z) = \sqrt{q} \varphi(z/\sqrt{q})$ ; we have  $\tilde{\varphi} \in H^2(\tilde{\beta})$  and, since  $\varphi(0) = 0$ ,  $\tilde{\varphi}$  maps  $\mathbb{D}$  into  $\mathbb{D}$ , by the Schwarz lemma, so  $\tilde{\varphi}$  is a symbol. We set now  $g(z) = f(z/\sqrt{q})$ ; then  $g \in H^2(\tilde{\beta})$  and, by 1), we have  $g \circ \tilde{\varphi} \in H^2(\tilde{\beta})$ . But

$$g[\tilde{\varphi}(z)] = f[\varphi(z/\sqrt{q})],$$

and saying that  $g \circ \tilde{\varphi} \in H^2(\tilde{\beta})$  is equivalent to saying that  $f \circ \varphi \in H^2(\beta)$ . Hence  $C_\varphi$  is bounded on  $H^2(\beta)$ .

3) By the parts 1) and 2), all the symbols  $\varphi \in H^2(\beta)$  with  $\varphi(0) = 0$  induce a bounded composition operator on  $H^2(\beta)$ , whatever  $\beta$ . Now, if  $\beta$  is slowly oscillating, by [4, Theorem 4.6], all the automorphisms  $T_a$  induce bounded composition operators on  $H^2(\beta)$ . It follows, as in 1), because  $T_a$  is analytic in an open neighborhood of  $\mathbb{D}$ , that then all symbols  $\varphi \in H^2(\beta)$  induce bounded composition operators (if  $a = -\varphi(0)$ , then  $\psi = T_a \circ \varphi = \sum_{n=0}^{\infty} \widehat{T}_a(n) \varphi^n$  is in  $H^2(\beta)$  and  $\psi(0) = 0$ ; the result follows since  $\varphi = T_{-a} \circ \psi$  and  $C_\varphi = C_\psi \circ C_{T_{-a}}$ ).  $\square$

The space  $\mathcal{M}(H^2(\beta))$  of multipliers of  $H^2(\beta)$  is, by definition, the set of all analytic functions  $h$  on  $\mathbb{D}$  such that  $hf \in H^2(\beta)$  for all  $f \in H^2(\beta)$ . It is easy to see (see [4, beginning of Section 6]) that  $\mathcal{M}(H^2(\beta)) \subseteq H^\infty$ . Actually, we have

$$\mathcal{M}(H^2(\beta)) \subseteq H^2(\beta) \cap H^\infty.$$

Clearly  $H^2(\beta) = \mathcal{M}(H^2(\beta))$  when  $H^2(\beta)$  is an algebra; hence

$$(2.6) \quad \text{If } H^2(\beta) \text{ is an algebra, then } H^2(\beta) \subseteq H^\infty,$$

**Proposition 2.8.** *We have  $H^2(\beta) \subseteq H^\infty$  if and only if  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ .*

Note that this last condition implies that  $H^2(\beta) \subseteq W^+(\mathbb{D})$ , where  $W^+(\mathbb{D})$  is the Wiener algebra of all analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that  $\sum_{n=0}^{\infty} |a_n| < +\infty$  if  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ .

*Proof.* Assume that  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ . Then, for  $f \in H^2(\beta)$  and  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ , we have, by the Cauchy-Schwarz inequality:

$$\sum_{n=0}^{\infty} |a_n| \leq \left( \sum_{n=0}^{\infty} |a_n|^2 \beta_n \right)^{1/2} \left( \sum_{n=0}^{\infty} \frac{1}{\beta_n} \right)^{1/2} < +\infty.$$

Hence  $f \in W^+(\mathbb{D})$ . Therefore  $H^2(\beta) \subseteq W^+(\mathbb{D}) \subseteq A(\mathbb{D}) \subseteq H^\infty$ .

Conversely, assume that  $H^2(\beta) \subseteq H^\infty$ . Let  $(a_n)_{n \geq 0}$  be a sequence such that  $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$ . Then, setting  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ , we have  $f \in H^2(\beta)$ . Now, if  $g(z) = \sum_{n=0}^{\infty} |a_n| z^n$ , we have  $g \in H^2(\beta)$  and  $\|g\| = \|f\|$ . By hypothesis, we hence have  $g \in H^\infty$ . Then

$$\sup_{|z| < 1} \left| \sum_{n=0}^{\infty} |a_n| z^n \right| = \|g\|_\infty$$

implies that  $\sum_{n=0}^{\infty} |a_n| \leq \|g\|_\infty$ . We have proved that  $\sum_{n=0}^{\infty} |a_n| < +\infty$  whenever  $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$ . That means that  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ .  $\square$

**Corollary 2.9.** *If  $H^2(\beta)$  is an algebra, then  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ .*

This corollary appeared already in [3, Theorem 3.2.7]. Moreover, [3, Theorem 3.2.7] states that the converse holds when  $\beta$  is “regular”, namely if  $\beta$  is log-concave,  $\beta_n \geq 1$ ,  $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$ , and, either  $\beta_n/n^c$  eventually decreases for some  $c > 0$ , or either  $\beta_n/n^4$  eventually increases and is log-concave. However, the converse does not hold in general, as the following result shows.

**Theorem 2.10.** *There exists a sequence  $\beta$  such that  $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$ , and for which:*

- 1)  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ ; so  $H^2(\beta) \subseteq H^\infty$ ;
- 2)  $H^2(\beta)$  is not an algebra.

*Proof.* Take  $m_k = 3^k$  and

$$\begin{cases} \beta_{m_k} &= m_k^2 \\ \beta_{2m_k} &= m_k^5, \end{cases}$$

and

$$\beta_{n+1} = \begin{cases} m_k^{3/m_k} \beta_n & \text{for } m_k \leq n \leq 2m_k - 1; \\ 9^{1/m_k} m_k^{-3/m_k} \beta_n & \text{for } 2m_k \leq n \leq m_{k+1} - 1. \end{cases}$$

Since  $\beta_n \gtrsim n^2$ , we have  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ . Since  $m_k^{1/m_k} \xrightarrow[k \rightarrow \infty]{} 1$ , we have  $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$ . Moreover, setting  $e_n(z) = z^n$ , we have

$$\frac{\|e_{m_k}^2\|}{\|e_{m_k}\|^2} = \frac{\|e_{2m_k}\|}{\|e_{m_k}\|^2} = \frac{\beta_{2m_k}^{1/2}}{\beta_{m_k}} = m_k^{1/2} = 3^{k/2} \xrightarrow[k \rightarrow \infty]{} +\infty;$$

hence  $H^2(\beta)$  is not an algebra, since otherwise, we would have a positive constant  $C$  for which  $\|fg\| \leq C\|f\|\|g\|$  for all  $f, g \in H^2(\beta)$ .  $\square$



Theorem 2.10 allows to answer negatively Zorboska's question 2).

**Theorem 2.11.** *There exist a sequence  $\beta$  such that  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$  and  $\beta_{n+1}/\beta_n \xrightarrow[n \rightarrow \infty]{} 1$ , and a symbol  $\varphi$  such that  $\varphi \in H^2(\beta)$  and  $\|\varphi\|_{\infty} < 1$ , but  $C_{\varphi}$  is not bounded on  $H^2(\beta)$ .*

*Proof.* We use the sequence obtained in Theorem 2.10. There exists  $\varphi \in H^2(\beta) \subset H^{\infty}$  such that  $\varphi^2 \notin H^2(\beta)$ ; we can assume that  $\|\varphi\|_{\infty} < 1$ . Clearly  $C_{\varphi}$  is not bounded on  $H^2(\beta)$  since  $z^2 \in H^2(\beta)$  and  $C_{\varphi}(z^2) = \varphi^2$ .  $\square$

### 2.3 Slowly oscillating sequences

In Corollary 2.9, we saw that  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < \infty$  is necessary for  $H^2(\beta)$  to be an algebra, and in [3, Theorem 3.2.7], it is shown that the converse holds when  $\beta$  satisfies some conditions. Here, we see that the converse holds with another regularity condition on  $\beta$ , namely when  $\beta$  is slowly oscillating (recall that we saw in Theorem 2.2, assuming that  $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$ , that  $\beta$  slowly oscillating is necessary in order that every symbol  $\varphi \in H^2(\beta)$  induces a bounded composition operator on  $H^2(\beta)$ ).

**Theorem 2.12.** *If  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$  and  $\beta$  is slowly oscillating, then  $H^2(\beta)$  is an algebra.*

Using Theorem 2.7, we get the following corollary.

**Theorem 2.13.** *If  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$  and  $\beta$  is slowly oscillating, then all symbols  $\varphi \in H^2(\beta)$  induce bounded composition operators on  $H^2(\beta)$ .*

Theorem 2.12 can be deduced from the condition (3.6.1) of [3, Lemma 3.6.3], but we give a direct proof for sake of completeness. We introduce the following notation:

$$(2.7) \quad B_n = \sum_{k=0}^n \frac{1}{\beta_k \beta_{n-k}}.$$

Note that  $B_n \geq \frac{1}{\beta_0 \beta_n}$  so  $\inf_{n \geq 0} B_n \beta_n > 0$ .

The proof will follow from the next two lemmas.

**Lemma 2.14.** *If  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$  and  $\beta$  is slowly oscillating, then  $\beta_n B_n = O(1)$ .*

*Proof.* Set  $M = \sum_{n=0}^{\infty} \frac{1}{\beta_n}$ . We have

$$B_n \leq 2 \sum_{0 \leq k \leq n/2} \frac{1}{\beta_k \beta_{n-k}}.$$

But  $\frac{1}{\beta_{n-k}} \leq C \frac{1}{\beta_n}$ , since  $n/2 \leq n-k \leq n$  for  $0 \leq k \leq n/2$ , and  $\beta$  is slowly oscillating. Hence

$$B_n \leq 2C \frac{1}{\beta_n} \sum_{0 \leq k \leq n/2} \frac{1}{\beta_k} \leq 2CM \frac{1}{\beta_n}. \quad \square$$

The second lemma is nothing else than the condition (3.2.2) of [3] (see also [7, Lemma, page 207]).

**Lemma 2.15.** *If  $\beta_n B_n = O(1)$ , then  $H^2(\beta)$  is an algebra.*

*Proof.* It suffices to show that  $f^2 \in H^2(\beta)$  for all  $f \in H^2(\beta)$ . Let  $f \in H^2(\beta)$  and write  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ . We have

$$[f(z)]^2 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k a_{n-k} \right) z^n.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_k a_{n-k} \right|^2 \beta_n &\leq \sum_{n=0}^{\infty} \left( \sum_{k=0}^n |a_k a_{n-k}|^2 \beta_k \beta_{n-k} \right) \left( \sum_{k=0}^n \frac{1}{\beta_k \beta_{n-k}} \right) \beta_n \\ &\lesssim \sum_{n=0}^{\infty} \left( \sum_{k=0}^n |a_k a_{n-k}|^2 \beta_k \beta_{n-k} \right) \\ &= \left( \sum_{j=0}^{\infty} |a_j|^2 \beta_j \right)^2 = \|f\|^4 < +\infty, \end{aligned}$$

which says that  $f^2 \in H^2(\beta)$ .  $\square$

As said, when  $\beta$  is “regular”,  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$  is also necessary in order that  $H^2(\beta)$  be an algebra ([3, Theorem 3.2.7]). In general, we have no necessary and sufficient condition. However, we can state the following necessary conditions.

**Proposition 2.16.** *If  $H^2(\beta)$  is an algebra, then*

- 1)  $\beta_n B_n = O(n)$ ;
- 2)  $\sum_{n=0}^{\infty} \beta_n B_n^2 < +\infty$ .

*Proof.* 1) By (2.4), we have  $\frac{1}{\beta_k \beta_l} \leq C \frac{1}{\beta_{k+l}}$ ; so

$$B_n = \sum_{k+l=n} \frac{1}{\beta_k \beta_l} \leq Cn \frac{1}{\beta_n}.$$

2) Let  $f(z) = \sum_{n=0}^{\infty} \frac{1}{\beta_n} z^n$ ; we have  $\|f\|^2 = \sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ , so  $f \in H^2(\beta)$ . Since  $H^2(\beta)$  is an algebra, we have  $f^2 \in H^2(\beta)$ . But  $[f(z)]^2 = \sum_{n=0}^{\infty} B_n z^n$ , so we get  $\sum_{n=0}^{\infty} B_n^2 \beta_n < +\infty$ .  $\square$

The necessary conditions of Corollary 2.9 and of Proposition 2.16 are not sufficient.

**Proposition 2.17.** *There exists a sequence  $\beta$  such that  $\sum_{n=0}^{\infty} \beta_n B_n^2 < +\infty$ , hence  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ , but for which  $H^2(\beta)$  is not an algebra.*

*Proof.* We take  $\beta_0 = 1$ , and, for  $n \geq 1$ ,  $\beta_n = n^\gamma$  for  $n$  even and  $\beta_n = n^{\gamma'}$  for  $n$  odd, where  $1 < \gamma' < \gamma$  and  $2\gamma' > \gamma + 1$  (for example,  $\gamma' = 3$  and  $\gamma = 4$ ). Since  $\beta_{n+1}/\beta_n$  is not bounded,  $H^2(\beta)$  is not an algebra. It is clear that  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ .

Now

$$\begin{aligned} B_{2n} &= \sum_{k=0}^{2n} \frac{1}{\beta_k \beta_{2n-k}} \approx \sum_{k \text{ even}} \frac{1}{k^\gamma (2n-k)^\gamma} + \sum_{k \text{ odd}} \frac{1}{k^{\gamma'} (2n-k)^{\gamma'}} \\ &\lesssim \frac{1}{n^\gamma} + \frac{1}{n^{\gamma'}} \lesssim \frac{1}{n^{\gamma'}}, \end{aligned}$$

so  $(2n)^\gamma B_{2n}^2 \lesssim (2n)^{\gamma-2\gamma'}$ .

Similarly  $(2n+1)^{\gamma'} B_{2n+1}^2 \lesssim (2n+1)^{-\gamma'}$ .

Since  $\gamma - 2\gamma' < -1$  and  $\gamma' > 1$ , we obtain the convergence of the series  $\sum_{n \geq 0} \beta_n B_n^2$ .  $\square$

Nevertheless,  $\beta$  slowly oscillating is not necessary for  $H^2(\beta)$  to be an algebra.

**Example 2.18.** *If  $\beta_n = e^{\sqrt{n}}$  for all  $n \geq 0$ , then  $H^2(\beta)$  is an algebra, though  $\beta$  is not slowly oscillating.*

That  $\beta$  is not slowly oscillating is clear. That  $H^2(\beta)$  is an algebra, follows from [3, Lemma 3.5.5].

As a consequence, using Theorem 2.2, we get the following result.

**Theorem 2.19.** *There exist a sequence  $\beta$  such that  $H^2(\beta)$  is an algebra, and a symbol  $\varphi \in H^2(\beta)$  for which  $C_\varphi$  is not bounded on  $H^2(\beta)$ .*

## 2.4 Hankel matrices

Actually the problem to know whether  $H^2(\beta)$  is an algebra or not can be formulated in terms of Schur multipliers acting on a family of Hankel matrices.

**Theorem 2.20.**  *$H^2(\beta)$  is an algebra if and only if the map*

$$\begin{aligned} \Psi: \ell^2 &\longrightarrow B(\ell^2) \\ u &\longmapsto \left( u_{k+l} \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}} \right)_{k,l} \end{aligned}$$

*is bounded.*

Let us point out that Lemma 2.15 also follows from this theorem. Actually, the condition  $\beta_n B_n = O(1)$  is equivalent to the fact that  $\Psi$  is bounded from  $\ell^2$  to  $HS(\ell^2) \subseteq B(\ell^2)$ , where  $HS(\ell^2)$  stands for the space of Hilbert-Schmidt operators on  $\ell^2$ .

Indeed, for every  $u \in \ell^2$ , we have  $\|\Psi(u)\|_{HS}^2 = \sum_n |u_n|^2 \beta_n B_n$ .

This remark gives a hint on the gap between the sufficient condition  $\beta_n B_n = O(1)$  and a potential characterization.

*Proof of Theorem 2.20.* The vector space  $H^2(\beta)$  is an algebra if and only if  $fg \in H^2(\beta)$  for all  $f, g \in H^2(\beta)$ . In other words,  $H^2(\beta)$  is an algebra if and only if

$$\sum_{n=0}^{\infty} \left| \sum_{p=0}^n a_p b_{n-p} \right|^2 \beta_n < +\infty$$

whenever  $\sum_{n=0}^{\infty} |a_n|^2 \beta_n < +\infty$  and  $\sum_{n=0}^{\infty} |b_n|^2 \beta_n < +\infty$ . Equivalently,

$$\sup_{\substack{a, b \in \ell^2 \\ \|a\| = \|b\| = 1}} \sum_{n=0}^{\infty} \left| \sum_{k+l=n} \frac{a_k}{\sqrt{\beta_k}} \frac{b_l}{\sqrt{\beta_l}} \sqrt{\beta_{k+l}} \right|^2 < +\infty$$

or, in other words,

$$\sup_{\substack{a, b, u \in \ell^2 \\ \|a\| = \|b\| = \|u\| = 1}} \left| \sum_{n=0}^{\infty} \sum_{k+l=n} a_k b_l u_n \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}} \right| < +\infty.$$

Let us point out that

$$\sum_{n=0}^{\infty} \sum_{k+l=n} a_k b_l u_n \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}} = \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} M_{k,l} b_l = \sum_{k=0}^{\infty} a_k \left( \Psi(u)(b) \right)_k$$

where  $M_{k,l} = u_{k+l} \sqrt{\frac{\beta_{k+l}}{\beta_k \beta_l}}$ .

Therefore,  $H^2(\beta)$  is an algebra if and only if

$$\sup_{\substack{b, u \in \ell^2 \\ \|b\| = \|u\| = 1}} \left\| \Psi(u)(b) \right\| < +\infty.$$

and the result follows.  $\square$

As an application of Theorem 2.20, we see that the equivalence between “ $H^2(\beta)$  is an algebra” and  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$  does not follow from the submultiplicativity of  $\beta$ .

**Theorem 2.21.** *There exists a submultiplicative sequence  $\beta$  such that  $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$  and  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$ , and for which  $H^2(\beta)$  is not an algebra.*

*Proof.* Let  $m_k = 3^k$  and  $\sigma(n)$  defined by  $\sigma(0) = 1$  and

$$\sigma(n) = k \quad \text{for } m_{k-1} \leq n < m_k \text{ and } k \geq 1.$$

We define

$$\beta_n = \exp(n/\sigma(n)).$$

- The submultiplicativity of  $\beta$  is obvious since  $\sigma$  is nondecreasing:

$$\frac{m+n}{\sigma(m+n)} = \frac{m}{\sigma(m+n)} + \frac{n}{\sigma(m+n)} \leq \frac{m}{\sigma(m)} + \frac{n}{\sigma(n)}.$$

- That  $\lim_{n \rightarrow \infty} \beta_n^{1/n} = 1$  is clear.
- That  $\sum_{n=0}^{\infty} \frac{1}{\beta_n} < +\infty$  is easy: since  $m_k - m_{k-1} = 2m_k/3$ , we get using geometric series:

$$\begin{aligned} \sum_{n=0}^{\infty} \exp[-n/\sigma(n)] &= \sum_{k=1}^{\infty} \left( \sum_{m_{k-1} \leq n < m_k} \exp(-n/k) \right) \lesssim \sum_{k=1}^{\infty} \frac{\exp(-2m_k/3)}{1 - e^{-1/k}} \\ &\lesssim \sum_{k=1}^{\infty} k \exp(-2m_k/3) < +\infty. \end{aligned}$$

- We give two proofs that  $H^2(\beta)$  is not an algebra.

– *First proof.* Let  $A_n$  be the  $(n \times n)$  self-adjoint matrix with all entries equal to 1, which satisfies  $A_n^2 = nA_n$ . Hence  $\|A_n\|^2 = \|A_n^2\| = n\|A_n\|$  and so  $\|A_n\| = n$ .

Let us be more specific. Consider the interval of integers  $I_k = [m_k/3, m_k/2)$  and let  $u = \mathbf{1}_{2I_k} = \mathbf{1}_{[2m_k/3, m_k)} \in \ell^2$ . We have  $\|u\|_{\ell^2} \approx \sqrt{m_k}$ .

Now, let  $\Psi(u)$  be the matrix  $\left( u_{i+j} \sqrt{\frac{\beta_{i+j}}{\beta_i \beta_j}} \right)_{i,j}$ . We recall that, with obvious notations:

$$|a_{i,j}| \leq b_{i,j} \quad \implies \quad \|A\| \leq \|B\|.$$

Therefore, since  $i, j \in I_k \implies i+j \in 2I_k$ :

$$\begin{aligned} \|\Psi(u)\| &\geq \left\| \left( u_{i+j} \sqrt{\frac{\beta_{i+j}}{\beta_i \beta_j}} \right)_{i,j \in I_k} \right\| = \left\| \left( \sqrt{\frac{\beta_{i+j}}{\beta_i \beta_j}} \right)_{i,j \in I_k} \right\| \\ &= \|(1)_{i,j \in I_k}\| = |I_k| \approx m_k. \end{aligned}$$

Indeed, for  $i, j \in I_k$ , we have  $i, j, i+j \in [m_k/3, m_k) = [m_{k-1}, m_k)$ , so that

$$\beta_{i+j} = \exp[(i+j)/k] = \exp(i/k) \exp(j/k) = \beta_i \beta_j,$$

and we use the norm of the matrix  $A_{|I_k|}$ .

Hence the quotient  $\frac{\|\Psi(u)\|_{B(\ell^2)}}{\|u\|_{\ell^2}}$  is unbounded as  $k \rightarrow \infty$ .

– *Second proof.* Actually, this proof was obtained thanks to the first one. Let  $f = f_k$  be the function defined by

$$f(z) = \sum_{m_k/3 \leq n < m_k/2} z^n = \sum_{n \in I_k} z^n.$$

We are going to prove that

$$(2.8) \quad \frac{\|f^2\|}{\|f\|^2} \xrightarrow{k \rightarrow \infty} +\infty.$$

For that, we first prove that

$$(2.9) \quad \|f\|^2 \lesssim k \exp\left(\frac{m_k}{2k}\right).$$

Indeed, setting again  $I_k = [m_k/3, m_k/2)$  and using  $1/3 + 1/6 = 1/2$ :

$$\|f\|^2 = \sum_{n \in I_k} \exp\left(\frac{n}{k}\right) = e^{m_k/3k} \sum_{0 \leq j \leq m_k/6} e^{j/k} \lesssim \frac{\exp\left(\frac{m_k}{2k}\right)}{\exp\left(\frac{1}{k}\right) - 1} \lesssim k \exp\left(\frac{m_k}{2k}\right).$$

It remains to show that

$$(2.10) \quad \|f^2\| \gtrsim k^{3/2} \exp\left(\frac{m_k}{2k}\right).$$

For that, setting  $J_k = [2m_k/3, m_k) = 2I_k$ , we have  $f^2 = \sum_{n \in J_k} c_n z^n$ , and

$$\|f^2\|^2 = \sum_{n \in J_k} c_n^2 \exp\left(\frac{n}{k}\right),$$

where  $c_n$  is the number of indices  $i \in I_k$  such that  $n - i \in I_k$ , that is

$$\max(m_k/3, n - m_k/2) \leq i \leq \min(m_k/2, n - m_k/3).$$

When  $(5/6)m_k \leq n < m_k$ , this amounts to  $n - m_k/2 \leq i < m_k/2$ , implying  $c_n \geq m_k - n$ .

Setting  $q = e^{1/k}$  and observing that  $m_k^2 q^{-m_k/6} \xrightarrow{k \rightarrow \infty} 0$ , we get

$$\|f^2\|^2 \geq \sum_{5/6m_k \leq n < m_k} (m_k - n)^2 q^n = q^{m_k} \sum_{0 \leq j \leq m_k/6} j^2 q^{-j} \approx q^{m_k} (1 - q^{-1})^{-3},$$

that is  $\|f^2\|^2 \gtrsim k^3 \exp(m_k/k)$ , since  $(1 - q^{-1})^{-1} \sim k$ , and that proves (2.10).  $\square$

**Remark.** Let us observe that, in the above proof,  $\sigma(n)$  behaves, roughly speaking, as  $\log n$ , so that  $\beta_n$  behaves as  $\gamma_n = \exp(n/\log n)$ , although  $H^2(\gamma)$  is an algebra, as said in the next proposition. The difference between  $\beta_n = \exp[n/\sigma(n)]$ , and  $\gamma_n$ , can be heuristically explained as follows. Set  $\tau(x) = \frac{x}{\log x}$  for  $x > 1$  and  $\tilde{\tau}_n = n/\sigma(n)$ . In the discrete case, we have

$$j, n \in [m_k/3, m_k/2] \implies \tilde{\tau}(n+j) - \tilde{\tau}(n) - \tilde{\tau}(j) = 0,$$

meaning that  $\tilde{\tau}'' = 0$ , while in the continuous case we have

$$\tau''(x) \sim \frac{-1}{x(\log x)^2} < 0.$$

Proposition 2.22 follows from [3, Theorem 3.2.7], but, for sake of completeness, we prefer to give a short elementary proof of this particular case.

**Proposition 2.22.** *For  $\gamma(n) = e^{n/\log n}$ ,  $n \geq 2$  (and  $\gamma_0 = \gamma_1 = 1$ , for instance), the space  $H^2(\gamma)$  is an algebra.*

*Proof.* Let

$$\alpha_{n,k} = \frac{n}{\log n} - \frac{k}{\log k} - \frac{n-k}{\log(n-k)}.$$

We have to prove, using Lemma 2.15, that

$$\sup_n \left( \sum_{k=2}^{n/2} \exp(\alpha_{n,k}) \right) < +\infty.$$

Observe that, for  $2 \leq k \leq \sqrt{n}$ , we have

$$\alpha_{n,k} \leq \frac{n}{\log n} - \frac{k}{\log k} - \frac{n-k}{\log n} = k \left( \frac{1}{\log n} - \frac{1}{\log k} \right) \leq -\frac{k}{2 \log k};$$

hence

$$\sum_{k=2}^{\sqrt{n}} \exp(\alpha_{n,k}) \leq \sum_{k=2}^{\infty} \exp \left( -\frac{k}{2 \log k} \right) := S < +\infty.$$

For  $\sqrt{n} \leq k \leq n/2$ , we have, in the same way:

$$\begin{aligned} \alpha_{n,k} &\leq k \left( \frac{1}{\log n} - \frac{1}{\log k} \right) \leq \sqrt{n} \left( \frac{1}{\log n} - \frac{1}{\log n - \log 2} \right) \\ &= -\frac{\sqrt{n} \log 2}{(\log n)(\log n - \log 2)}; \end{aligned}$$

hence

$$\sum_{\sqrt{n} \leq k \leq n/2} \exp(\alpha_{n,k}) \leq \frac{n}{2} \exp \left( -\frac{\sqrt{n} \log 2}{(\log n)(\log n - \log 2)} \right) \xrightarrow{n \rightarrow \infty} 0. \quad \square$$

### 3 Composition operators induced by the automorphisms on $h^p(\beta)$

For  $p = 2$ , it is easy to check that all the composition operators  $C_{T_a}$ , for  $a \in \mathbb{D}$ , generated by the automorphisms  $T_a$ , defined in (2.3), (and actually all composition operators, by Littlewood's subordination principle) are bounded

on  $H^2$ . We recently studied [4] the weighted version  $H^2(\beta)$  of this space and gave a complete characterization of those weights  $\beta = (\beta_n)$  for which either  $C_\varphi$  is bounded on  $H^2(\beta)$  for all automorphisms, or for all symbols. In this section, we show that this is never the case for  $p \neq 2$ . We mention in passing that this boundedness issue was previously considered by Blyudze and Shimorin [1] in the case when the initial space is  $h^1$  and  $h^p$  is the target space. In that case, the authors show that  $C_{T_a} : h^1 \rightarrow h^p$  is bounded if and only if  $p \geq 2$ . This has been made much more precise in [10].

We will prove the following result.

**Theorem 3.1.** *Let  $p \in [1, +\infty]$ , with  $p \neq 2$ . Then  $C_{T_a}$  is unbounded on  $h^p$  for all  $a \in \mathbb{D} \setminus \{0\}$ .*

*Moreover,  $C_{T_a}$  is never bounded on  $h^p(\beta)$ , whatever the choice of  $\beta$  satisfying (1.1).*

S. Charpentier, N. Espouiller and R. Zarouf informed us that they proved, independently, Theorem 3.1 for  $\beta \equiv 1$ , using [10] (see [2]).

*Proof.* Set

$$I = \left[ \frac{1}{2}, \frac{2}{3} \right] \quad \text{and} \quad J = [\alpha^{-1}, \alpha],$$

( $\alpha = 5/4$  for instance).

As in [4, Proposition 4.2], the rotation invariance of  $h^p(\beta)$  allows us to claim that, if  $C_{T_a}$  is bounded on  $h^p(\beta)$  for some  $a \in \mathbb{D}$ ,  $a \neq 0$ , then there exists a constant  $K$  such that

$$(3.1) \quad \|C_{T_a}\| \leq K, \quad \forall a \in I.$$

Now, everything will rely on the matrix  $A = (a_{m,n})_{m,n}$ , where

$$a_{m,n} = \widehat{(T_a)^n}(m) \left( \frac{\beta_m}{\beta_n} \right)^{1/p},$$

which represents  $C_{T_a}$  on the canonical (Schauder) basis of  $h^p(\beta)$ . If this matrix defines a bounded operator, its columns and rows (the columns of the transposed operator) are respectively uniformly bounded on  $\ell^p$  and on the dual space  $\ell^q$  ( $q$  the conjugate exponent of  $p$ ), that is (with some  $R$  independent of  $a \in I$ ):

$$(3.2) \quad C_n = \sum_{m=0}^{\infty} |a_{m,n}|^p \frac{\beta_m}{\beta_n} \quad \text{satisfies} \quad \sup_n C_n \leq R < +\infty,$$

and

$$(3.3) \quad L_m = \sum_{n=0}^{\infty} |a_{m,n}|^q \left( \frac{\beta_m}{\beta_n} \right)^{q/p} \quad \text{satisfies} \quad \sup_m L_m \leq R < +\infty.$$

We will show that, for  $p \neq 2$ , one of the necessary conditions (3.2) or (3.3) fails. We will hence separate two cases.

We need an auxiliary result. Recall first the following elementary lemma (see [4, Lemma 4.16]).



**Lemma 3.2.** Let  $f: [A, B] \rightarrow \mathbb{R}$  be a  $\mathcal{C}^2$ -function such that  $|f'| \geq \delta$  and  $|f''| \leq M$ . Then

$$\left| \int_A^B e^{if(x)} dx \right| \leq \frac{2}{\delta} + \frac{M(B-A)}{\delta^2}.$$

This lemma implies the following extended version of [4, Proposition 4.12].

**Proposition 3.3.** If  $r \in J = [\alpha^{-1}, \alpha]$  and  $s \geq 1$ , it holds, when  $r = m/n$ :

$$(3.4) \quad \int_I |\widehat{T}_a^n(m)|^s da \gtrsim \left( \int_I |\widehat{T}_a^n(m)| da \right)^s \geq \delta n^{-s/2}.$$

*Proof.* The first inequality in (3.4) is just Hölder's inequality. For the second one, we proved in [4, Proposition 4.14] that

$$(3.5) \quad |\widehat{T}_a^n(m)| \geq \delta n^{-1/2} |\cos(n\psi_r(a) + \pi/4)| + O(n^{-3/5}),$$

where  $\psi_r = f$  satisfies the assumptions of Lemma 3.2, and the constant controlling the error term  $O(n^{-3/5})$  does not depend on  $a$ . Moreover, we have the classical Fourier expansion

$$|\cos x| = c + \sum_{l=1}^{\infty} \delta_l \cos lx,$$

with  $c > 0$  and  $\delta_l = O(l^{-2})$ . Hence (actually  $\sum_{l=1}^{\infty} |\delta_l| < +\infty$  would suffice):

$$(3.6) \quad |\widehat{T}_a^n(m)| \geq \delta n^{-1/2} \left( c + \sum_{l=1}^{\infty} \delta_l \cos(l(n\psi_r(a) + \pi/4)) \right) + O(n^{-3/5}).$$

We now apply Lemma 3.2 with  $f(a) = l(n\psi_r(a) + \pi/4)$ . Here, for given  $l$ , we have  $|f'| \geq \delta nl$  and  $|f''| \leq Mnl$  on  $I$ , so that

$$\left| \int_I \cos(l(n\psi_r(a) + \pi/4)) da \right| \lesssim \left( \frac{1}{nl} + \frac{nl}{n^2 l^2} \right) \approx \frac{1}{nl}.$$

It now follows from (3.5) that

$$\begin{aligned} \int_I |\widehat{T}_a^n(m)| da &\geq \delta n^{-1/2} \left( c + O\left( \sum_{l=1}^{\infty} \frac{|\delta_l|}{nl} \right) \right) + O(n^{-3/5}) \\ &\geq \delta c n^{-1/2} + O(n^{-3/5}) \geq \delta' n^{-1/2}, \end{aligned}$$

and this ends the proof.  $\square$

We now come back to the proof of Theorem 3.1. We will reason by contradiction, and separate two cases.

We set  $J_l = [l/\sqrt{\alpha}, \sqrt{\alpha} l]$ .

- Case  $p < 2$ .

It follows from (3.2) that, for  $n \in J_l$ , we have

$$\sum_{m \in J_l} |\widehat{T}_a^n(m)|^p \frac{\beta_m}{\beta_n} \leq R.$$

Integrating on  $I$  and using Proposition 3.3 give, since  $m, n \in J_l$  (hence  $m/n \in J$ ) have the same size as  $l$ :

$$l^{-p/2} \sum_{m \in J_l} \frac{\beta_m}{\beta_n} \lesssim R.$$

Now, summing up over  $n \in J_l$  further gives

$$l^{-p/2} \left( \sum_{k \in J_l} \beta_k \right) \left( \sum_{k \in J_l} \beta_k^{-1} \right) \lesssim Rl.$$

But, by the Cauchy-Schwarz inequality:

$$l^2 \lesssim |J_l|^2 \leq \left( \sum_{k \in J_l} \beta_k \right) \left( \sum_{k \in J_l} \beta_k^{-1} \right),$$

and we get  $l^{2-p/2} \lesssim Rl$  or  $R \gtrsim l^{1-p/2}$ .

Since  $1 - p/2 > 0$ , we have a contradiction for large  $l$ .

- *Case  $p > 2$ .*

The proof is nearly the same. It now follows from (3.3) that, for  $m \in J_l$ :

$$\sum_{n \in J_l} |\widehat{T}_a^n(m)|^q \left( \frac{\beta_m}{\beta_n} \right)^{q/p} \leq R.$$

Set  $\gamma_k = \beta_k^{q/p}$  and proceed as in the case  $p < 2$  to get

$$l^{-q/2} \left( \sum_{k \in J_l} \gamma_k \right) \left( \sum_{k \in J_l} \gamma_k^{-1} \right) \lesssim Rl,$$

or, again by the Cauchy-Schwarz inequality,  $l^{2-q/2} \lesssim Rl$  or  $R \gtrsim l^{1-q/2}$ .

Since this time  $1 - q/2 > 0$ , we have again a contradiction for large  $l$ .  $\square$

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