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ABSTRACT. Let F be a field of characteristic 2. Our aim in this paper is to prove that the extension given by the function field of a singular conic is excellent for bilinear forms. We also give examples showing that in general this extension is not excellent for quadratic forms of arbitrary dimension.

1. INTRODUCTION

Throughout this paper F denotes a field of charactersitic 2. By considering some problems related to the behavior of F-quadratic forms under a field extension K/F, we are sometimes led to consider the difficult question of whether the extension K/F is excellent or not. Recall that K/F is *excellent* for quadratic (bilinear) forms if for any quadratic (bilinear) form φ over F, the anisotropic part $(\varphi_K)_{\text{an}}$ of the K-form φ_K is defined over F, i.e., there exists a quadratic (bilinear) form ψ over F such that $(\varphi_K)_{\text{an}}$ is isometric to ψ_K.

For example, the excellence property holds for totally singular quadratic forms and any field extension of F [5, Proposition 8.1(iii)]. Moreover, the following examples of extensions are excellent for quadratic and bilinear forms:

(1) A purely transcendental extension. This is because any anisotropic F-form stays anisotropic over a purely transcendental extension of F.
(2) An algebraic extension of odd degree. This is a consequence of Springer's theorem [1, Corollary 18.5].
(3) A quadratic extension. This is due to Hoffmann and Laghribi [6, Lemma 5.4].
(4) More generally, multiquadratic purely inseparable extensions. This is due to Hoffmann for quadratic and bilinear forms [3], [4].

Recently the authors studied the excellence of quartic extensions. Among others, they proved that a mixed biquadratic extension is excellent for quadratic forms [12]. It is not difficult to see that this extension is also excellent for bilinear forms.

For extensions given by function fields of quadrics, the case of quadratic forms of dimension 2 is covered by quadratic extensions which are excellent. In this paper we discuss the excellence property for function fields of conics, i.e., functions fields $F(Q)$ of projective quadrics given F-quadratic forms Q of dimension 3. Recall that such a field extension is excellent in characteristic not 2. This was first proved by Arason [2, Appendix II] and later by Rost who produced an elementary proof that presents the advantage of being extendable to characteristic 2 [13]. In characteristic 2, the form Q could be of type $(1,1)$ or $(0,3)$ (see below for the definition of the type of a quadratic form). When Q is of type $(1,1)$, the extension $F(Q)$ is excellent for quadratic forms as well as bilinear forms. For quadratic forms, this is due to Hoffmann and Laghribi who easily adapted Rost's proof to characteristic 2 [6, Corollary 5.7], and for bilinear forms as a consequence of the result [7, Corollary 3.3] stating that an anisotropic totally singular form (and thus an anisotropic bilinear form) stays anisotropic over the function field of a form which is not totally singular. Now we address the situation when Q is of type $(0,3)$. Generally, in this case the extension $F(Q)$ is not excellent for quadratic forms (see Corollaries 10 and 11). For bilinear forms we will prove that such an extension is excellent.

MSC: 11E04, 11E81
Date: August 2, 2023.
Key words and phrases. Quadratic (bilinear) form, Isotropy, excellence, function field of a conic.

Theorem 1. *Let B be an anisotropic bilinear form over F and Q an anisotropic F-quadratic form of type $(0,3)$. Then, there exists an F-bilinear form C such that $(B_{F(Q)})_{\mathrm{an}} \simeq C_{F(Q)}$. Hence, the extension $F(Q)/F$ is excellent for bilinear forms.*

This theorem played a crucial role for the descent of bilinear forms recently studied by the two authors [11]. Our proof of this theorem is inspired from Rost's proof in characteristic not 2, but we will adapt many arguments to our situation. The complications come from the fact that we consider inseparable quadratic extensions in characteristic 2. More precisely, it is known in characteristic 2 that if an anisotropic F-bilinear form becomes isotropic over an inseparable quadratic extension $F(\sqrt{a})$, then B contains, up to a nonzero scalar, a subform isometric to $\langle 1, a+x^2\rangle_b$ for some $x \in F$ [8, Lemma 3.4], but not the form $\langle 1, a\rangle_b$ in general (see the comments after Lemma 2). Moreover, some arguments given by Rost based on the trace map for quadratic extensions fail in characteristic 2 for quadratic inseparable extensions.

2. Background on quadratic and bilinear forms

Recall that any quadratic form φ over F can be written up to isometry as follows:

$$\varphi \simeq [a_1, b_1] \perp [a_2, b_2] \perp \ldots \perp [a_r, b_r] \perp \langle c_1, \ldots, c_s\rangle, \tag{1}$$

where $\perp$ denotes the orthogonal sum of quadratic forms, and $[a_i, b_i]$ (*resp.* $\langle c_1, \ldots, c_s\rangle$) denotes the quadratic form $a_ix^2 + xy + b_iy^2$ (*resp.* $\sum_{i=1}^{s} c_ix_i^2$). In this case, we say that φ is of type (r, s). As in equation (1), the form φ is called:

- nonsingular (*resp.* singular) if $s = 0$ (*resp.* $s > 0$),
- totally singular if $r = 0$,

A quadratic form φ of underlying vector space V is called isotropic if there exists a nonzero vector $v \in V$ such that $\varphi(v) = 0$, otherwise φ is called anisotropic. Note that φ is isotropic iff φ contains the form $[0, 0]$ or $\langle 0\rangle$ as a subform. A scalar $\alpha \in F$ is represented by φ if there exists $v \in V$ such that $\varphi(v) = \alpha$. We denote by $D_F(\varphi)$ the set of scalars in $F^* := F \setminus \{0\}$ represented by φ.

Any quadratic form φ uniquely decomposes as follows:

$$\varphi \simeq i \times [0, 0] \perp j \times \langle 0\rangle \perp \varphi_{\mathrm{an}},$$

for some integers i and j, where φ_{an} is anisotropic that we call the anisotropic part of φ.

All bilinear forms considered in this paper are supposed to be symmetric, regular and of finite dimension. The regularity means that the radical of the bilinear form is trivial. Such a bilinear form of dimension n is isometric to the diagonal bilinear form:

$$((x_1, \cdots, x_n), (y_1, \cdots, y_n)) \mapsto \sum_{i=1}^{n} a_ix_iy_i$$

for some $a_1, \cdots, a_n \in F^*$. This diagonal bilinear form is denoted by $\langle a_1, \cdots, a_n\rangle_b$.

To any bilinear form B of underlying F-vector space V, we attach a totally singular quadratic form $\widetilde{B}$ defined on V by: $\widetilde{B}(v) = B(v, v)$ for all $v \in V$. This quadratic form depends only on the isometry class of B. A scalar is said to be represented by B when it is represented by $\widetilde{B}$. We say that B is isotropic when $\widetilde{B}$ is isotropic, otherwise B is called anisotropic. Note that B is isotropic iff B contains as a subform a metabolic plane $\mathbb{M}(a) := \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ for some $a \in F$.

As for quadratic forms, any bilinear form B uniquely decomposes as follows:

$$B \simeq \mathbb{M}(a_1) \perp \cdots \perp \mathbb{M}(a_n) \perp B_{\mathrm{an}},$$

for suitable $a_1, \cdots, a_n \in F$, where B_{an} is an anisotropic bilinear form that we call the anisotropic part of B.

Any quadratic form φ of underlying F-vector space V and type (r, s) can be viewed as a homogeneous polynomial P_φ of degree 2 after choosing an F-basis of V. This polynomial is reducible if ($r = 1$ and $\dim \varphi_{\mathrm{an}} = 0$) or ($r = 0$ and $\dim \varphi_{\mathrm{an}} = 1$) [5]. If P_φ is irreducible, we denote by $F(\varphi)$ the function field of the projective quadric given by φ, otherwise we take $F(\varphi) = F$.

The function field of a bilinear form B is defined to be the function field of the quadratic form $\widetilde{B}$.

3. Proof of Theorem 1

We fix $Q = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form over F. We take for $F(Q)$ the function field of the projective conic given by Q. Then, $F(Q)$ is isomorphic to the quotient field of the ring $R = F[x, y]/(y^2 + ax^2 + b)$. Note that $R = F[x] \oplus yF[x]$ as F-vector space. Define $d : \mathbb{R} \to \mathbb{N} \cup \{-\infty\}$ by:

$$d(P + yQ) = \max\{\deg P, 1 + \deg Q\} \quad \text{for all} \quad P, Q \in F[x]$$

(here $\deg 0 = -\infty$). Moreover, let $R_n = \{r \in R \mid d(r) \leq n\}$. Clearly, R_n is an F-vector subspace of R and we have $R_0 = F$ and $R_n.R_m \subset R_{n+m}$.

Let B be an anisotropic F-bilinear form of underlying vector space V. The key result for the proof of Theorem 1 is the following lemma which extends [13, Lemma] to the case of bilinear forms in characteristic 2:

Lemma 2. *Suppose that for some $n \geq 1$ there exists*

$$v \in (V \otimes_F R_n) \setminus (V \otimes_F R_{n-1})$$

such that $\widetilde{B}(v) = 0 \in R$. Then, there exists a subspace $W \subset V$ of dimension 2 such that:

(1) $B|_W \simeq c\,\langle 1, a + \epsilon^2\rangle_b$ for some $c \in F^*$ and $\epsilon \in F$.

(2) there exist a nonzero $v' \in V \otimes_F R_{n-1}$ and $b' \in D_F(Q)$ such that $\widetilde{B'}(v') = 0$, where

$$B' = b'(B|_W) \perp (B|_{W^\perp}).$$

Before proving this lemma, let us mention two crucial changes comparing to the original version [13, Lemma] due to Rost: In statement (1), we take the 2-dimensional bilinear forms $\langle 1, a + \epsilon^2\rangle_b$ for $\epsilon \in F$ instead of the form $\langle 1, a\rangle_b$, and in statement (2) we consider the form $b'(B|_W)$ for $b' \in D_F(Q)$ instead of $b(B|_W)$.

Proof of Lemma 2. We write

$$v = v_0 + \sum_{i=1}^{n} (v_i y x^{i-1} + w_i x^i); \quad v_i, w_i \in V.$$

Since $\widetilde{B}(v) = 0 \in R$, we get

$$\begin{aligned} 0 &= \widetilde{B}(v_n) y^2 x^{2(n-1)} + \widetilde{B}(w_n) x^{2n} \qquad \pmod{R_{2n-1}} \\ &= (\widetilde{B}(v_n) a + \widetilde{B}(w_n)) x^{2n} \qquad \pmod{R_{2n-1}}. \end{aligned}$$

Thus, $\widetilde{B}(v_n)a = \widetilde{B}(w_n)$ (i.e., $\widetilde{B}$ is isotropic over $F(\sqrt{a})$). Note that $v_n, w_n \neq 0$ since B is anisotropic and $v \notin V \otimes_F R_{n-1}$. Moreover, since $a \notin F^2$, the vectors v_n and w_n are linearly independent.

Let $c = \widetilde{B}(v_n)$, $\alpha = \sqrt{a}$ and W the 2-dimensional vector subspace of V generated by the vectors v_n, w_n. Since $\widetilde{B}|_W \simeq c\,\langle 1, a\rangle$, it follows from [10, Lemma 3.7] that $B|_W \simeq c\,\langle 1, a + \epsilon^2\rangle_b$ for a suitable $\epsilon \in F$.

Identifying W with $F(\alpha) = F \oplus F\alpha$ (by $1 \mapsto v_n$ and $\alpha \mapsto w_n$), we get $\widetilde{B}(w) = cw^2$ for all $w \in W$.

Now we write $v = p + q$ with $p \in W \otimes_F R$ and $q \in W^\perp \otimes_F R$. We can express

$$p = (y + x\alpha)x^{n-1} + (y + x\alpha)\mu x^{n-2} + \lambda x^{n-1} + \overline{p}$$

for some $\lambda, \mu \in W$ such that $\overline{p} \in W \otimes_F R_{n-2}$ and $q \in W^\perp \otimes_F R_{n-1}$ (if $n = 1$, we have $\mu = \overline{p} = 0$).

Let $b' = \lambda^2 + b \in F$ and put $v' = (b')^{-1}(\lambda + y + x\alpha)p + q$. Then, $b' \in D_F(Q)$ and v' is a zero of the form $B' = b'(B|_W) \perp (B|_{W^\perp})$ because:

$$\begin{aligned}
\widetilde{B}'(v') &= b'\widetilde{B}((b')^{-1}(\lambda + y + x\alpha)p) + \widetilde{B}(q)\\
&= b'c(b')^{-2}(\lambda + y + x\alpha)^2 p^2 + \widetilde{B}(q)\\
&= c(b')^{-1}(\lambda^2 + y^2 + x^2 a)p^2 + \widetilde{B}(q)\\
&= c(b')^{-1}(\lambda^2 + b)p^2 + \widetilde{B}(q)\\
&= \widetilde{B}(p) + \widetilde{B}(q)\\
&= \widetilde{B}(v)\\
&= 0.
\end{aligned}$$

Now to show that $v' \in V \otimes_F R_{n-1}$, we just need to verify that $(b')^{-1}(\lambda + y + x\alpha)p \in W \otimes_F R_{n-1}$. We can express $(b')^{-1}(\lambda + y + x\alpha)p$ as:

$$\begin{aligned}
(b')^{-1}(\lambda + y + x\alpha)p &= (b')^{-1}(\lambda + y + x\alpha)\Big((\lambda + y + x\alpha)x^{n-1} + (\lambda + y + x\alpha)\mu x^{n-2} + \lambda\mu x^{n-2} + \overline{p}\Big)\\
&= (b')^{-1}(\lambda^2 + y^2 + x^2 a)x^{n-1} + (b')^{-1}(\lambda^2 + y^2 + x^2 a)\mu x^{n-2}\\
&\quad +(b')^{-1}(\lambda + y + x\alpha)(\lambda\mu x^{n-2} + \overline{p})\\
&= x^{n-1} + \mu x^{n-2} + (b')^{-1}(\lambda + y + x\alpha)(\lambda\mu x^{n-2} + \overline{p}).
\end{aligned}$$

Therefore, $(b')^{-1}(\lambda + y + x\alpha)p \in W \otimes_F R_{n-1}$ and thus $(b')^{-1}(\lambda + y + x\alpha)p + q = v' \in V \otimes_F R_{n-1}$. □

Here we would like to stress that although the main idea of the proof is heavily inspired from [13], certain changes were required for our setting. The important one is the choice of the vector v', we correct it by the factor λ, this is very helpful to show that $v' \in V \otimes_F R_{n-1}$ and then avoid the use of trace form that does not apply in our case. A consequence of this change is the appearance of the scalar $b' \in D_F(Q)$.

A consequence of the previous lemma is the following proposition which is the analogue of [13, Proposition]. Its proof is the same as that done by Rost.

Proposition 3. *Let B be a bilinear form over F. Then, there exist an integer $p \geq 0$, bilinear forms B_i, C_i for $0 \leq i \leq p$ and elements $c_i \in F^*, \epsilon_i \in F$ for $0 \leq i \leq p-1$ such that $B = B_0$ and*

(1) $B_i \simeq c_i \langle 1, a + \epsilon_i^2\rangle_b \perp C_i$ for $0 \leq i \leq p-1$.
(2) $B_{i+1} \simeq c_i b_i \langle 1, a + \epsilon_i^2\rangle_b \perp C_i$ and $b_i \in D_F(Q)$ for $0 \leq i \leq p-1$.
(3) $((B_p)_{F(Q)})_{\mathrm{an}} \simeq ((B_p)_{\mathrm{an}})_{F(Q)}$.

Proof. We use induction on the dimension of B_{an}. Thus, we assume that B is anisotropic and $B_{F(Q)}$ is isotropic. Then, there exist $n \geq 0$ and a nonzero $v \in V \otimes_F R_n$ such that $\widetilde{B}(v) = 0$. We proceed by induction on n.

If $n = 0$, then $v \in V$ and B would be isotropic over F. Hence, $n \geq 1$. We may also assume $v \notin V \otimes_F R_{n-1}$ and take $B_1 = B'$, where B' is the bilinear form defined in the Lemma 2. If B' is anisotropic, we apply induction hypothesis for $n-1$ and if B' is isotropic we apply the induction hypothesis for $\dim B'_{\mathrm{an}} < \dim B$. In any case we find forms $B' = B'_0, \ldots, B'_p$ as in the proposition and $B = B_0, B_i = B'_{i-1} (i = 1, \ldots, p+1)$ is the required sequence. $\square$

Now we are able to give the proof of Theorem 1 including some explanations proper to our setting.

Proof of Theorem 1. Let B be an anisotropic bilinear form over F. We keep the same notations as in Proposition 3. All the bilinear forms B_i, for $0 \leq i \leq p$, are isometric over $F(Q)$ because $b \in D_{F(Q)}(\langle 1, a\rangle)$ implies $b_i \in D_{F(Q)}(\langle 1, a\rangle) = D_{F(Q)}(\langle 1, a + \epsilon_i^2\rangle)$ for any $b_i \in D_F(Q)$ and $\epsilon_i \in F$, meaning that $(b_i \langle 1, a + \epsilon_i^2\rangle_b)_{F(Q)} \simeq (\langle 1, a + \epsilon_i^2\rangle_b)_{F(Q)}$ by the roundness of a bilinear Pfister form. In particular, $(B_i)_{F(Q)} \simeq (B_{i+1})_{F(Q)}$. Now the bilinear form C needed in the theorem is $(B_p)_{\mathrm{an}}$. $\square$

4. Non-excellence of function fields of singular conics for quadratic forms

We finish this paper by some examples (Corollaries 10 and 11) showing the non-excellence of function fields of singular conics for quadratic forms. To this end we recall some definitions and facts on quadratic forms.

For a bilinear form B defined on an F-vector space V, and a quadratic form φ defined on an F-vector space W, we associate a quadratic form $B \otimes \varphi$ defined on $V \otimes_F W$ by:

$$B \otimes \varphi(v \otimes w) = \widetilde{B}(v)\varphi(w) \text{ for any } (v, w) \in V \times W \tag{2}$$

and whose polar form is $B \otimes B_\varphi$, where B_φ is the polar form of φ. Note that $B \otimes \varphi$ is nonsingular when φ is nonsingular.

An n-fold bilinear Pfister form is a bilinear form of type $\langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_n\rangle_b$ for some $a_i \in F^*$. An $(n+1)$-fold quadratic Pfister form is a nonsingular quadratic form of type $\langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_n\rangle_b \otimes [1, b]$ for some $a_i \in F^*$, $b \in F$. The set of forms isometric (*resp.* similar) to n-fold quadratic Pfister forms will be denoted by P_nF (*resp.* GP_nF). Recall that a quadratic Pfister form is isotropic iff it is hyperbolic. The same result is true for bilinear Pfister forms using the metabolicity, but we will not use this property.

Let IF be the fundamental ideal of the Witt ring $W(F)$ of F-bilinear forms. For any integer $n \geq 1$, let I^nF (*resp.* I_q^nF) be the n-th power of IF (*resp.* the subgroup $I^{n-1}F \otimes W_q(F)$ of the Witt group of nonsingular forms $W_q(F)$), we take $I^0F = W(F)$.

We denote by $\wp(F) = \{\alpha^2 + \alpha \mid \alpha \in F\}$. The Arf invariant $\Delta(\varphi)$ of a nonsingular form $\varphi \simeq [a_1, b_1] \perp [a_2, b_2] \perp \ldots \perp [a_r, b_r]$ is defined to be $\sum_{i=1}^r a_ib_i + \wp(F)$ in $F/\wp(F)$. Recall that a nonsingular quadratic form φ belongs to I_q^2F iff $\Delta(\varphi) = 0$.

Since we are in characteristic 2, we need to consider a refinement of the definition of excellence in the case of quadratic forms. More precisely, an extension K/F is called *(r, s)-excellent* if for any F-quadratic form φ of type (r, s), the K-form $(\varphi_K)_{\mathrm{an}}$ is defined over F. Obviously, an extension which is excellent is necessarily (r, s)-excellent for any pair of positive integers r and s.

Proposition 4. *Any field extension of F is (r, s)-excellent for $r \leq 1$.*

Proof. (1) As we mentioned in the introduction, the excellence property holds for totally singular quadratic forms, meaning that any field extension of F is $(0, s)$-excellent.

(2) Let K/F be a field extension and φ an F-quadratic form of type $(1, s)$. Let us write $\varphi = [a, b] \perp \langle c_1, \cdots, c_s\rangle$. There exists a totally singular form ψ over F such that $(\langle c_1, \cdots, c_s\rangle_K)_{\mathrm{an}} \simeq \psi_K$. Moreover, the anisotropic part of φ_K is isometric to $([a, b] \perp \psi)_K$ or ψ_K according as $[a, b] \perp \psi$ is anisotropic over K or not. $\square$

Concerning the (r, s)-excellence property for $r > 1$, we restrict ourselves to the extensions given by function fields of singular conics.

Proposition 5. *The function fields of singular conics are* $(2, 0)$*-excellent.*

Proof. Let Q be an anisotropic totally singular F-quadratic form of dimension 3, and let φ be an anisotropic F-quadratic form of type $(2, 0)$. Let $K = F(Q)$.

– If $\Delta(\varphi) = 0$, then $\varphi \in GP_2F$. Hence, φ_K is anisotropic or hyperbolic.

– If $\Delta(\varphi) \neq 0$, then φ_K is anisotropic [7, Th. 1.3].

This proves that $(\varphi_K)_{\mathrm{an}}$ is defined over F. □

Proposition 6. *In general, the function field of a singular conic is not* $(3, 0)$*-excellent.*

Proof. We produce an example showing the non-excellence of function fields of singular conics for forms of type $(3, 0)$.

Let $\mathbb{F}_2$ be the finite field with two elements. Let $\gamma = [t_1, t_2] \perp [t_3, t_4] \perp [1, t_1t_2 + t_3t_4]$ be an Albert quadratic form over the rational function field $k := \mathbb{F}_2(t_1, t_2, t_3, t_4)$ in the variables t_1, t_2, t_3, t_4 over $\mathbb{F}_2$. Let Q be the totally singular quadratic form $\langle 1, t_1, t_3\rangle$.

(1) The form $\gamma_{k(Q)}$ is isotropic because Q is dominated by γ. Then, there exists $\tau \in GP_2(k(Q))$ such that $\gamma_{k(Q)} \sim \tau$.

(2) Suppose that τ is defined over k and let δ be a k-quadratic form of dimension 4 such that $\gamma_{k(Q)} \sim \delta_{k(Q)}$. Note that $\delta \in GP_2(k)$ because if $c \in k$ satisfies $\triangle(\delta) = c + \wp(k) \in k/\wp(k)$, then $[1, c]_{k(Q)} \sim 0$ because $\triangle(\delta)_{k(Q)} = \Delta(\tau) = 0$. But then [7, Proposition 1.1] implies $[1, c] \sim 0$, that is, $\triangle(\delta) = 0 \in k/\wp(k)$.

Now the form $\gamma \perp \delta$ belongs to I_q^2k and becomes hyperbolic over $k(Q)$. Hence, we get $\gamma \perp \delta \in I_q^3k$ [9, Corollary 4.11]. Passing to $k(\delta)$, we get $\gamma_{k(\delta)} \in I_q^3k(\delta)$. By the Hauptsatz, $\gamma_{k(\delta)} \sim 0$. Using [8, Theorem 1.2], we get that $\dim \gamma_{\mathrm{an}}$ is divisible by 4, and thus γ is isotropic, a contradiction. □

Using the same arguments as in the proof of Proposition 6, we get:

Proposition 7. *In general, the function field of a singular conic is not* $(2, 1)$*-excellent.*

Proof. Let $\mathbb{F}_2$ be the finite field with two elements. Let $\gamma = [t_1, t_2] \perp [t_3, t_4] \perp \langle 1\rangle$ be a form of type $(2, 1)$ over the rational function field $k := \mathbb{F}_2(t_1, t_2, t_3, t_4)$ in the variables t_1, t_2, t_3, t_4 over $\mathbb{F}_2$. Let Q be the totally singular quadratic form $\langle 1, t_1, t_3\rangle$.

(1) The form $\gamma_{k(Q)}$ is isotropic because Q is dominated by γ. Then, there exists γ' an $F(Q)$-quadratic form of type $(1, 1)$ such that $\gamma_{k(Q)} \sim \gamma'$.

(2) Suppose that γ' is defined over k and let δ be a k-quadratic form of type $(1, 1)$ such that $\gamma_{k(Q)} \sim \delta_{k(Q)}$. We write $\delta = a[1, b] \perp \langle 1\rangle$ for $a, b \in k$ and $a \neq 0$. Using the completion lemma [5, Lem. 3.9], we get

$$([t_1, t_2] \perp [t_3, t_4] \perp [1, t_1t_2 + t_3t_4])_{k(Q)} \sim (a[1, b] \perp [1, b])_{k(Q)}.$$

Now we conclude as in the proof of Proposition 6 that the Albert form $[t_1, t_2] \perp [t_3, t_4] \perp [1, t_1t_2 + t_3t_4]$ is isotropic over k, a contradiction. □

To reach the non-excellence of function fields of singular conics for forms of type $(3 + r, s)$ with $r, s \geq 0$, and for forms of type $(2, s + 1)$ for $s \geq 0$, we use a generic argument based on the following two lemmas.

Lemma 8. *Let L/F be a field extension, t a variable over L and φ an anisotropic F-quadratic form such that φ_L is isotropic and $(\varphi_L)_{\mathrm{an}}$ is not defined over F. Then, the $F(t)$-quadratic form $\psi := \varphi \perp \langle t\rangle$ is isotropic over $L(t)$ but $(\psi_{L(t)})_{\mathrm{an}}$ is not defined over $F(t)$.*

Proof. Let $\varphi_1 = (\varphi_L)_{\mathrm{an}}$ and suppose that φ_1 is not defined over F. We have $\varphi_L \simeq i \times [0,0] \perp j \times \langle 0\rangle \perp \varphi_1$ with $i+j>0$. Hence, $(\varphi \perp \langle t\rangle)_{L(t)} \simeq i \times \mathbb{H} \perp j \times \langle 0\rangle \perp (\varphi_1 \perp \langle t\rangle)_{L(t)}$. Since φ_1 is anisotropic over L, then $\varphi_1 \perp \langle t\rangle$ is anisotropic over $L(t)$. Hence, $((\varphi \perp \langle t\rangle)_{L(t)})_{\mathrm{an}} \simeq (\varphi_1 \perp \langle t\rangle)_{L(t)}$.

Suppose that $(\varphi_1 \perp \langle t\rangle)_{L(t)}$ is defined over $F(t)$. Hence, there exists an $F(t)$-quadratic form θ such that

$$(\varphi_1 \perp \langle t\rangle)_{L(t)} \simeq \theta_{L(t)}$$

We extend scalars to the field of Laurent series $L((t))$, we get

$$(\varphi_1 \perp \langle t\rangle)_{L((t))} \simeq \theta_{L((t))} \tag{3}$$

Note that the form $(\varphi_1 \perp \langle t\rangle)_{L((t))}$ remains anisotropic. Let θ' be the first residue form of θ with respect to the t-adic valuation of $F((t))$. We take the first residue form in equation (3) with respect to the t-adic valuation of $L((t))$ to get $\varphi_1 \simeq \theta'_L$, that is, φ_1 is defined over F which is contrary to our assumption. □

Lemma 9. *Let L/F be a field extension and φ an anisotropic F-quadratic form such that φ_L is isotropic and $(\varphi_L)_{\mathrm{an}}$ is not defined over F. Let x_1, y_1 be variables over L and $\psi := \varphi \perp x_1[1, y_1^{-1}]$ over $F_1 := F(x_1, y_1)$. Then, ψ is isotropic over $L_1 := L(x_1, y_1)$ and $(\psi_{L_1})_{\mathrm{an}}$ is not defined over F_1.*

Proof. The form $\psi = \varphi \perp x_1[1, y_1^{-1}]$ is anisotropic over $F_1 := F(x_1, y_1)$. Let $\varphi_1 = (\varphi_L)_{\mathrm{an}}$ and suppose that φ_1 is not defined over F. We have $\varphi_L \simeq i \times [0,0] \perp j \times \langle 0\rangle \perp \varphi_1$ such that $i+j>0$. Hence, $(\varphi \perp x_1[1, y_1^{-1}])_{L_1} \simeq i \times [0,0] \perp j \times \langle 0\rangle \perp (\varphi_1 \perp x_1[1, y_1^{-1}])_{L_1}$. Since φ_1 is anisotropic over L, the form $\varphi_1 \perp x_1[1, y_1^{-1}]$ is also anisotropic over L_1. Hence, $((\varphi \perp x_1[1, y_1^{-1}])_{L_1})_{\mathrm{an}} \simeq (\varphi_1 \perp x_1[1, y_1^{-1}])_{L_1}$.

Suppose that $((\varphi \perp x_1[1, y_1^{-1}])_{L_1})_{\mathrm{an}}$ is defined over F_1. Hence, there exists an F_1-quadratic form θ such that

$$(\varphi_1 \perp x_1[1, y_1^{-1}])_{L_1} \simeq \theta_{L_1}.$$

We extend scalars to the field of iterated Laurent series $\widetilde{L_1} = L((x_1))((y_1))$, we get:

$$(\varphi_1 \perp x_1[1, y_1^{-1}])_{\widetilde{L_1}} \simeq \theta_{\widetilde{L_1}}. \tag{4}$$

Note that the form $(\varphi_1 \perp x_1[1, y_1^{-1}])_{\widetilde{L_1}}$ is anisotropic. Let θ' be the first residue form of θ with respect to the y_1-adic valuation of $\widetilde{F_1} := F((x_1))((y_1))$. Taking the first residue form in equation (4) with respect to the y_1-adic valuation of $\widetilde{L_1}$, we get:

$$(\varphi_1 \perp \langle x_1\rangle)_{L((x_1))} \simeq \theta'_{L((x_1))}.$$

We reproduce the same arguments as in the proof of Lemma 8 to conclude that φ_1 is defined over F, which is contrary to our assumption. □

We get the following corollaries that prove the non-excellence of function fields of conics for quadratic forms of arbitrary dimension.

Corollary 10. *For any $m, s \geq 0$, there exists a function field of a singular conic which is not $(3+m, s)$-excellent.*

Proof. It follows from Proposition 6 that there exists a field k of characteristic 2, a nonsingular k-quadratic form φ of dimension 6, and a totally singular k-quadratic Q of dimension 3 such that $\varphi_{k(Q)}$ is isotropic but $(\varphi_{k(Q)})_{\mathrm{an}}$ is not defined over k. Now we start with φ and $k(Q)$ and iterate s times Lemma 8 to get a form φ' of type $(3, s)$ over $k' := k(t_1, \ldots, t_s)$ such that φ' becomes isotropic over $k'(Q)$ but the anisotropic part $(\varphi'_{k'(Q)})_{\mathrm{an}}$ is not defined over k'. Now we take φ' and $k'(Q)$, and we iterate r times Lemma 9 to get a form φ'' of type $(3+r, s)$ over

$k'' := k'(x_1, y_1, \ldots, x_r, y_r)$ such that φ'' becomes isotropic over $k''(Q)$ but the anisotropic part $(\varphi''_{k''(Q)})_{\mathrm{an}}$ is not defined over k''. □

Corollary 11. *For any $s \geq 0$, there exists a function field of a singular conic which is not $(2, s+1)$-excellent.*

Proof. We reproduce the same arguments as in the proof of Corollary 10 by combining Proposition 7 with Lemmas 8 and 9. □

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