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ABSTRACT. This paper is devoted to the study of the descent problem in the spirit of conjectures proposed by Kahn in characteristic different from 2 [15]. Descent problem seeks conditions under which a K-form (quadratic or bilinear) is defined over F for a field extension K/F. We study this problem in a complete manner for K-quadratic and bilinear forms up to dimension 4 when F is a field of characteristic 2 and K is the function field of a projective quadric.

1. INTRODUCTION

Throughout this paper F denotes an infinite field of characteristic 2. Let K be a field extension of F. A general problem in the algebraic theory of quadratic (bilinear) forms consists in studying the behavior of F-quadratic (bilinear) forms after extending scalars to K. Another problem, which we may consider as an opposite to the behavior study, is the descent problem that can be formulated in a general setting as follows: *Let φ be a quadratic (bilinear) form over K. Under which conditions the form φ is defined over F up to isometry (or up to Witt equivalence)?* Recall that the K-form φ is defined over F up to isometry (*resp.* up to Witt equivalence) if there exists an F-form ψ such that $\varphi \simeq \psi_K$ (*resp.* $\varphi \sim \psi_K$), where ψ_K is the form ψ considered as a form over K after scalar extension, and $\simeq$ (*resp.* $\sim$) denotes the isometry (*resp.* the Witt equivalence) of quadratic forms.

The descent problem is very difficult to study for an arbitrary field extension K of F. The case of an extension given by the function field of a quadric is very important because it is related to other problems, mainly the isotropy problem and the classification of forms by height and degree in the sense of Knebusch's splitting theory. In [15] Kahn formulated some conjectures on the descent problem over function fields of quadrics in characteristic not 2. The philosophy of his conjectures is that when $K = F(Q)$, the function field of the quadric given by the F-quadratic form Q, then the K-quadratic form φ is defined over F once it is Witt equivalent to θ_K for some F-quadratic form θ, and the dimension of Q is large enough than the dimension of φ. Up to now only some positive answers to Kahn's conjectures have been obtained (see [15], [18] and [14, Theorem 3.9]).

In this paper we study the descent for quadratic and bilinear forms in characteristic 2 in the spirit of Kahn's conjectures. Taking inspiration by the conjecture [15, Conjecture 2], we give the following analogue for quadratic forms in characteristic 2 that takes into account the case of singular forms:

Conjecture 1.1. *Let Q be an anisotropic F-quadratic form and $K = F(Q)$. Let φ be a quadratic form over K such that $\dim Q > 2\dim\varphi$ and one of the two conditions holds:*

(1) If φ is nonsingular, we suppose that $\varphi + I_q^{n+1}K \in \mathrm{Im}(W_q(F)/I_q^{n+1}F \longrightarrow W_q(K)/I_q^{n+1}K)$ and $2^n > \dim\varphi$ for some integer $n \geq 1$.

(2) If φ is singular, we suppose that $\varphi \sim \theta_K$ for some F-quadratic form θ.

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Then, φ is defined over F.

Conjecture 1.1 is obviously true when φ is totally singular, this is due to the fact that the anisotropic part of any totally singular F-quadratic form over any field extension K/F is defined over F [10, Proposition 8.1(iii)]. Henceforth, we consider Conjecture 1.1 for φ not totally singular. Moreover, sometimes it happens that the condition $\dim Q > 2\dim\varphi$ in Conjecture 1.1 is not optimal as we will see in Theorem 1.2. In fact, this condition can be written as $2r + s > 2\dim\varphi$, where (r, s) denotes the type of Q. For some K-forms φ of dimension ≤ 4 we will take the weaker condition $r + s > \dim\varphi$ as the following result shows:

Theorem 1.2. *Let Q be an anisotropic F-quadratic form of type (r, s). Let φ be an anisotropic K-quadratic form which is not totally singular. Suppose that $2 \leq \dim\varphi \leq 4$ and one of the following conditions holds:*

(1) $\dim\varphi = 2$*,* $\varphi + I_q^3K \in \mathrm{Im}(W_q(F)/I_q^3F \longrightarrow W_q(K)/I_q^3K)$ *and* $r + s > 2$.

(2) $\dim\varphi = 3$*,* $\varphi \sim \theta_K$ *for some F-quadratic form θ, and either ($r = 0$ and $s > 4$) or ($r \geq 1$ and $r + s > 3$).*

(3) φ is nonsingular of dimension 4, $\varphi + I_q^4K \in \mathrm{Im}(W_q(F)/I_q^4F \longrightarrow W_q(K)/I_q^4K)$ *and* $r + s > 4$.

(4) φ is of type $(1, 2)$, $\varphi \sim \delta_K$ for some F-quadratic form δ and $\dim Q > 8$.

Then, φ is defined over F.

For bilinear forms, the situation is more subtle comparing to that of quadratic forms. First, for the field $K = F(Q)$ we have to distinguish between the cases where Q is totally singular or not. In fact, for Q not totally singular, we get the descent of an $F(Q)$-bilinear form B without the hypothesis $\dim Q > 2\dim B$ as given in the following proposition.

Proposition 1.3. *Let Q be an anisotropic F-quadratic form of dimension ≥ 2 which is not totally singular, and $K = F(Q)$. Let B be an anisotropic bilinear form over K such that $B + I^{n+1}K \in \mathrm{Im}(W(F)/I^{n+1}F \longrightarrow W(K)/I^{n+1}K)$ for some integer $n \geq 1$ satisfying $2^n > \dim B$. Then, B is defined over F.*

This proposition was noticed before by Laghribi and Rehmann when they tried to continue on the descent results obtained in [27]. They used a sophisticated proof based on a lifting argument to reduce the question to the descent in characteristic not 2. Here we propose a direct argument based on transfer and the Hauptsatz.

However, for the field $F(Q)$, where Q is a totally singular form, we must not restrict ourselves to the analogues of the conditions given by Conjecture 1.1. Otherwise, Proposition 8.2 produces a counterexample. In fact, by this proposition we should take into account the notion of the norm degree of Q. In this case, we formulate the following question:

Question 1.4. *Let Q be an anisotropic totally singular quadratic form over F, and $K = F(Q)$. Let B be an anisotropic bilinear form over K such that $\dim Q > 2\dim B$ and $B + I^{n+1}K \in \mathrm{Im}(W(F)/I^{n+1}F \longrightarrow W(K)/I^{n+1}K)$ for some integer $n \geq 1$ satisfying $2^n > \dim B$ and $\mathrm{ndeg}_F(Q) > 2^{n+1}$. Is it true that B is defined over F?*

Our answers to Question 1.4 are given in the following theorem.

Theorem 1.5. *Let Q be an anisotropic totally singular F-quadratic form, $K = F(Q)$ and B an anisotropic bilinear form over K. Suppose that $\dim B \in \{2, 3, 4\}$, $\dim Q > 2\dim B$, and one of the following conditions holds:*

(1) $\dim B = 2$ *and* $B + I^3K \in \mathrm{Im}(W(F)/I^3F \longrightarrow W(K)/I^3K)$.
(2) $\dim B = 3$*,* $\mathrm{ndeg}_F(Q) > 8$ *and* $B \in \mathrm{Im}(W(F) \longrightarrow W(K))$.
(3) $B \in GBP_2(K)$ *and* $B \in \mathrm{Im}(W(F) \longrightarrow W(K))$.

Then, B *is defined over* F *in cases (1) and (2). In case (3), there exists* $\rho \in BP_2(F)$ *such that* B *is similar to* ρ*. In particular,* B *is defined over* F *when* $B \in BP_2(K)$*.*
(here $GBP_m(F)$ *denotes the set of* F*-bilinear forms similar to* m*-fold bilinear Pfister forms, see below.)*

Note that for the descent in dimension 2, we don't need the hypothesis $\mathrm{ndeg}_F(Q) > 8$, just the condition $\dim Q > 4$ suffices. Moreover, the descent in dimensions 2 and 4 mentioned in Theorem 1.5 was considered in [27, Theorem 1.3], but the proof there is incomplete as we will explain at the beginning of Section 8. We propose new arguments and results to complete the proof.

We finish this section by a result giving a relation between Conjecture 1.1 and Question 1.4, this is due to the fact that the Witt group $W_q(F)$ of nonsingular F-quadratic forms is endowed with a structure of module over the Witt ring $W(F)$ of regular symmetric F-bilinear forms.

Proposition 1.6. *Let* $s \geq 1$ *be an integer,* Q *an anisotropic totally singular* F*-quadratic form of dimension* ≥ 2*, and* $K = F(Q)$*. Suppose that Conjecture 1.1 is true for nonsingular* K*-quadratic forms* φ *of dimension* $\leq 2s$ *when* $\dim Q > 4s$*. Then, Question 1.4 is true for* K*-bilinear* B *of dimension* $\leq s$ *when* $\dim Q > 4s$ *without the hypothesis* $\mathrm{ndeg}_F(Q) > 2^{n+1}$*.*

The rest of this paper is organized as follows. The next section is devoted to some backgrounds on quadratic forms. In Section 3 we summarize all the cohomological kernels needed for the proofs. After that we present the proofs of the two main results of the paper, Theorems 1.2 and 1.5. This requires intensive tools, inspiration from the work of Kahn [15], and will be done case by case into two main parts talking about descent for quadratic forms and bilinear forms separately. In the last section we prove that the function field of a conic is excellent for bilinear forms. This result will play a crucial role in the study of the descent for bilinear forms.

2. Background

Recall that any quadratic form φ over F can be written up to isometry as follows:

$$\varphi \simeq [a_1, b_1] \perp [a_2, b_2] \perp \ldots \perp [a_r, b_r] \perp \langle c_1, \ldots, c_s\rangle, \tag{2.1}$$

where $\perp$ denotes the orthogonal sum of quadratic forms, and $[a, b]$ (*resp.* $\langle a\rangle$) denotes the quadratic form $ax^2 + xy + by^2$ (*resp.* ax^2). Obviously, $\dim \varphi = 2r + s$ (the dimension of φ). The quadratic form $\langle c_1, \ldots, c_s\rangle$ is unique up to isometry, we call it the quasilinear part of φ, and denote it by $\mathrm{ql}(\varphi)$. As in equation (2.1), the form φ is called:

- nonsingular (*resp.* singular) if $s = 0$ (*resp.* $s > 0$),
- totally singular if $r = 0$,
- semisingular if $r > 0$ and $s > 0$.

Any quadratic form φ uniquely decomposes as follows: $\varphi \simeq \varphi_{an} \perp i \times [0, 0] \perp j \times \langle 0\rangle$. The form φ_{an} is called the anisotropic part of φ, and the integer i (*resp.* j) is called the Witt index and denoted by $i_W(\varphi)$ (*resp.* the defect index and denoted by $i_d(\varphi)$). The total index of φ, denoted by $i_t(\varphi)$, is the integer $i_W(\varphi) + i_d(\varphi)$.

Let φ and ψ be quadratic forms of underlying vector spaces V and W, respectively. We say that φ is dominated by ψ, denoted by $\varphi \prec \psi$, if there exists an injective F-linear map $\sigma : V \longrightarrow W$ such that $\varphi(v) = \psi(\sigma(v))$ for all $v \in V$. We refer to [10, Th. 3.4] for an explicit formulation of this relation.

We mention the completion lemma due to Hoffmann and Laghribi which will play a crucial role in many proofs.

Lemma 2.1. ([10, Lemma 3.9]) *Let R and R' be nonsingular quadratic forms over F, and $c_i, c'_i, d_i \in F$, $1 \leq i \leq n$. Suppose that $R \perp \langle c_1, \ldots, c_n\rangle \simeq R' \perp \langle c'_1, \ldots, c'_n\rangle$. Then, there exist $d'_1, \ldots, d'_n \in F$ such that $R \perp [c_1, d_1] \perp \ldots \perp [c_n, d_n] \simeq R' \perp [c'_1, d'_1] \perp \ldots \perp [c'_n, d'_n]$.*

For $a, b \in F$ with $b \neq 0$, we denote by $[a, b)$ the quaternion F-algebra generated by two elements i and j subject to the relations: $i^2 + i = a$, $j^2 = b$ and $jij^{-1} = i + 1$. Quaternion F-algebras satisfy the following relations in the Brauer group $\mathrm{Br}(F)$ of F:

$$[a_1 + a_2, b) \sim [a_1, b) \otimes_F [a_2, b),$$
$$[a, b_1b_2) \sim [a, b_1) \otimes_F [a, b_2).$$

Let φ be an F-quadratic form of underlying vector space V. The Cliford algebra of φ, denoted by $C(\varphi)$, is the algebra

$$C(\varphi) = T(V)/I,$$

where $T(V)$ is the tensor algebra of V, and I is the two-sided ideal of $T(V)$ generated by $v \otimes v - \varphi(v)$ for all $v \in V$. The algebra $C(\varphi)$ is central simple over F and admits a $\mathbb{Z}/2\mathbb{Z}$-grading $C(\varphi) = C_0(\varphi) \oplus C_1(\varphi)$.

The subalgebra $C_0(\varphi)$ of $C(\varphi)$ is called the even Clifford algebra of φ. Recall that $\dim_F C(\varphi) = 2^{\dim \varphi}$ and $\dim_F C_0(\varphi) = 2^{\dim \varphi - 1}$.

If φ is nonsingular, then $C(\varphi)$ is a central simple F-algebra. Its class in $\mathrm{Br}(F)$, denoted by $c(\varphi)$, is called the Clifford invariant of φ. Moreover, the center of $C_0(\varphi)$ is a separable quadratic algebra $F[x]/\langle x^2 + x + \delta\rangle$ for some $\delta \in F$. The class $\delta + \wp(F)$ in $F/\wp(F)$ is called the Arf invariant of φ and denoted by $\triangle(\varphi)$. Explicitly, if $\varphi \simeq a_1[1, b_1] \perp a_2[1, b_2] \perp \ldots \perp a_n[1, b_n]$ for $a_1, \ldots, a_n \in F^*$ and $b_1, \ldots, b_n$, then we have $C(\varphi) \simeq \otimes_{i=1}^n [b_i, a_i)$ and $\triangle(\varphi) = \sum\limits_{i=1}^n b_i + \wp(F)$.

We will prove some results on bilinear forms after reducing to nonsingular quadratic forms defined over the Laurent series field $F((t))$. More specifically, we will need the notion of residue forms that we recall below.

Let K be a field with a discrete Henselian valuation v. We denote by A its ring, π a uniformizer and $k = A/\pi A$ the residue field. Let φ be an anisotropic K-quadratic form (possibly singular) of underlying vector space V. For each integer $i = 0, 1, 2$, we attach the set $V_i = \{v \in V \mid \varphi(v) \in \pi^i A\}$, this is an A-module [29, page 342]. The induced residue forms $\overline{\varphi_0}$ and $\overline{\varphi_1}$ are the k-forms given by:

$$\begin{array}{rccl} \overline{\varphi_i}: & V_i/V_{i+1} & \longrightarrow & k \\ & v + V_{i+1} & \mapsto & \overline{\pi^{-i}\varphi(v)}. \end{array}$$

The form $\overline{\varphi_0}$ (*resp.* $\overline{\varphi_1}$) is called the first residue form (*resp.* the second residue form) of φ. Clearly, these forms are anisotropic and could be singular. If φ is nonsingular, then $\dim \varphi = \dim \overline{\varphi_0} + \dim \overline{\varphi_1}$ [29, Theorem 1].

We recall some results needed for the proofs. The first one is the "Hauptsatz" of Arason-Pfister:

Theorem 2.2. *(*[7, Theorem 23.7]*)*
(1) Let $\varphi \neq 0$ be an anisotropic quadratic form lying in I_q^nF. Then, $\dim\varphi \geq 2^n$.
(2) Let $B \neq 0$ be an anisotropic bilinear form lying in I^nF. Then, $\dim B \geq 2^n$.

We recall some results on the isotropy over function fields of quadrics.

Proposition 2.3. *(*[19, Corollary 3.1]*) Let ψ and ψ' be an anisotropic quadratic forms over F of dimension ≥ 2. If ψ is totally singular and ψ' is not totally singular, then $\psi_{F(\psi')}$ is anisotropic.*

As a corollary we get:

Corollary 2.4. *Let ψ and ψ' be an anisotropic quadratic forms over F of dimension ≥ 2. Suppose that ψ' is not totally singular. If $\psi_{F(\psi')}$ is isotropic then $i_W(\psi_{F(\psi')}) \geq 1$.*

Proof. Suppose that $\psi_{F(\psi')}$ is isotropic. The corollary is trivial if ψ is nonsingular. So suppose that ψ is singular. Since ψ' is not totally singular, it follows from Proposition 2.3 that ψ is not totally singular and $\mathrm{ql}(\psi)_{F(\psi')}$ is anisotropic. By the uniqueness of the quasilinear part, we get $i_W(\psi_{F(\psi')}) \geq 1$. □

Theorem 2.5. *(*[11, Theorem 1.1]*) Let φ and ψ be anisotropic quadratic forms over F such that $\dim\varphi \leq 2^n < \dim\psi$ for some integer $n \geq 0$. Then, $\varphi_{F(\psi)}$ is anisotropic.*

Lemma 2.6. *(*[11, Lemma 2.11]*) Let φ be an isotropic quadratic form over F. Then, any form ψ satisfying $\psi \prec \varphi$ and $\dim\psi \geq \dim\varphi - i_t(\varphi) + 1$ is also isotropic.*

Recall that an Albert quadratic form is a nonsingular quadratic form of dimension 6 and trivial Arf invariant. We will often use the following result on the isotropy of an Albert form:

Proposition 2.7. *Let γ be an anisotropic Albert form over F, ψ an anisotropic quadratic form of type (r, s) such that either ($r = 0$ and $s > 4$) or ($r \geq 1$ and $r + s > 3$). Then $\gamma_{F(\psi)}$ is anisotropic.*

Proof. (1) Suppose $r \geq 1$ and $r + s > 3$. Then, ψ dominates a form ψ' of dimension 5 and type $(1, 3)$. Moreover, $i_W(\psi_{F(\psi')}) \geq 1$ (Corollary 2.4). Hence $F(\psi')(\psi)/F(\psi')$ is purely transcendental. Since, $\gamma_{F(\psi')}$ is anisotropic by [19, Theorem 1.1], we conclude that $\gamma_{F(\psi)}$ is anisotropic.

(2) Suppose that $r = 0$ and $s > 4$. Again, [19, Theorem 1.1] implies that $\gamma_{F(\psi)}$ is anisotropic. □

Corollary 2.8. *Let D be a division biquaternion algebra over F and ψ an anisotropic F-quadratic form of type (r, s) such that either ($r = 0$ and $s > 4$) or ($r \geq 1$ and $r + s > 3$). Then, $D \otimes_F F(\psi)$ is a division algebra.*

Proof. Let γ be an Albert quadratic form over F such that $C(\gamma) \sim D$. By Proposition 2.7, $\gamma_{F(\psi)}$ is anisotropic. It follows from [30] that $D_{F(\psi)}$ is a division algebra. □

Nonsingular forms in I_q^2F and I_q^3F are classified by their Arf and Clifford invariants.

Theorem 2.9. *(*[33, Theorem 2]*) Let φ be a nonsingular quadratic form over F. Then, we have:*
(1) $\varphi \in I_q^2F$ iff $\triangle(\varphi) = 0$.
(2) If $\varphi \in I_q^2F$, then $\varphi \in I_q^3F$ iff $C(\varphi) \sim 0$.

Lemma 2.10. *Let Q be an anisotropic F-quadratic form, and $K = F(Q)$. Let $\varphi \in W_q(K)$ and $\psi \in W_q(F)$ be such that $\varphi \perp \psi_K \in I_q^2 K$. Then, $\triangle(\varphi) = r + \wp(K)$, where $r \in F$ is a representative of $\triangle(\psi)$.*

Proof. This is because $\varphi \perp \psi_K \in I_q^2 K$ implies $\triangle(\varphi) = \triangle(\psi_K)$. $\square$

The following lemma will be helpful for computing the even Clifford algebra.

Lemma 2.11. *(*[28, Lemma 2]*) Let φ be a nonsingular F-quadratic form and $a \in F^*$. Then, $C_0(\varphi \perp \langle a \rangle) \simeq C(a\varphi)$.*

3. Cohomological kernels

Let $W_q(F)$ be Witt group of nonsingular quadratic forms over F, and $W(F)$ the Witt ring of regular bilinear forms over F. It is well known that $W_q(F)$ is endowed with a $W(F)$-module structure induced by tensor product [5]. For any integer $m \geq 1$, let $I^m F$ denote the m-th power of the fundamental ideal IF of $W(F)$, and $I_q^{m+1} F = I^m F \otimes W_q(F)$ (we take $I^0 F = W(F)$). The quotients $I_q^{m+1} F / I_q^{m+2} F$ and $I^m F / I^{m+1} F$ are denoted by $\overline{I}_q^{m+1} F$ and $\overline{I}^m F$.

For $c_1, \cdots, c_m \in F^*$, let $\langle c_1, \cdots, c_m \rangle_b$ denote the diagonal bilinear form $\sum_{i=1}^m c_i x_i y_i$. For $n \geq 1$ an integer, an n-fold bilinear Pfister form is a bilinear form isometric to $\langle 1, a_1 \rangle_b \otimes \cdots \otimes \langle 1, a_n \rangle_b$ for $a_1, \cdots, a_n \in F^*$. We denote this form by $\langle\langle a_1, \cdots, a_n \rangle\rangle_b$. The form $\langle 1 \rangle_b$ is called the 0-fold bilinear Pfister form. An $(n+1)$-fold quadratic Pfister form is a form isometric to $\langle\langle a_1, \cdots, a_n \rangle\rangle_b \otimes [1, b]$ for some $a_1, \cdots, a_n \in F^*$ and $b \in F$, we denote it by $\langle\langle a_1, \cdots a_n; b]]$. Let $P_n(F)$ (*resp.* $BP_n(F)$) be the set of forms isometric to n-fold quadratic Pfister forms (*resp.* the set of forms isometric to n-fold bilinear Pfister forms). We take $GP_n(F) = F^* \cdot P_n(F)$ and $GBP_n(F) = F^* \cdot BP_n(F)$.

To any bilinear form B of underlying vector space V, we associate a totally singular quadratic form $\widetilde{B}$ defined on V as follows: $v \mapsto B(v, v)$ for all $v \in V$. A quasi-Pfister form is a totally singular quadratic form isometric to $\widetilde{B}$, where B is a bilinear Pfister form. Recall that a quadratic (bilinear) Pfister form is isotropic iff it is hyperbolic (metabolic). A (quasi-)Pfister form φ is round which means that φ represents a scalar $\alpha \in F^*$ iff $\varphi \simeq \alpha\varphi$.

A quadratic form φ is called a Pfister neighbor if there exists a quadratic Pfister form π such that $\varphi \prec \pi$ and $2 \dim \varphi > \dim \pi$. In this case, the form φ is not totally singular and for any field extension L/F, one knows that φ_L is isotropic iff π_L is isotropic. We define in the same way the notion of quasi-Pfister neighnor for totally singular quadratic forms[10, Section 8]. The notion of Pfister neighbor also exists for bilinear forms but we don't need it here.

We recall the famous Kato's isomorphisms [16] that will play a crucial role in some results.

$$\begin{array}{rcl} \alpha_m : \overline{I}^m F & \to & \nu_F(m) \\ \overline{\langle\langle a_1, \ldots, a_m \rangle\rangle_b} & \mapsto & \dfrac{\mathrm{d}a_1}{a_1} \wedge \ldots \wedge \dfrac{\mathrm{d}a_m}{a_m} \end{array} \tag{3.1}$$

and

$$\begin{array}{rcl} e_{m+1} : \overline{I}_q^{m+1} F & \to & H_2^{m+1}(F) \\ \overline{\langle\langle a_1, \ldots, a_m; b]]} & \mapsto & \overline{b\dfrac{\mathrm{d}a_1}{a_1} \wedge \ldots \wedge \dfrac{\mathrm{d}a_m}{a_m}} \end{array} \tag{3.2}$$

where $H_2^{m+1}(F)$ (*resp.* $\nu_F(m)$) is the cokernel (*resp.* the kernel) of the Artin-Schreier operator $\wp : \Omega_F^m \longrightarrow \Omega_F^m/\mathrm{d}\Omega_F^{m-1}$ such that Ω_F^m is the space of m-differential forms over F and d is the differential operator.

For any field extension K/F, let $H_2^{m+1}(K/F)$ denote the kernel of the restriction map $H_2^{m+1}(F) \longrightarrow H_2^{m+1}(K)$. Similarly, we define the kernels $\overline{I}_q^{m+1}(K/F)$ and $\overline{I}^{m+1}(K/F)$.

We summarize in this section the cohomological kernels that will be needed. We start with an injectivity result in the setting of Kato cohomology due to Aravire and Baeza.

Proposition 3.1. *(*[2, Lemma 2.17]*) Let L/F be a purely transcendental extension. Then, $H_2^{m+1}(L/F) = 0$.*

We prove a general result that is helpful for the descent over function fields of singular quadratic forms.

Proposition 3.2. *Let L be an arbitrary field extension of F, and φ an anisotropic F-quadratic form of dimension ≥ 3 such that*

(1) $\varphi \simeq [1, a] \perp \varphi'$, or

(2) $\varphi \simeq \langle 1 \rangle \perp \varphi'$,

for a suitable quadratic form φ' of dimension n. Let $(\alpha_1, \alpha_2, \beta_1, \cdots, \beta_n) \in L^{n+2}$ be such that:

- *In case (1) we suppose $(\alpha_1, \alpha_2) \neq (0, 0)$ and $\alpha_1^2 + \alpha_1\alpha_2 + a\alpha_2^2 + \varphi'(\beta_1, \cdots, \beta_n) = 0$.*
- *In case (2) we suppose $\alpha_1^2 + \varphi'(\beta_1, \cdots, \beta_n) = 0$ and $p := X^2 + \varphi'(\beta_1, \cdots, \beta_n) \in F(\beta_1, \cdots, \beta_n)[X]$ irreducible.*

Then, $H_2^{m+1}(F(\varphi)/F) \subseteq H_2^{m+1}(L/F)$ for any integer $m \geq 0$.

Proof. In case (1) we see that $i_W(\varphi_L) \geq 1$ because $(\alpha_1, \alpha_2) \neq 0$. Hence, the extension $L(\varphi)/L$ is purely transcendental. Now if $w \in \Omega_F^m$ satisfies $\overline{w} \in H_2^{m+1}(F(\varphi)/F)$, then $\overline{w}_L \in H_2^{m+1}(L(\varphi)/L)$. It follows from Proposition 3.1 that $\overline{w} \in H_2^{m+1}(L/F)$.

Now suppose we are in case (2). We propose another argument since we don't know if the condition $i_W(\varphi_L) \geq 1$ is satisfied. The function field of the affine quadric given by φ is as follows: $F(\varphi) = F(z_1, \cdots, z_n)(\sqrt{\varphi'(z_1, \cdots, z_n)})$, where $z_1, \cdots, z_n$ are independent variables over F. Let $K = F(z_1, \cdots, z_n)$.

Let $w \in \Omega_F^m$ be such that $\overline{w} \in H_2^{m+1}(F(\varphi)/F))$. Since $\overline{w}_K \in H_2^{m+1}(K(\sqrt{\varphi'(z_1, \cdots, z_n)})/K)$, we get by the norm theorem for differential forms [31] that

$$\overline{w \wedge \frac{\mathrm{d}(X^2 + \varphi'(z_1, \cdots, z_n))}{X^2 + \varphi'(z_1, \cdots, z_n)}} = 0 \in H_2^{m+2}(K(X)).$$

In particular, we have

$$\overline{w \wedge \frac{\mathrm{d}(X^2 + \varphi'(z_1, \cdots, z_n))}{X^2 + \varphi'(z_1, \cdots, z_n)}} = 0 \in H_2^{m+2}(K(X, \beta_n)). \tag{3.3}$$

Consider on the field $K(X, \beta_n)$ the $(z_n - \beta_n)$-adic valuation and take $\pi = z_n - \beta_n$ as an uniformizer. Applying the residue map χ_π given in [6, Page 15] to equation (3.3), we get

$$\overline{w \wedge \frac{\mathrm{d}(X^2 + \varphi'(z_1, \cdots, z_{n-1}, \beta_n))}{X^2 + \varphi'(z_1, \cdots, z_{n-1}, \beta_n)}} = 0 \in H_2^{m+2}(F(z_1, \cdots, z_{n-1}, X, \beta_n)). \tag{3.4}$$

Now repeating the same argument successively for the variables $z_1, \cdots, z_{n-1}$, we obtain

$$\overline{w \wedge \frac{\mathrm{d}p}{p}} = 0 \in H_2^{m+2}(F(\beta_1, \cdots, \beta_n, X)). \tag{3.5}$$

Since the polynomial $p \in F(\beta_1, \cdots, \beta_n)[X]$ is irreducible, we get by the norm theorem [31] that $\overline{w} = 0$ over the field $F(\beta_1, \cdots, \beta_n)(p)$, which is isomorphic to $F(\alpha_1, \beta_1, \cdots, \beta_n)$, a subfield of L. Hence, $\overline{w} \in H_2^{m+1}(L/F)$. □

As a consequence, we get the following corollary:

Corollary 3.3. *Let φ be an anisotropic F-quadratic form of dimension ≥ 3, and ψ an F-quadratic form of dimension ≥ 2 dominated by φ. Then, $H_2^{m+1}(F(\varphi)/F) \subseteq H_2^{m+1}(F(\psi)/F)$ for any integer $m \geq 0$.*

Proof. If $i_W(\varphi_{F(\psi)}) \geq 1$, then we are in case (1) of Proposition 3.2, and thus $H_2^{m+1}(F(\varphi)/F) \subseteq H_2^{m+1}(F(\psi)/F)$. So suppose that $i_W(\varphi_{F(\psi)}) = 0$. Then, ψ is necessarily dominated by $\mathrm{ql}(\varphi)$. In particular, we have $\varphi \simeq \psi \perp \varphi'$ for some quadratic form φ'. Without loss of generality we may suppose $1 \in D_F(\psi)$, and put $\psi = \langle 1, c_1, \cdots, c_s \rangle$. The function field of the affine quadric given by ψ is $F(\psi) = F(x_1, \cdots, x_s)(\alpha)$, where $\alpha = \sqrt{c_1x_1^2 + \cdots + c_sx_s^2}$. Obviously, $v := (\alpha, x_1, \cdots, x_s, 0, \cdots, 0)$ is an isotropy vector of $\varphi_{F(\psi)}$. Since ψ is anisotropic over F, the polynomial $X^2 + \sum_{i=1}^{s} c_ix_i^2 \in F(x_1, \cdots, x_s)[X]$ is irreducible. Consequently, we are in the conditions of case (2) of Proposition 3.2, and thus $H_2^{m+1}(F(\varphi)/F) \subseteq H_2^{m+1}(F(\psi)/F)$. □

The following corollary is an immediate consequence of Corollary 3.3 and the Kato's isomorphism (3.2).

Corollary 3.4. *Let φ be an anisotropic F-quadratic form of dimension ≥ 3, and ψ an F-quadratic form of dimension ≥ 2 dominated by φ. Then, $\overline{I}_q^{m+1}(F(\varphi)/F) \subseteq \overline{I}_q^{m+1}(F(\psi)/F)$ for any integer $m \geq 0$.*

Another result that will play an important role is the following proposition.

Proposition 3.5. *(*[22, Corollary 4.11]*) Let $\varphi \in I_q^{m+1}F$ for some $m \geq 0$, and ψ an anisotropic totally singular F-quadratic form of dimension $> 2^m$. If $\varphi_{F(\psi)} \in I_q^{m+2}F(\psi)$, then $\varphi \in I_q^{m+2}F$. In other words, $\overline{I}_q^{m+1}(F(\psi)/F) = \{0\}$.*

We deduce the following corollary:

Corollary 3.6. *Let $\varphi \in I_q^{m+1}F$ for some $m \geq 0$, and ψ an anisotropic F-quadratic form of type (r, s) such that $r + s > 2^m$. If $\varphi_{F(\psi)} \in I_q^{m+2}F(\psi)$, then $\varphi \in I_q^{m+2}F$.*
In other words, $\overline{I}_q^{m+1}(F(\psi)/F) = \{0\}$.

Proof. Since ψ is of type (r, s), there exists ψ' a totally singular form of dimension $r + s$ such that $\psi' \prec \psi$. Corollary 3.4 implies that $I_q^{m+1}(F(\psi)/F) \subset I_q^{m+1}(F(\psi')/F)$. Since $\dim \psi' = r + s > 2^m$, it follows from Proposition 3.5 that $I_q^{m+1}(F(\psi')/F) = \{0\}$. This proves the corollary. □

We extend the previous corollary to the case of a compositum of two function fields of quadrics.

Corollary 3.7. *Let ψ be an anisotropic singular F-quadratic form of type (r, s) such that $r+s > 2^m$ for some integer $m \geq 0$. Let θ be an anisotropic F-quadratic form satisfying one of the following conditions:*

(1) θ is not totally singular.

(2) θ is totally singular and the type (r', s') of $(\psi_{F(\theta)})_{an}$ satisfies $r' + s' > 2^m$.

Then, $\overline{I}_q^{m+1}(F(\theta)(\psi)/F) \subset \overline{I}_q^{m+1}(F(\theta)/F)$.

Proof. (1) Suppose that θ is not totally singular. If $\psi_{F(\theta)}$ is isotropic, then $i_W(\psi_{F(\theta)}) \geq 1$ (Corollary 2.4). Hence, $F(\theta)(\psi)/F(\theta)$ is purely transcendental, and thus the corollary is verified. If $\psi_{F(\theta)}$ is anisotropic, then the corollary is a consequence of Corollary 3.6.

(2) Suppose that θ is totally singular and the type (r', s') of $(\psi_{F(\theta)})_{an}$ satisfies $r' + s' > 2^m$.

If $i_W(\psi_{F(\theta)}) \geq 1$ or $\psi_{F(\theta)}$ is anisotropic, then we conclude as in case (1). So suppose that $\psi_{F(\theta)}$ is isotropic and $i_W(\psi_{F(\theta)}) = 0$, then $i_d(\psi_{F(\theta)}) > 0$ and thus $F(\theta)(\psi)$ is purely transcendental over $F(\theta)(\psi')$, where $\psi' = (\psi_{F(\theta)})_{an}$. Hence, $\overline{I}_q^{m+1}(F(\theta)(\psi)/F) \subset \overline{I}_q^{m+1}(F(\theta)(\psi')/F)$. With the hypothesis on the type of ψ', we get by Corollary 3.6 $\overline{I}_q^{m+1}(F(\theta)(\psi')/F) \subset \overline{I}_q^{m+1}(F(\theta)/F)$. $\square$

Below we give another kernel that will be needed to study the descent for quadratic forms of dimension 4.

Theorem 3.8. *(*[1, (6.2)] *for $k = 1$, and* [3, Theorem 1.5] *for $k = 2$)*
Let $\tau \in P_kF$ anisotropic such that $k = 1$ or 2. Then, for any integer $m \geq 1$, we have

$$\overline{I}_q^{m+1}(F(\tau)/F) = \overline{I^{m+1-k}F \otimes \tau}.$$

In the case $k = 2$, this theorem is mentioned in [3] for a function field of a Pfister neighbor φ of τ of dimension 3. Our formulation of Theorem 3.8 involving the field $F(\tau)$ is possible using Proposition 3.1 and the fact that the extensions $F(\tau)(\varphi)/F(\tau)$ and $F(\varphi)(\tau)/F(\varphi)$ are purely transcendental because $\tau_{F(\varphi)}$ and $\varphi_{F(\tau)}$ are isotropic.

Using the Kato's isomorphism (3.2), we deduce from [13, Theorem 5.6] the following:

Theorem 3.9. *Let τ be an anisotropic F-quadratic form of dimension ≥ 3. Then, we have*

$$\overline{I}_q^{3}(F(\tau)/F) = \{\rho + I_q^4F \mid \rho \in P_3F \text{ and } x\tau \subset \rho \text{ for some } x \in F^*\}.$$

In particular, this kernel is trivial if $\dim \tau > 8$, or $5 \leq \dim \tau \leq 8$ and τ is not a Pfister neighbor.

We recall the kernel of the function field of a bilinear Pfister form, it is a consequence of [2, Theorem 4.1] and the Kato's isomorphism (3.2).

Theorem 3.10. *Let $B = \langle\langle a_1, \cdots, a_n\rangle\rangle_b$ be an anisotropic bilinear Pfister form over F. Then, for any integer $m \geq 1$, we have*

$$\overline{I}_q^{m+1}(F(B)/F) = \begin{cases} \overline{B \otimes I_q^{m-n+1}F} & \text{if } m \geq n \\ 0 & \text{if } m < n. \end{cases}$$

We finish this section with some kernels in the setting of bilinear forms. Recall that for L/F a purely transcendental extension, the natural map $\Omega_F^m \longrightarrow \Omega_L^m$ is injective by [2, Lemma 2.2]. Similarly, this map is also injective for L/F separable because a 2-basis of F stays a 2-basis of L. In particular, in both cases, we get the injectivity of the natural map $\nu_F(m) \longrightarrow \nu_L(m)$. Using the Kato's isomorphism (3.1), we deduce that

$$\overline{I}^m(L/F) = \{0\}. \tag{3.6}$$

This injectivity result applies to the case of quasi-Pfister neighbors as follows:

Proposition 3.11. *Let π be an anisotropic quasi-Pfister form over F, and Q a quasi-Pfister neighbor of π. Then, $\overline{I}^m(F(Q)/F) = \overline{I}^m(F(\pi)/F)$ for any integer $m \geq 0$.*
In other words, if $B \in I^m F$ then $B_{F(\pi)} \in I^{m+1}F(\varphi)$ iff $B_{F(Q)} \in I^{m+1}F(Q)$.

Proof. Since Q is a quasi-Pfister neighbor of π, the forms $\pi_{F(Q)}$ and $Q_{F(\pi)}$ are isotropic. It follows from [4, Proposition 7.7] that $F(\pi)(x_1, \cdots, x_s) = F(Q)(y_1, \cdots, y_t)$ for some independent variables $x_1, \cdots, x_s, y_1, \cdots, y_t$ over F. Now it is clear from equation (3.6) that $\overline{I}^m(F(Q)/F) = \overline{I}^m(F(\pi)/F)$. □

As for Theorem 3.10, the kernel of the function field of a bilinear Pfister forms was also computed by Aravire and Baeza in the setting of the graded-Witt group of bilinear forms. We need this kernel in the following specific case:

Theorem 3.12. *(*[2, Theorem 2.2]*) Let $B = \langle\langle a_1, \cdots, a_n\rangle\rangle_b$ be an anisotropic bilinear Pfister form over F. Then, we have*

$$\overline{I}^n(F(B)/F) = \{\overline{\langle\langle x_1, \cdots, x_n\rangle\rangle_b} \mid x_1, \cdots, x_n \in F(a_1, \cdots, a_n)^*\} \cup \{0\}.$$

In the setting of Witt ring, we have the following theorem:

Theorem 3.13. *(*[20, Theorem 1.2]*) Let Q be an anisotropic totally singular F-quadratic form of dimension ≥ 2, and $\mathrm{ndeg}_F(Q) = 2^d$. Then, an anisotropic F-bilinear form B becomes metabolic over $F(Q)$ iff $B \simeq \alpha_1\pi_1 \perp \cdots \perp \alpha_s\pi_s$ for some integer $s \geq 1$, where $\alpha_i \in F^*$ and $\pi_i \in BP_dF$ such that $\widetilde{\pi_i} \simeq \theta$ for all $1 \leq i \leq s$, where θ is the quasi-Pfister form associated to the norm field of Q. In particular, $\dim B$ is divisible by 2^d.*

Similarly to Proposition 3.5, we have:

Proposition 3.14. *(*[22, Proposition 4.13]*) Let Q be an anisotropic totally singular F-quadratic form of dimension $> 2^m$. Then, $\overline{I}^m(F(Q)/F) = \{0\}$.*

4. Proof of Theorem 1.2: The case of types $(1,0)$ and $(1,1)$

Let Q be an anisotropic F-quadratic form of type (r, s), and $K = F(Q)$. Let φ be an anisotropic K-quadratic form which is not totally singular such that $\dim \varphi \in \{2, 3\}$.

1. Suppose $\dim \varphi = 2$ **and** $r + s > 2$. Set $\varphi \simeq a[1, b]$ for $a, b \in K^*$ and suppose we have

$$\varphi \perp \psi_K \in I_q^3 K \tag{4.1}$$

for some nonsingular F-form ψ. By Lemma 2.10, we may suppose $b \in F$. Moreover, using Theorem 2.9, we get $C(\varphi \perp \psi_K) \sim 0$, that is, $C(\psi_K) \sim [b, a)$. It follows that $C(\psi)_{K(\alpha)} \sim 0$, where $\alpha^2 + \alpha = b$. Taking the Clifford invariant, this means $c(\psi) \in \overline{I}_q^2(K(\alpha)/F)$. Since $r + s > 2$, Corollary 3.7 implies that $c(\psi) \in \overline{I}_q^2(F(\alpha)/F)$. Hence, there exists $u \in F^*$ such that $c(\psi)$ is the class of $[b, u)$ in $\mathrm{Br}(F)$. Theorem 2.9 implies that $\psi \perp u[1, b] \in I_q^3 F$. Extending this relation to K, and combining with equation (4.1) yields $\varphi \perp u[1,b]_K \in I_q^3 K$. It follows by the Hauptsatz that $\varphi \simeq u[1,b]_K$.

2. Suppose $\dim \varphi = 3$ **and either** ($r = 0$ **and** $s > 4$) **or** ($r \geq 1$ **and** $r + s > 3$)**.** Let us write $\varphi \simeq a[1,b] \perp \langle c \rangle$ for $a, b, c \in K^*$, and suppose that $\varphi \sim \psi_K$ for some F-quadratic form ψ. By the uniqueness of the quasilinear part, we may suppose $c \in F^*$ and $\mathrm{ql}(\psi) = \langle c \rangle$. Without loss of generality, we may suppose $c = 1$. Taking the even Clifford algebra in the equivalence $\varphi \sim \psi_K$, we get by Lemma 2.11

$$C_0(\psi_K) \sim C_0(a[1,b] \perp \langle 1 \rangle) \sim C(a[1,b]) \sim [b,a).$$

Hence $\mathrm{ind} C_0(\psi_K) \leq 2$. We get by Corollary 2.8 $\mathrm{ind} C_0(\psi) \leq 2$. Thus, there exist scalars $e, f \in F^*$ such that $C_0(\psi) \sim [e,f)$. So we get $[b,a) \simeq [e,f)_K$. Since the two quaternion algebras are isomorphic if and only if their norm forms are isometric, we get:

$$a[1,b] \perp [1,b] \simeq ([1,e] \perp f[1,e])_K.$$

Adding $\langle 1 \rangle$ to the previous equation and canceling $[0,0]$, we get:

$$a[1,b] \perp \langle 1 \rangle \simeq (f[1,e] \perp \langle 1 \rangle)_K.$$

5. Proof of Theorem 1.2: The case of type $(2,0)$

Let Q be an anisotropic F-quadratic form of type (r,s), and $K = F(Q)$. Let φ be an anisotropic K-quadratic form of type $(2,0)$. We suppose $r + s > 4$ and $\varphi + I_q^4 F \in \mathrm{Im}(W_q(F)/I_q^4 F \longrightarrow W_q(K)/I_q^4 K)$. Let ψ be an F-quadratic form such that

$$\varphi \perp \psi_K \in I_q^4 K. \tag{5.1}$$

We treat two cases according to $\triangle(\varphi) = 0$ or not.

1. Suppose $\dim \varphi = 4$ **and** $\triangle(\varphi) = 0$**.** Taking the Clifford algebra in equation (5.1) yields $C(\varphi) \sim C(\psi_K)$. Since, $\varphi \in GP_2(K)$ anisotropic, we have $\mathrm{ind} C(\varphi) = 2$. Thus, we get $\mathrm{ind} C(\psi)_K = 2$. By using Corollary 2.8, we get $\mathrm{ind} C(\psi) = 2$. Hence, there exists $\varphi_0 \in P_2(F)$ such that $C(\psi) \sim C(\varphi_0)$. Theorem 2.9 implies that

$$\psi \perp \varphi_0 \in I_q^3 F. \tag{5.2}$$

Passing to K, we obtain $\psi_K \perp (\varphi_0)_K \in I_q^3 K$. Combining with equation (5.1), we get

$$\varphi \perp (\varphi_0)_K \in I_q^3 K. \tag{5.3}$$

Further for any scalar $x \in K^*$, we have $\varphi_0 \perp x\varphi_0 \in I_q^3 K$. We choose a scalar $x \in D_K(\varphi)$, adding $\varphi_0 \perp x\varphi_0 \in I_q^3 K$ to equation (5.3), we get:

$$\varphi \perp (x\varphi_0)_K \in I_q^3 K.$$

Note that since φ and $x\varphi_0$ both represent x, thus $\varphi \perp x\varphi_0$ is isotropic. Hence, we get by the Hauptsatz $\varphi \simeq (x\varphi_0)_K$. Consequently, $K(\varphi) = K(\varphi_0)$.

Now extending equation (5.1) to $K(\varphi_0)$, we get

$$\varphi_{K(\varphi_0)} \perp \psi_{K(\varphi_0)} \in I_q^4 K(\varphi_0).$$

Since $K(\varphi) = K(\varphi_0)$ and $\varphi \in GP_2(K)$, we have $\varphi_{K(\varphi_0)} \sim 0$ and thus $\psi_{K(\varphi_0)} \in I_q^4 K(\varphi_0)$. Since $(\varphi_0)_{K(\varphi_0)} \sim 0$, it follows $(\psi \perp \varphi_0)_{K(\varphi_0)} \sim \psi_{K(\varphi_0)} \in I_q^4 K(\varphi_0)$. To sum up, we have

$\psi \perp \varphi_0 \in I_q^3F$ and $(\psi \perp \varphi_0)_{K(\varphi_0)} \in I_q^4K(\varphi_0)$. Hence,

$$\psi \perp \varphi_0 + I_q^4F \in \overline{I_q}^3(K(\varphi_0)/F). \tag{5.4}$$

Recall that $K(\varphi_0) = F(\varphi_0)(Q)$ (since the polynomial given by Q stays irreducible over $F(\varphi_0)$, see [10, Section 4.2] for details). Since $r + s > 4$, we apply Corollary 3.7(1) to get

$$\psi \perp \varphi_0 + I_q^4F \in \overline{I_q}^3(F(\varphi_0)/F).$$

Hence, by Theorem 3.8, there exists $\rho \in IF$ such that

$$\psi \perp \varphi_0 \perp \rho \otimes \varphi_0 \in I_q^4F. \tag{5.5}$$

Let $r \in F^*$ be a representative of $\det \rho$. Since $\rho \perp \langle 1, r\rangle_b \in I^2F$, it follows from equation (5.5):

$$\psi \perp \varphi_0 \perp \langle 1, r\rangle_b \otimes \varphi_0 \in I_q^4F.$$

Consequently, $\psi \perp r\varphi_0 \in I_q^4F$. Passing to K and combining with equation (5.1), we get $\varphi \perp (r\varphi_0)_K \in I_q^4K$. The Hauptsatz implies $\varphi \simeq (r\varphi_0)_K$, as desired.

2. Suppose $\dim \varphi = 4$ **and** $\triangle(\varphi) \neq 0$**.** Recall that we have $\varphi \perp \psi_K \in I_q^4F$ by equation (5.1).

By Lemma 2.10, we have $\triangle(\varphi) = r + \wp(K)$ for some $r \in F \setminus \wp(F)$ satisfying $\triangle(\psi) = r + \wp(F)$. Let $L = K(\alpha)$ be such that $\alpha^2 + \alpha = r$. Taking Clifford algebra in equation (5.1), we get $C(\varphi) \sim C(\psi_K)$. In particular, $C(\varphi_L) \sim C(\psi_L)$. Note that $\varphi_L \in GP_2(L)$ because $r \in \wp(L)$. Hence, $\mathrm{ind}C(\psi_L) \leq 2$. Moreover, we have $L = F(\alpha)(Q)$. By Corollary 2.8, we get $\mathrm{ind}C(\psi_{F(\alpha)}) \leq 2$. Thus, either $\mathrm{ind}C(\psi) \leq 2$ or $\mathrm{ind}C(\psi) = 4$. We carefully consider the two cases separately:

(1) If $\mathrm{ind}C(\psi) \leq 2$, then there exists $a, b \in F^*$ such that $C(\psi) \sim [b, a)$. In particular, $C(\psi) \sim [r, 1) \otimes_F [b, a)$.

(2) If $\mathrm{ind}C(\psi) = 4$, then there exists an Albert form γ such that $C(\psi) \sim C(\gamma)$. Since $\mathrm{ind}C(\gamma_{F(\alpha)}) = \mathrm{ind}C(\psi_{F(\alpha)}) \leq 2$, the form $\gamma_{F(\alpha)}$ is isotropic. We get, by [19, Théorème 1.1(2)], $\gamma \simeq x[1, r] \perp y[1, t] \perp z[1, r + t]$ for some scalars $x, y, z, t \in F^*$. Now we write $C(\gamma)$ explicitly:

$$\begin{aligned} C(\gamma) &= C(x[1, r] \perp y[1, t] \perp z[1, r + t]) \\ &\sim [r, x) \otimes_F [t, y) \otimes_F [r + t, z) \\ &\sim [r, x) \otimes_F [t, y) \otimes_F [r, z) \otimes_F [t, z) \\ &\sim [r, xz) \otimes_F [t, yz). \end{aligned}$$

Thus, in both cases we can write $C(\psi) \sim [r, s) \otimes_F [b, a)$ for suitable scalars $a, b, s \in F^*$. We now consider $\tau = s(a[1, b] \perp [1, b + r])$ and compute its Clifford algebra:

$$\begin{aligned} C(\tau) &= C(as[1, b] \perp s[1, b + r]) \\ &\sim [b, as) \otimes_F [b + r, s) \\ &\sim [b, a) \otimes_F [b, s) \otimes_F [b, s) \otimes_F [r, s) \\ &\sim [b, a) \otimes_F [r, s). \end{aligned}$$

Hence, we have $C(\psi) \sim C(\tau)$, and thus, by Theorem 2.9, we get

$$\psi \perp \tau \in I_q^3F. \tag{5.6}$$

Passing to K, we get $\psi_K \perp \tau_K \in I_q^3 K$, and adding to equation (5.1) we get:

$$\varphi \perp \tau_K \in I_q^3 K. \tag{5.7}$$

We now have an F-quadratic form τ of dimension 4 such that $\varphi \perp \tau_K \in I_q^3 K$ and $\triangle(\varphi) = \triangle(\psi_K) = \triangle(\tau_K) = r + \wp(K)$. We pass equation (5.7) to $K(\tau)$ to get:

$$\varphi_{K(\tau)} \perp \tau_{K(\tau)} \in I_q^3 K(\tau).$$

Since $\dim(\tau_{K(\tau)})_{\mathrm{an}} < 4$, we obtain by the Hauptsatz

$$(\varphi \perp \tau)_{K(\tau)} \sim 0 \tag{5.8}$$

that is, $\varphi_{K(\tau)} \simeq \tau_{K(\tau)}$. Hence, $\varphi_{K(\tau)}$ is isotropic, and thus by [19, Theorem 1.3], $\varphi \simeq x\tau_K$ for some scalar $x \in K^*$. Consequently, $K(\varphi) = K(\tau)$.

Summarizing, we have our initial hypothesis $\varphi \perp \psi_K \in I_q^4 K$ (equation (5.1)) for some F-quadratic form ψ, another F-quadratic form τ of dimension 4 such that $\psi \perp \tau \in I_q^3 F$ (equation (5.6)) and $E := K(\varphi) = K(\tau)$.

We extend equation (5.1) to E, to get $(\varphi \perp \psi)_E \in I_q^4 E$. In other words, $\psi_E \sim \varphi_E \perp \eta$ for some form $\eta \in I_q^4 E$. In particular, we have $(\psi \perp \tau)_E \sim \varphi_E \perp \tau_E \perp \eta$. But we have seen before that $\varphi_E \perp \tau_E \sim 0$ (equation (5.8)), hence $(\psi \perp \tau)_E \in I_q^4 E$. Since $\psi \perp \tau \in I_q^3 F$, we conclude that

$$\psi \perp \tau + I_q^4 F \in \overline{I}_q^3(E/F). \tag{5.9}$$

Note that $E = K(\tau) = F(\tau)(Q)$. Since $r + s > 4$, it follows from Corollary 3.7(1)

$$\psi \perp \tau \in \overline{I}_q^3(F(\tau)/F).$$

By Theorem 3.9, there exists $\rho \in P_3(F)$ such that

$$\psi \perp \tau \perp \rho \in I_q^4 F, \tag{5.10}$$

and $x\tau \subset \rho$ for some $x \in F^*$. We write $\rho \simeq x\tau \perp \theta$ for some F-quadratic form θ. Since $\tau_{F(\tau)}$ is isotropic, the Pfister form $\rho_{F(\tau)}$ is hyperbolic, and thus $\theta_{F(\tau)}$ is isotropic. Again using [19, Theorem 1.3], we get a scalar $y \in F^*$ such that $\theta \simeq y\tau$, and thus $\rho \simeq x\tau \perp y\tau$. Hence, we derive from equation (5.10) the following

$$\psi \perp xy\tau \in I_q^4 F.$$

Extending this last equation to K, and combining with equation (5.1), we get:

$$\varphi \perp xy\tau_K \in I_q^4 K.$$

It follows from the Hauptsatz $\varphi \simeq (xy\tau)_K$, as desired.

6. Proof of Theorem 1.2: The case of type $(1, 2)$

The proof we will give is quite trickier than the previous cases and we will use a different line of approach. We recall a preliminary result.

Proposition 6.1. ([21, Corollary 2.13]) *Let $p \in F[X]$ be an irreducible polynomial. For any anisotropic totally singular F-quadratic form γ, we have $i_d(\gamma_{F(p)}) \leq [\frac{\dim \gamma}{2}]$, where $[m]$ denotes the integer part of m.*

Corollary 6.2. *Let Q be an anisotropic F-quadratic form of dimension > 8. Let $p \in F[X]$ be an irreducible polynomial, and (r', s') the type of $(Q_{F(p)})_{an}$. If $i_W(Q_{F(p)}) = 0$, then $r' + s' > 4$.*

Proof. Let $L = F(p)$ and (r, s) the type of Q. We have $i_d(Q_L) = i_d(\mathrm{ql}(Q)_L)$. Let (r', s') be the type of $(Q_L)_{an}$. Since $i_W(Q_L) = 0$, we get $r = r'$.

(1) Suppose that $s = 2k$. By Proposition 6.1, we have $i_d(\mathrm{ql}(Q)_L) \leq [\frac{s}{2}] = k$. So, we have $s' = \dim(\mathrm{ql}(Q)_L)_{\mathrm{an}} = s - i_d(\mathrm{ql}(Q)_L) \geq k$. By our hypothesis $\dim Q = 2r + s = 2r + 2k > 8$, and thus in this situation $r' + s' \geq r + k > 4$.

(2) Suppose that $s = 2k + 1$. Again, as we did in (1), $i_d(\mathrm{ql}(Q)_L) \leq [\frac{s}{2}] = k$ and thus $s' = s - i_d(\mathrm{ql}(Q)_L) \geq s - k = k + 1$. Moreover, $\dim Q = 2r + s = 2r + 2k + 1 > 8$, that is, $r + k \geq 4$. We thus have $r' + s' \geq r + k + 1 > 4$. $\square$

Definition 6.3. *A singular quadratic form φ is called quasi-hyperbolic if $i_t(\varphi) \geq \frac{\dim \varphi}{2}$.*

We also need the following lemma:

Lemma 6.4. *(*[26, Corollary 2.7]*) Let $\varphi = R \perp \mathrm{ql}(\varphi)$ be an anisotropic quadratic form over F, and π a quasi-Pfister form such that $\mathrm{ql}(\varphi) \simeq \pi \otimes \gamma$ for some totally singular form γ. Let $c_1, \cdots, c_s \in D_F(\pi)$ be such that $\langle c_1, \cdots, c_s \rangle \prec \varphi$. Suppose that $1 \in D_F(R)$. Then, there exist $d_1, \cdots, d_s \in F$ and R' a nonsingular form such that $\varphi \simeq [c_1, d_1] \perp \cdots \perp [c_s, d_s] \perp R' \perp \mathrm{ql}(\varphi)$. In particular,* $\dim R \geq 2s$.

The descent for quadratic forms of type $(1, 2)$ will be based on the following corollary that answers the quasi-hyperbolicity over inseparable quadratic extensions.

Corollary 6.5. *Let $\varphi = R \perp \mathrm{ql}(\varphi)$ be an anisotropic semisingular F-quadratic form. Let $d \in F \setminus F^2$ and suppose that φ is quasi-hyperbolic over $F(\sqrt{d})$. Then,* $\dim R \geq 4$, *and for any scalar $\alpha \in D_F(R)$, there exist a nonsingular form R' of dimension* $\dim R - 2$ *and $a \in F$ such that*

$$\varphi \sim \alpha \langle 1, d \rangle_b \otimes [1, a] \perp R' \perp \mathrm{ql}(\varphi). \tag{6.1}$$

Proof. Let $L = F(\sqrt{d})$ and V the underlying vector space of φ. We may suppose $\alpha = 1$. Since φ_L is quasi-hyperbolic, then the uniqueness of the quaslinear part implies that $\mathrm{ql}(\varphi)_L$ is quasi-hyperbolic. Hence, $\mathrm{ql}(\varphi) \simeq \langle 1, d \rangle \otimes \rho$ for some totally singular form ρ [9, Theorem 7.7]. Moreover, $x^2 + d$ is a norm of $\varphi_{F(x)}$ [25]. Hence, $\varphi_{F(x)}(v) = x^2 + d$ for some vector $v \in V \otimes F(x)$. We may suppose $v \in V \otimes F[x]$ [7, Theorem 17.3]. Since φ is anisotropic, we may write $v = v_0 + xv_1$ for $v_0, v_1 \in V$. Hence, the condition $\varphi_{F(x)}(v) = x^2 + d$ implies the following relations: $\varphi(v_0) = d$, $\varphi(v_1) = 1$ and $B_\varphi(v_0, v_1) = 0$, meaning that $\langle 1, d \rangle \prec \varphi$. It follows from Corollary 6.4 that $\dim R \geq 4$, and we have the following isometry:

$$\varphi \simeq [1, a] \perp d[1, b] \perp \theta \perp \mathrm{ql}(\varphi),$$

where $a, b \in F$ and θ is a nonsingular quadratic form of dimension $\dim R - 4$. The previous isometry can be re-written as follows:

$$\varphi \sim \langle 1, d \rangle_b \otimes [1, a] \perp R' \perp \mathrm{ql}(\varphi),$$

where $R' = d[1, a + b] \perp \theta$ is of dimension $\dim R - 2$. Hence the corollary. $\square$

For the rest of this section, we fix Q an anisotropic F-quadratic form of dimension > 8, $K = F(Q)$, and φ an anisotropic K-quadratic form of type $(1, 2)$ such that $\varphi \sim \psi_K$ for some F-quadratic form ψ. We will prove that φ is defined over F in many steps.

Set $\varphi \simeq a[1, b] \perp \langle 1, x\rangle$ for some scalars $a, b, x \in K^*$. By the uniqueness of the quasilinear part, we have $\mathrm{ql}(\psi)_K \simeq \langle 1, x\rangle$, and thus we may suppose $x \in F^*$ and write $\psi \simeq \theta \perp \langle 1, x\rangle$ for some nonsingular F-quadratic form θ. So we have

$$\varphi \sim (\theta \perp \langle 1, x\rangle)_K. \tag{6.2}$$

Note that $K(\sqrt{x}) = F(\sqrt{x})(Q)$ as the polynomial given by Q remains irreducible over $F(\sqrt{x})$.

Proposition 6.6. *We keep the same notations as mentioned before. Then, there exist $k, l \in F^*$ such that*

$$\varphi_{K(\sqrt{x})} \simeq (k[1, l] \perp \langle 1, x\rangle)_{K(\sqrt{x})}. \tag{6.3}$$

Proof. **(1) Suppose $i_W(Q_{F(\sqrt{x})}) > 0$.** Then, the extension $K(\sqrt{x})/F(\sqrt{x})$ is purely transcendental, so we write $K(\sqrt{x}) = F(\sqrt{x})(t_1, \cdots, t_n)$ for some independent variables $t_1, \cdots, t_n$ over $F(\sqrt{x})$. Let $L = F(t_1, \cdots, t_n)$.

Extending equation (6.2) to $K(\sqrt{x})$, canceling the form $\langle 0\rangle$, and using the excellence of the extension $L(\sqrt{x})/L$, we get a nonsingular L-quadratic form δ of dimension 2 such that

$$(\theta \perp \langle 1\rangle)_{L(\sqrt{x})} \sim (\delta \perp \langle 1\rangle)_{L(\sqrt{x})}. \tag{6.4}$$

Since in equation (6.4) we have nondefective forms and F is infinite, we can apply [17, Proposition 2.9] by specializing $t_1, \cdots, t_n$ to suitable scalars in F, getting scalars $k, l \in F^*$ such that

$$(\theta \perp \langle 1\rangle)_{F(\sqrt{x})} \sim (k[1, l] \perp \langle 1\rangle)_{F(\sqrt{x})}. \tag{6.5}$$

After extending equation (6.5) to $K(\sqrt{x})$ and adding the form $\langle 0\rangle$, we get

$$(\theta \perp \langle 1, x\rangle)_{K(\sqrt{x})} \sim (k[1, l] \perp \langle 1, x\rangle)_{K(\sqrt{x})}. \tag{6.6}$$

Now combining with equation (6.2) yields $\varphi_{K(\sqrt{x})} \simeq (k[1, l] \perp \langle 1\rangle)_{K(\sqrt{x})}$, as desired.

(2) Suppose $i_W(Q_{F(\sqrt{x})}) = 0$. Extending equation (6.2) to $K(\sqrt{x})$, and canceling the zero form $\langle 0\rangle$, we obtain

$$(a[1, b] \perp \langle 1\rangle)_{K(\sqrt{x})} \sim (\theta \perp \langle 1\rangle)_{K(\sqrt{x})}. \tag{6.7}$$

We will apply to equation (6.7) the descent for forms of type $(1, 1)$. Let $Q' = (Q_{F(\sqrt{x})})_{an}$. Since $i_W(Q_{F(\sqrt{x})}) = 0$, the extension $F(\sqrt{x})(Q)/F(\sqrt{x})(Q')$ is purely transcendental when $Q_{F(\sqrt{x})}$ is isotropic. Moreover, by Corollary 6.2, the type (r', s') of Q' satisfies $r' + s' > 4$. So without loss of generality, we may suppose that Q is anisotropic over $F(\sqrt{x})$. By the descent for forms of type $(1, 1)$, there exists a nonsingular $F(\sqrt{x})$-quadratic form η of dimension 2 such that

$$\begin{aligned}(a[1, b] \perp \langle 1\rangle)_{K(\sqrt{x})} &\simeq (\eta \perp \langle 1\rangle)_{K(\sqrt{x})} \\ &\sim (\theta \perp \langle 1\rangle)_{K(\sqrt{x})}.\end{aligned} \tag{6.8}$$

By abuse of notations, we identify $\triangle(\eta)$ and $\triangle(\theta)$ with their representatives. Applying the completion lemma to equation (6.8), we get $(\eta \perp [1, \triangle(\eta)])_{K(\sqrt{x})} \sim (\theta \perp [1, \triangle(\theta)])_{K(\sqrt{x})}$, and thus

$$(\eta \perp \theta \perp [1, \triangle(\eta) + \triangle(\theta)])_{K(\sqrt{x})} \sim 0.$$

This implies that $\eta \perp \theta \perp [1, \triangle(\eta) + \triangle(\theta)] + I_q^3 F(\sqrt{x})$ belongs to $\overline{I_q}^2(K(\sqrt{x})/F(\sqrt{x}))$. Applying Corollary 3.7(2), we deduce

$$\eta \perp \theta \perp [1, \triangle(\eta) + \triangle(\theta)] \in I_q^3 K(\sqrt{x}).$$

Hence, $C(\eta) \sim C(\theta)_{F(\sqrt{x})}$, and thus $\mathrm{ind}C(\theta)_{F(\sqrt{x})} \leq 2$. Then, there exists $\tau_1 := \langle\langle k, l]] \in P_2F$ and $e \in F$ such that

$$C(\theta) \sim C(\tau_1) \otimes [e, x).$$

Passing to $F(\sqrt{x})$, we obtain

$$C(\eta) \sim C(\eta \perp [1, \triangle(\eta)]) \sim [l, k)_{F(\sqrt{x})}.$$

Hence, $\eta \perp [1, \triangle(\eta)] \simeq (k[1, l] \perp [1, l])_{F(\sqrt{x})}$. Adding the form $\langle 1\rangle_{F(\sqrt{x})}$ and canceling the hyperbolic plane yields

$$\eta \perp \langle 1\rangle \simeq (k[1, l] \perp \langle 1\rangle)_{F(\sqrt{x})}. \tag{6.9}$$

Now extending equation (6.9) to $K(\sqrt{x})$, and combining it with equation (6.8), we get $(a[1, b] \perp \langle 1\rangle)_{K(\sqrt{x})} \simeq (k[1, l] \perp \langle 1\rangle)_{K(\sqrt{x})}$. In particular, we obtain

$$(a[1, b] \perp \langle 1, x\rangle)_{K(\sqrt{x})} \simeq (k[1, l] \perp \langle 1, x\rangle)_{K(\sqrt{x})}.$$

This proves the proposition. □

From equation (6.3), it is clear that we have

$$(a[1, b] \perp k[1, l] \perp \langle 1, x\rangle)_{K(\sqrt{x})} \simeq 2 \times \mathbb{H} \perp \langle 1, 0\rangle_{K(\sqrt{x})}, \tag{6.10}$$

which means that the form $a[1, b] \perp (k[1, l] \perp \langle 1, x\rangle)_K$ becomes quasi-hyperbolic over $K(\sqrt{x})$. Moreover, over K, we have

$$\varepsilon := i_W(a[1, b] \perp k[1, l]_K \perp \langle 1, x\rangle_K) \in \{0, 1, 2\}.$$

We will discuss on each value of ε to descent φ to F.

Lemma 6.7. *We keep the same notations as mentioned before. Then, we have $\varepsilon \neq 1$.*

Proof. Suppose $\varepsilon = 1$. This implies

$$a[1, b] \perp (k[1, l] \perp \langle 1, x\rangle)_K \simeq [0, 0] \perp a_1[1, b_1] \perp \langle 1, x\rangle_K, \tag{6.11}$$

for some scalars $a_1, b_1 \in K^*$. It is clear from (6.11) and (6.10) that $i_W((a_1[1, b_1] \perp \langle 1, x\rangle)_{K(\sqrt{x})}) = 1$, meaning that $\lambda := a_1[1, b_1] \perp \langle 1, x\rangle$ is quasi-hyperbolic over $K(\sqrt{x})$. But this is not possible by Corollary 6.5 because the regular part of λ has dimension smaller that 4. □

Lemma 6.8. *We keep the same notations as mentioned before. If $\varepsilon = 2$, then φ is defined over F.*

Proof. The condition $\varepsilon = 2$ means that $a[1, b] \perp k[1, l]_K \perp \langle 1, x\rangle_K \simeq 2 \times \mathbb{H} \perp \langle 1, x\rangle_K$. Adding on both sides the form $k[1, l]_K$ and canceling the form $2 \times \mathbb{H}$, we get $\varphi \simeq (k[1, l] \perp \langle 1, x\rangle)_K$. □

Proposition 6.9. *We keep the same notations as mentioned before. If $\varepsilon = 0$, then we have:*

(1) There exist $u \in K^*$ and $\gamma \in I_q^3 F$ such that

$$\begin{aligned} a[1, b] \perp (k[1, l] \perp \langle 1, x\rangle)_K &\simeq k\,\langle 1, x\rangle_b \otimes [1, u] \perp \langle 1, x\rangle_K \\ &\sim (\gamma \perp \langle 1, x\rangle)_K. \end{aligned}$$

(2) If $\langle\langle x, k, u]]$ is isotropic, then φ is defined over F.

(3) Let L/F be a field extension and $\eta \in GP_n(L)$ with $n \geq 3$. Suppose we have

$$(\eta \perp \langle 1, x\rangle)_{K.L} \sim (k \langle 1, x\rangle_b \otimes [1, u] \perp \langle 1, x\rangle)_{K.L}.$$

Then, $\eta_{K.L}$ is isotropic iff $\langle\langle x, k, u]]_{K.L}$ is isotropic. When $\eta_{K.L}$ is anisotropic, we necessarily have $\dim \eta = 8$, i.e., $\eta \in GP_3(L)$.

Proof. (1) Since $a[1, b] \perp (k[1, l] \perp \langle 1, x\rangle)_K$ is anisotropic over K, and becomes quasi-hyperbolic over $K(\sqrt{x})$, it follows from Corollary 6.5

$$a[1, b] \perp (k[1, l] \perp \langle 1, x\rangle)_K \sim k \langle 1, x\rangle_b \otimes [1, u] \perp v[1, w] \perp \langle 1, x\rangle_K , \tag{6.12}$$

for some scalars $u, v, w \in K^*$. Passing to $K(\sqrt{x})$, we conclude that $(v[1, w] \perp \langle 1, x\rangle)_{K(\sqrt{x})}$ is quasi-hyperbolic. Hence, using again Corollary 6.5 and taking into account the dimension of the regular part, we necessary have $v[1, w] \perp \langle 1, x\rangle_K$ isotropic. Consequently, $v[1, w] \perp \langle 1, x\rangle_K \sim \langle 1, x\rangle_K$ because $\langle 1, x\rangle_K$ is anisotropic. Now equation (6.12) could be written as:

$$a[1, b] \perp (k[1, l] \perp \langle 1, x\rangle)_K \simeq k \langle 1, x\rangle_b \otimes [1, u] \perp \langle 1, x\rangle_K . \tag{6.13}$$

Recall our initial hypothesis $a[1, b] \perp \langle 1, x\rangle \sim (\theta \perp \langle 1, x\rangle)_K$, and substitute in equation (6.13), we obtain:

$$(\theta \perp k[1, l] \perp \langle 1, x\rangle)_K \sim k \langle 1, x\rangle_b \otimes [1, u] \perp \langle 1, x\rangle . \tag{6.14}$$

Passing to $K(\sqrt{x})$ and cancelling $\langle 0\rangle$, gives us

$$(\theta \perp k[1, l] \perp \langle 1\rangle)_{K(\sqrt{x})} \sim \langle 1\rangle .$$

Now we obtain by completion lemma:

$$(\theta \perp k[1, l] \perp [1, l + \triangle(\theta)])_{K(\sqrt{x})} \sim 0.$$

Hence, $\theta \perp k[1, l] \perp [1, l + \triangle(\theta)] + I_q^3F \in \overline{I_q}^2 K(\sqrt{x})$. Recall that we have either $i_W(Q_{F(\sqrt{x})}) > 0$ or ($i_W(Q_{F(\sqrt{x})}) = 0$ and the type (r', s') of $(Q_{F(\sqrt{x})})_{an}$ satisfies $r' + s' > 4$ (Corollary 6.2)). Hence, we get by Corollary 3.7(2)

$$\theta \perp k[1, l] \perp [1, l + \triangle(\theta)] + I_q^3F \in \overline{I_q}^2 F(\sqrt{x}).$$

By Theorem 3.10, there exists a nonsingular F-quadratic form ρ such that

$$\theta \perp k[1, l] \perp [1, l + \triangle(\theta)] \perp \langle 1, x\rangle_b \otimes \rho \in I_q^3F.$$

Because $\rho \perp [1, \triangle(\rho)] \in I_q^2F$, we have

$$\theta \perp k[1, l] \perp [1, l + \triangle(\theta)] \sim \langle 1, x\rangle_b \otimes [1, \triangle(\rho)] \perp \gamma,$$

where $\gamma \in I_q^3F$. We now pass to K, add $\langle 1, x\rangle$ and cancel the hyperbolic planes, we obtain:

$$(\theta \perp k[1, l] \perp \langle 1, x\rangle)_K \sim (\gamma \perp \langle 1, x\rangle)_K.$$

Combining this equation with (6.13) and (6.14) yields

$$\begin{aligned} a[1, b] \perp (k[1, l] \perp \langle 1, x\rangle)_K &\simeq k \langle 1, x\rangle_b \otimes [1, u] \perp \langle 1, x\rangle_K \\ &\sim (\gamma \perp \langle 1, x\rangle)_K. \end{aligned} \tag{6.15}$$

This completes the proof of statement (1). For the rest of the proof, let π denotes the form $\langle\langle x, k, u]] \in P_3(K)$.

(2) Suppose that π is isotropic, then it is hyperbolic. Hence, $k\langle 1,x\rangle \otimes [1,u] \simeq \langle 1,x\rangle \otimes [1,u]$. This implies that $k\langle 1,x\rangle \otimes [1,u] \perp \langle 1,x\rangle_K \sim \langle 1,x\rangle_K$, and by equation (6.15)

$$a[1,b] \perp (k[1,l] \perp \langle 1,x\rangle)_K \sim \langle 1,x\rangle_K .$$

Hence, $\varphi \simeq (k[1,l] \perp \langle 1,x\rangle)_K$ is defined over F.

(3) Let L be a field extension of F and $\eta \in GP_n(L)$ with $n \geq 3$. Suppose we have

$$(\eta \perp \langle 1,x\rangle)_{K.L} \sim (k\langle 1,x\rangle_b \otimes [1,u] \perp \langle 1,x\rangle)_{K.L}.$$

If $\eta_{K.L}$ is isotropic, then it is hyperbolic. Hence, we get

$$\langle 1,x\rangle_{K.L} \sim (k\langle 1,x\rangle_b \otimes [1,u] \perp \langle 1,x\rangle)_{K.L}.$$

Consequently, $\pi_{K.L}$ is isotropic because $k\langle 1,x\rangle_b \otimes [1,u] \perp \langle 1,x\rangle \prec \pi$. Conversely, if $\pi_{K.L}$ is isotropic, then the same argument as in the proof of (2) gives $(k\langle 1,x\rangle_b \otimes [1,u] \perp \langle 1,x\rangle)_{K.L} \sim \langle 1,x\rangle_{K.L}$, and thus

$$(\eta \perp \langle 1,x\rangle)_{K.L} \sim \langle 1,x\rangle_{K.L} .$$

Then, $i_W((\eta \perp \langle 1,x\rangle)_{K.L}) = \frac{\dim \eta}{2}$. Lemma 2.6 implies that any form dominated by $(\eta \perp \langle 1,x\rangle)_{K.L}$ and having dimension $\geq \frac{\dim\eta}{2} + 3$ is isotropic. Hence, $\eta_{K.L}$ is isotropic because $\dim\eta \geq \frac{\dim\eta}{2} + 3$ since $\dim\eta \geq 8$.

Suppose that $\eta_{K.L}$ is anisotropic. Hence, $\pi_{K.L}$ is anisotropic, and thus its Pfister neighbor $(k\langle 1,x\rangle_b \otimes [1,u] \perp \langle 1,x\rangle)_{K.L}$ is also anisotropic. Then, $j := i_t((\eta \perp \langle 1,x\rangle)_{K.L}) = i_W((\eta \perp \langle 1,x\rangle)_{K.L}) = \frac{\dim\eta}{2} - 2$. Since $\eta_{K.L}$ is anisotropic and dominated by $(\eta \perp \langle 1,x\rangle)_{K.L}$, it follows from Lemma 2.6 that $\dim\eta < \dim\eta + 2 - j + 1 = \frac{\dim\eta}{2} + 5$, i.e., $\dim\eta = 8$. □

Proposition 6.10. *We keep the same notations as mentioned before. If $\varepsilon = 0$, then φ is defined over F.*

Proof. Suppose $\varepsilon = 0$. We will prove that φ is defined over F. We have by Proposition 6.9

$$\begin{aligned} a[1,b] \perp (k[1,l] \perp \langle 1,x\rangle)_K &\simeq k\langle 1,x\rangle_b \otimes [1,u] \perp \langle 1,x\rangle_K \\ &\sim (\gamma \perp \langle 1,x\rangle)_K, \end{aligned} \tag{6.16}$$

for some $u \in K^*$ and $\gamma \in I_q^3F$. Without loss of generality, we may suppose γ anisotropic, and thus $\dim\gamma \geq 8$ by the Hauptsatz. Also, we may suppose, by Proposition 6.9(2), that $\pi := \langle\langle x,k,u]]$ is anisotropic. Let $\lambda = \langle\langle x,k\rangle\rangle$ and $\mu = k\langle 1,x\rangle_b \otimes [1,u] \perp \langle 1,x\rangle_K$.

Let $(F_i, \gamma_i)_{0\leq i\leq h}$ be the generic splitting tower of γ, where h is the height of γ. Let d be the degree of γ and $\widetilde{\pi} \in P_d(F_{h-1})$ its leading form. So we have $\gamma_{F_{h-1}} \sim z\widetilde{\pi}$ for some $z \in F_{h-1}^*$. We discuss on the height h of γ.

(1) Suppose $h = 1$. This means that γ is similar to $\widetilde{\pi}$. By Proposition 6.9, $\dim\gamma = 8$ since π is anisotropic. Moreover, $\pi_{K(\sqrt{x})}$ isotropic implies that $\gamma_{K(\sqrt{x})}$ is isotropic (Proposition 6.9), and thus $\gamma_{K(\sqrt{x})}$ is hyperbolic. In particular, $\gamma + I_q^4F \in \overline{I_q}^3(K(\sqrt{x})/F)$. Corollary 3.7(2) implies that $\gamma_{F(\sqrt{x})} \in I_q^4F(\sqrt{x})$, and by the Hauptsatz $\gamma_{F(\sqrt{x})} \sim 0$. Consequently, $\gamma \simeq y\langle 1,x\rangle_b \otimes \tau$ for some $y \in F^*$ and $\tau = \langle\langle p,q]] \in P_2(F)$. Now since $i_W((\gamma \perp \langle 1,x\rangle)_K) = 2$ (by equation (6.16)), the form $(\gamma \perp \langle 1\rangle)_K$ is isotropic. Using the roundness of a Pfister form, we get

$$(\gamma \perp \langle 1\rangle)_K \simeq (\langle 1,x\rangle_b \otimes \tau \perp \langle 1\rangle)_K.$$

Consequently,

$$(\gamma \perp \langle 1,x\rangle)_K \sim (p[1,q] \perp px[1,q] \perp \langle 1,x\rangle)_K.$$

Substituting in equation (6.16), we obtain

$$a[1,b] \perp \langle 1,x\rangle \sim (k[1,l] \perp p[1,q] \perp px[1,q] \perp \langle 1,x\rangle)_K.$$

Since $\dim Q > 8$, it follows from Theorem 2.5 that $a[1,b] \perp \langle 1,x\rangle$ is defined over F.

(2) Suppose $h > 1$. Extending (6.16) to $K.F_{h-1}$, we get

$$(z\widetilde{\pi} \perp \langle 1,x\rangle)_{K.F_{h-1}} \sim (k\,\langle 1,x\rangle_b \otimes [1,u] \perp \langle 1,x\rangle)_{K.F_{h-1}}, \tag{6.17}$$

for some $z \in F_{h-1}^*$. Moreover, π remains anisotropic over $K.F_{h-1}$ because $\dim \gamma_i > 8$ for all $i < h-1$. Hence, equation (6.17) and Proposition 6.9 imply that $\widetilde{\pi}_{K.F_{h-1}}$ is anisotropic and $\dim \widetilde{\pi} = 8$. Moreover, since $\pi_{K(\pi)}$ is isotropic, the form $\widetilde{\pi}_{K.F_{h-1}(\pi)}$ is isotropic (again by equation (6.17) and Proposition 6.9). Hence, $\widetilde{\pi}_{K.F_{h-1}} \simeq \pi_{K.F_{h-1}}$. This implies

$$(\gamma \perp \pi)_{K.F_{h-1}} \in I_q^4 K.F_{h-1}$$

because $\gamma_{F_{h-1}} \sim z\widetilde{\pi}$. In other words, $\gamma \perp \pi + I_q^4 K \in \overline{I}_q^3(K.F_{h-1}/K)$. We claim that

$$\gamma_K \perp \pi \in I_q^4 K. \tag{6.18}$$

In fact, if γ_{h-2} is isotropic over $K.F_{h-2}$, then $K.F_{h-1}/K.F_{h-2}$ is purely transcendental, and hence $\gamma \perp \pi + I_q^4 K \in \overline{I}_q^3(K.F_{h-2}/K)$. If γ_{h-2} is anisotropic over $K.F_{h-2}$, then we get the same conclusion by Corollary 3.6. Repeating the same argument for the forms γ_i for $i < h-2$, we get the desired claim.

Now equation (6.18) gives $\gamma_{K(\lambda)} \in I_q^4 K(\lambda)$. Corollaries 3.7(2) and 6.2 imply that $\gamma_{F(\lambda)} \in I_q^4 F(\lambda)$, that is, $\gamma + I_q^4 F \in \overline{I}_q^3(F(\lambda)/F)$. Hence, there exists by Theorem 3.10 a scalar $v \in F$ such that

$$\gamma \perp \lambda \otimes [1,v] \in I_q^4 F. \tag{6.19}$$

We extend (6.19) to K and we substitute in equation (6.18), we obtain $\pi \perp \lambda \otimes [1,v] \in I_q^4 K$. We get by the Hauptsatz

$$\pi \simeq (\lambda \otimes [1,v])_K. \tag{6.20}$$

Adding on both sides of equation (6.20) the form $\langle 1,x\rangle_K$, and canceling the hyperbolic planes, we deduce

$$(k\,\langle 1,x\rangle_b \otimes [1,v] \perp \langle 1,x\rangle)_K \simeq k\,\langle 1,x\rangle_b \otimes [1,u] \perp \langle 1,x\rangle\,.$$

We combine this isometry with equation (6.16) to get

$$a[1,b] \perp \langle 1,x\rangle \sim (k[1,l] \perp k\,\langle 1,x\rangle_b \otimes [1,v] \perp \langle 1,x\rangle)_K.$$

Since $\dim Q > 8$, it follows from Theorem 2.5 that φ is defined over F. $\square$

7. Proof of Proposition 1.3

Let Q be an anisotropic F-quadratic form which is not totally singular, and $K = F(Q)$. Let B be a bilinear form over K such that $B + I^{n+1}K \in \operatorname{Im}(W(F)/I^{n+1}F \longrightarrow W(K)/I^{n+1}K)$ for some integer $n \geq 1$ satisfying $2^n > \dim B$. We will prove that B is defined over F.

Since Q is not totally singular, we may write $K = L(\alpha)$, where L/F is purely transcendental and $\alpha^2 + \alpha \in L \setminus \wp(L)$. Let $s : K \longrightarrow L$ be an L-linear map satisfying $s(1) = 0$, and let $s_* : W(K) \longrightarrow W(L)$ be the transfer map induced by s. By [7, Corollary 34.17], we have $s_*(I^{n+1}K) \subset I^{n+1}L$. Let $C \in W(F)$ be such that

$$B \perp C_K \in I^{n+1}K. \tag{7.1}$$

Applying the map s_* to equation (7.1), we get $s_*(B) \in I^{n+1}L$. Since $\dim s_*(B) = 2\dim B < 2^{n+1}$, we conclude by the Hauptsatz that $s_*(B) \sim 0$. By [7, Corollary 34.15], there exists an L-bilinear form D such that $B \sim D_K$. Since K/L is excellent and B anisotropic, we may write $B \simeq D_K$. We then have $D_K \perp C_K \in I^{n+1}K$. Moreover, for any integer $i \geq 1$, we have $\overline{I}^i(K/L) = \{0\}$ by equation (3.6) because K/L is separable. So, after iterating this kernel, we obtain from $D_K \perp C_K \in I^{n+1}K$ the following

$$D \perp C_L \in I^{n+1}L.$$

Let $\rho \in I^{n+1}L$ be such that $D \perp C_L \sim \rho$. The field F is infinite, so we specialize D and ρ to suitable F-bilinear forms D_0 and ρ_0, respectively (we specialize the variables defining L to suitable scalars in F so that our bilinear forms D and ρ have "very good reduction" in the sense of Knebusch [17, Proposition 2.2]). Hence, we get

$$(D_0)_L \perp C_L \sim \rho_0 \in I^{n+1}L.$$

Extending scalars to K, and combining with equation (7.1) yields

$$(D_0)_K \perp B \in I^{n+1}K.$$

It follows from the Hauptsatz that $B \simeq (D_0)_K$.

8. Proof of Theorem 1.5

The descent for bilinear forms of dimension 2 or 4 has been studied in [27], but it turns out that the proof for these two cases is incomplete. In fact, it was considered that any element α of $\overline{I}^2(F(\sqrt{x})/F)$ belongs to $\overline{IF \otimes \langle 1, x+a^2\rangle_b}$ for some scalar $a \in F$, while α must be in a finite sum $\sum_i \overline{I^{k-1}F \otimes \langle 1, x+a_i^2\rangle_b}$. Here we will proceed in a different way. For the descent in dimension 2 (Proposition 8.1), we use a generic argument to reduce to nonsingular forms, and thus apply Theorem 1.2. This reduction has the advantage to conclude the descent taking the group $\mathrm{Im}(W(F)/I^3F \longrightarrow W(K)/I^3K)$ instead of $\mathrm{Im}(W(F) \longrightarrow W(K))$. For dimensions 3 and 4 (Propositions 8.2 and 8.4), to avoid the obstacle caused by the finite sum mentioned before, we work with a kernel $\overline{I}^2(F(Q')/F)$ for Q' a totally singular F-quadratic form of dimension 3, so in this case each nonzero element of the kernel is reduced to a symbol (Theorem 3.12). Moreover, we prove in Section 10 that the extension $F(Q')/F$ is excellent for bilinear forms (Theorem 10.1), and we work with the group $\mathrm{Im}(W(F) \longrightarrow W(K))$ instead of the group $\mathrm{Im}(W(F)/I^4F \longrightarrow W(K)/I^4K)$. Taking into account these new changes, the other arguments that we will use are similar to those used by Laghribi and Rehmann, which are themselves inspired from Kahn's method in characteristic not 2 [15].

Proposition 8.1. *Let Q be an anisotropic totally singular F-quadratic form of dimension > 4, and $K = F(Q)$. Let B be an anisotropic bilinear form over K of dimension 2. If $B + I^3K \in \mathrm{Im}(W(F)/I^3F \longrightarrow W(K)/I^3K)$, then B is defined over F.*

Proof. We keep the same notations and hypotheses as in the proposition. Let t be a variable over K and $\varphi = B \otimes [1, t^{-1}]$ defined over $\widetilde{K} := K((t))$ the field of Laurent series over K. The form φ is anisotropic over $\widetilde{K}$. Let $\widetilde{F}$ denote the field $F((t))$.

The condition $B + I^3K \in \mathrm{Im}(W(F)/I^3F \longrightarrow W(K)/I^3K)$ implies that $\det B$ is defined over F by a scalar $d \in F^*$. Hence, $B \simeq \alpha \langle 1, d\rangle_b$ for some $\alpha \in K^*$. Moreover, we have $\varphi + I^4\widetilde{K} \in \mathrm{Im}(W(\widetilde{F})/I^4\widetilde{F} \longrightarrow W(\widetilde{K})/I^4\widetilde{K})$. It follows from Theorem 1.2 that $\varphi \simeq \psi_{\widetilde{K}}$ for some quadratic form ψ over $\widetilde{F}$. Clearly $\alpha \langle 1, d\rangle$ is the first residue form of φ with respect to the t-adic valuation of $\widetilde{K}$. Similarly, the first residue form of $\psi_{\widetilde{K}}$ is given by ψ'_K, where ψ' is the first residue form of ψ with respect to the t-adic valuation of $\widetilde{F}$. Since $\alpha\langle 1, d\rangle \simeq \psi'_K$, we deduce that $\alpha\langle 1, d\rangle$ represents a scalar $\beta \in F^*$. Using the roundness of a Pfister form, we get $B \simeq (\beta\langle 1, d\rangle_b)_K$, and thus B is defined over F. $\square$

Proposition 8.2. *Let Q be an anisotropic totally singular F-quadratic form of dimension > 6 such that $\mathrm{ndeg}_F(Q) > 8$. Let $K = F(Q)$ and B an anisotropic K-bilinear form of dimension 3 such that $B \in \mathrm{Im}(W(F) \longrightarrow W(K))$. Then, B is defined over F.*

Proof. We keep the same notations and hypotheses as in the proposition. We can write $F(Q) = L(Q')$, where L/F is purely transcendental and Q' is an anisotropic totally singular L-form of dimension 3.

Let $C \in W(F)$ be such that $B \sim C_K$. In particular, $B \simeq (C_K)_{\mathrm{an}}$. The extension $L(Q')/L$ is excellent for bilinear forms (Theorem 10.1), hence there exists an L-bilinear form C' of dimension 3 such that $(C_K)_{\mathrm{an}} \simeq C'_K$. Consequently, $(C \perp C')_{L(Q')} \sim 0$. Since $\dim Q' > 2$, we get $\det C_L = \det C'$, and thus $C_L \perp C' \in I^2L$. Consequently, we have

$$C_L \perp C' + I^3L \in \overline{I}^2(L(Q')/L). \tag{8.1}$$

Let $\rho \in BP_2(L)$ be such that Q' is similar to a subform of ρ (i.e., $\widetilde{Q'}$ is a quasi-Pfister neighbor of $\widetilde{\rho}$). By Proposition 3.11, we get $C_L \perp C' + I^3L \in \overline{I}^2(L(\rho)/L)$. Hence there exists, by Theorem 3.12, a form $C'' \in BP_2(L)$ such that

$$C_L \perp C' \perp C'' \in I^3L. \tag{8.2}$$

Now since L/F is purely transcendental and F is infinite, we specialize the equation (8.2), as we did in the proof of Proposition 1.3, to get

$$C \perp C'_1 \perp C''_1 \in I^3F,$$

for suitable F-bilinear forms C'_1 and C''_1 such that $\dim C'_1 = 3$ and $C''_1 \in BP_2(F)$. Let $x \in D_F(C'_1)$. Since $C''_1 \perp xC''_1 \in I^3F$, we deduce

$$C \perp C'_1 \perp xC''_1 \in I^3F.$$

Let $D = (C'_1 \perp xC''_1)_{\mathrm{an}}$. Since we chose $x \in D_F(C'_1)$, we have $\dim D \leq 5$. Passing to K and keeping in mind our initial hypothesis $B \sim C_K$, we get

$$B \perp D_K \in I^3K. \tag{8.3}$$

If $\dim D = 3$, then simply using the Hauptsatz on equation (8.3), we get that $B \simeq D_K$, as desired. So suppose $\dim D = 5$.

Let $\det D = d$, $\gamma = D \perp \langle d\rangle_b$ (an Albert bilinear form) and $\theta = B \perp \langle d\rangle_b \in GBP_2(K)$. Then, we have:

$$\theta \perp \gamma_K \in I^3K. \tag{8.4}$$

Passing to $K(\theta)$, we get by the Hauptsatz $\gamma_{K(\theta)} \sim 0$. Since θ is anisotropic, we have $\mathrm{ndeg}_K(\widetilde{\theta}) = 4$. It follows from Theorem 3.13 that $\dim(\gamma_K)_{\mathrm{an}}$ is divisible by 4. Hence, γ_K is isotropic.

(1) Suppose that γ is isotropic. Then, $\gamma \sim \rho$ for some $\rho \in GBP_2(F)$. Let $y \in D_F(\rho)$. Since $\rho \perp dy\rho \in I^3F$, it follows from equation (8.4)

$$B \perp (\langle d\rangle_b \perp dy\rho)_K \in I^3K.$$

Since $\lambda := (\langle d\rangle_b \perp dy\rho)_{\mathrm{an}}$ has dimension < 5, we conclude by the Hauptsatz that $B \simeq \lambda_K$.

(2) Suppose that γ is anisotropic. Note that $\mathrm{ndeg}_F(\widetilde{\gamma}) = \{8, 16\}$ because γ is anisotropic. Moreover, we have $\mathrm{ndeg}_F(\widetilde{\gamma}) \geq \mathrm{ndeg}_F(Q) > 8$ because γ_K is isotropic (a consequence of [10, Proposition 8.13]). Hence, $\mathrm{ndeg}_F(\widetilde{\gamma}) = 16$. Consequently, γ_K isotropic implies that Q is similar to a subform of $\widetilde{\gamma}$ [24, Theorem 1.1], which is not possible since $\dim Q > 6$. $\square$

Below we present an example showing that Proposition 8.2 fails in general when $\dim Q > 6$ but $\mathrm{ndeg}_F(Q) = 8$.

Example 8.3. *Take $F = F_0(x, y, z)$ the rational function field in the variables x, y, z over a field F_0 of characteristic 2. Let $\eta = \langle x, y, xy, 1 + x, z, (1 + x)z\rangle_b \perp \langle yz\rangle_b$. Then, we have the following:*

(1) η is anisotropic over F, and $\mathrm{ndeg}_F(\widetilde{\eta}) = 8$.
(2) $\dim(\eta_{F(\eta)})_{\mathrm{an}} = 3$ and $(\eta_{F(\eta)})_{\mathrm{an}}$ is not defined over F.

Proof. (1) Using the standard isometry $\langle a, b\rangle \simeq \langle a, a + b\rangle$, we deduce

$$\begin{aligned}\widetilde{\eta} &= \langle x, y, xy, 1 + x, z, (1 + x)z\rangle \perp \langle yz\rangle \\ &\simeq \langle x, y, xy, 1, z, xz\rangle \perp \langle yz\rangle \\ &= \langle\langle x, y\rangle\rangle \perp z\,\langle 1, x, y\rangle\,.\end{aligned}$$

Thus, $\widetilde{\eta}$ is anisotropic over F and it is a quasi-Pfister neighbor of the quasi-Pfister form $\langle\langle x, y, z\rangle\rangle$. The norm field of $\widetilde{\eta}$ is $F^2(x, y, z)$, hence $\mathrm{ndeg}_F(\widetilde{\eta}) = 8$.

(2) Let γ denote the Albert bilinear form $\langle x, y, xy, 1 + x, z, (1 + x)z\rangle_b$. Since $\widetilde{\eta}$ is a quasi-Pfister neighbor of dimension 7, it follows that $i_d(\widetilde{\eta}_{F(\eta)}) = 3$ [10, Theorem 8.11(ii)]. But, by [12, Proposition 1], we know $2i_W(\eta_{F(\eta)}) \geq i_d(\eta_{F(\eta)})$, thus we obtain $i_W(\eta_{K(\eta)}) \geq 2$.

(i) Suppose $i_W(\eta_{F(\eta)}) = 2$. Then, $B := (\eta_{F(\eta)})_{\mathrm{an}}$ is an $F(\eta)$-bilinear form of dimension 3. Suppose that B is defined over F. Then, there exists an F-bilinear form C of dimension 3 such that $B \simeq C_{F(\eta)}$. In particular, $(\eta \perp C)_{F(\eta)} \sim 0$, that is

$$(\gamma \perp \langle yz\rangle_b \perp C)_{F(\eta)} \sim 0.$$

Moreover, the condition $\mathrm{ndeg}_K(\widetilde{\eta}) = 8$ implies, by Theorem 3.13, that $\dim(\gamma \perp \langle yz\rangle_b \perp C)_{an}$ is divisible by 8. The form $\gamma \perp \langle yz\rangle_b \perp C$ is not metabolic, otherwise we would get

$$\gamma \sim \langle yz\rangle_b \perp C$$

and thus γ would be isotropic over F. Hence, $\dim(\gamma \perp \langle yz\rangle_b \perp C)_{an} = 8$, and we have by Theorem 3.13

$$\gamma \sim \langle yz\rangle_b \perp C \perp \pi,$$

for some $\pi \in GBP_3(F)$. Since $\dim \gamma = 6$, we get $i_W(\langle yz\rangle_b \perp C \perp \pi) = 3$. This is a contradiction, since $\langle yz\rangle_b \perp C \in GBP_2(F)$ and $\pi \in GBP_3(F)$, this Witt index must be a power of 2 by [23, Theorem 3.7].

(ii) Suppose $i_W(\eta_{F(\eta)}) = 3$. Then, comparing determinants yields $\eta_{F(\eta)} \sim (\langle yz\rangle_b)_{F(\eta)}$. Adding $\langle yz\rangle_b$ on both sides, we get:

$$\gamma_{F(\eta)} \sim 0.$$

This is not possible by Theorem 3.13 because $\dim \gamma$ is not divisible by $8 = \mathrm{ndeg}_F(\widetilde{\eta})$.

Hence, we deduce from the two cases (i) and (ii) that $\dim(\eta_{F(\eta)})_{\mathrm{an}} = 3$ and $(\eta_{F(\eta)})_{\mathrm{an}}$ is not defined over F. $\square$

Proposition 8.4. *Let Q be an anisotropic totally singular F-quadratic form of dimension > 8, and $K = F(Q)$. Let $B \in GBP_2(K)$ be anisotropic such that $B \in \mathrm{Im}(W(F) \longrightarrow W(K))$. Then, we have:*

(1) There exists $\rho \in BP_2(F)$ such that B is similar to ρ_K.

(2) If $B \in BP_2(K)$, then B is defined over F.

Proof. We write $B = x\delta$, where $x \in K^*$ and $\delta \in BP_2K$. Let $C \in W(F)$ be such that $B \sim C_K$. We write $K = L(Q')$ such that the extension L/F is purely transcendental and Q' is an anisotropic L-bilinear form of dimension 3. Using the fact that K/L is excellent for bilinear forms and following the same arguments as in the proof of Proposition 8.2, we prove the existence of $C_1' \in GBP_2(F)$ and $C_1'' \in BP_2(F)$ such that

$$C \perp C_1' \perp yC_1'' \in I^3F, \tag{8.5}$$

where $y \in D_F(C_1')$. Let $D = (C_1' \perp yC_1'')_{\mathrm{an}}$. We have $D \in I^2F$ of dimension ≤ 6. Extending equation (8.5) to K and using $B \sim C_K$, we get

$$x\delta \perp D_K \in I^3K. \tag{8.6}$$

It follows from the Hauptsatz that $D_{K(\delta)} \sim 0$. We have two cases:

(i) If $\dim D < 6$, then necessary $\dim D = 4$, otherwise by the Hauptsatz and equation (8.6) δ would be isotropic. Hence, $D \simeq z\rho$ for some $z \in F^*$ and $\rho \in BP_2(F)$. Then, from equation (8.6), we get $\delta \perp \rho_K \in I^3K$. By the Hauptsatz $\delta \simeq \rho_K$.

(ii) If $\dim D = 6$, then the condition $D_{K(\delta)} \sim 0$ implies that $\mathrm{ndeg}_K(\widetilde{\delta}) = 4$ divides $\dim D$ (Theorem 3.13), which is not possible.

This concludes the proof of statement (1). This proof also shows that B is defined over F when $B \in BP_2(K)$. □

9. Proof of Proposition 1.6

Let $s \geq 1$ be an integer, Q an anisotropic totally singular F-quadratic form of dimension ≥ 2, and $K = F(Q)$. Suppose we have the following hypothesis: *"Conjecture 1.1 is true for nonsingular $F(Q)$-quadratic forms φ of dimension $\leq 2s$ when Q is a totally singular F-quadratic form such that* $\dim Q > 4s$*"*.

Let B be an anisotropic K-bilinear form such that $\dim B \leq s$. Suppose $\dim Q > 4s$ and $B + I^{n+1}K \in \mathrm{Im}(W(F)/I^{n+1}F \longrightarrow W(K)/I^{n+1}K)$ such that $2^n > \dim B$. Our aim is to prove that B is defined over F.

Let t be a variable over K, $\widetilde{K} = K((t))$, $\widetilde{F} = F((t))$ and $\varphi = B \otimes [1, t^{-1}]$. The form φ is anisotropic over $\widetilde{K}$ and satisfies: $\varphi + I_q^{n+2}\widetilde{F} \in \mathrm{Im}(W_q(\widetilde{F})/I_q^{n+2}\widetilde{F} \longrightarrow W_q(\widetilde{K})/I_q^{n+2}\widetilde{K})$, $\dim \varphi \leq 2s$ and $2^{n+1} > \dim \varphi$.

Since $\dim Q > 4s$ and $2^{n+1} > \dim \varphi$, we deduce from our hypothesis that $\varphi \simeq \psi_{\widetilde{K}}$ for some $\widetilde{F}$-quadratic form ψ. On the one hand, the first residue form of φ with respect to the t-adic valuation of $\widetilde{K}$ is $\widetilde{B}$, the totally singular form associated to B. On the other hand, the first residue form of $\psi_{\widetilde{K}}$ is give by θ_K, where θ is the first residue form of ψ with respect to the t-adic valuation of $\widetilde{F}$.

Since θ is defined over F, we deduce that $\widetilde{B}$ represents a scalar $c \in F^*$. Hence, $B \simeq \langle c \rangle_b \perp C$ for some K-bilinear form C. The form C satisfies the same conditions as B, and thus we apply to C the same argument as we did for B. We continue step by step until we prove that B is defined over F.

10. Excellence of function fields of conics

In this section we discuss the excellence property for function fields of conics, i.e., extensions of the shape $F(Q)$, where Q is an F-quadratic form of dimension 3. Recall that such a field extension is excellent in characteristic not 2. This was first proved by Arason [8, Appendix II] and later by Rost who produced an elementary proof that presents the advantage to be extended to characteristic 2 [32]. In characteristic 2, the form Q could be of type $(1,1)$ or $(0,3)$. When Q is of type $(1,1)$, the extension $F(Q)$ is excellent for quadratic forms and bilinear forms as well. For quadratic forms, this is due to Hoffmann and Laghribi who easily adapted Rost's proof to characteristic 2 [11, Corollary 5.7], and for bilinear forms as a consequence of the result [19, Corollary 3.3] stating that an anisotropic totally singular form (and thus an anisotropic bilinear form) stays anisotropic over the function field of a form which is not totally singular. Now we address the situation when Q is of type $(0,3)$. In general in this case, the extension $F(Q)$ is not excellent for quadratic forms (see Example 10.4). For bilinear forms we will prove the following theorem which played a crucial role in the proof of Theorem 1.5:

Theorem 10.1. *Let B be an anisotropic bilinear form over F and Q an anisotropic F-quadratic form of type $(0,3)$. Then, there exists an F-bilinear form C such that $(B_{F(Q)})_{\mathrm{an}} \simeq C_{F(Q)}$. Hence, the extension $F(Q)/F$ is excellent for bilinear forms.*

Our proof of this theorem is inspired from Rost's proof in characteristic not 2, but we will adapt many arguments to our situation. The complications come from the fact that we will consider inseparable quadratic extensions. More precisely, it is known in characteristic 2 that if an anisotropic F-bilinear form becomes isotropic over an inseparable quadratic extension $F(\sqrt{a})$, then B contains, up to a nonzero scalar, a subform isometric to $\langle 1, a+x^2\rangle_b$ for some $x \in F$ ([20, Lemma 3.4]), but not the form $\langle 1, a\rangle_b$ in general (see the comments after Lemma 10.2). Moreover, some arguments given by Rost based on the trace map for quadratic extensions fail in characteristic 2 for quadratic inseparable extensions.

From now on we fix $Q = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form over F. We take for $F(Q)$ the function field of the projective conic given by Q. Then, $F(Q)$ is isomorphic to the quotient field of the ring $R = F[x,y]/(y^2 + ax^2 + b)$. Note that $R = F[x] \oplus yF[x]$ as F-vector space. Define $d : \mathbb{R} \to \mathbb{N} \cup \{-\infty\}$ by:

$$d(P + yQ) = \max\{\deg P, 1 + \deg Q\} \quad \text{for all} \quad P, Q \in F[x]$$

(here $\deg 0 = -\infty$). Moreover, let $R_n = \{r \in R \mid d(r) \leq n\}$. Clearly, R_n is an F-vector subspace of R and we have $R_0 = F$ and $R_n.R_m \subset R_{n+m}$.

Let B be an anisotropic F-bilinear form of underlying vector space V. We denote by $\widetilde{B}$ the totally singular form corresponding to B, it is given by: $\widetilde{B}(v) = B(v,v)$ for all $v \in V$. The key result for the proof of Theorem 10.1 is the following lemma which extends [32, Lemma] to the case of bilinear forms in characteristic 2:

Lemma 10.2. *Suppose that for some $n \geq 1$ there exists*

$$v \in (V \otimes_F R_n) \setminus (V \otimes_F R_{n-1})$$

such that $\widetilde{B}(v) = 0 \in R$. Then, there exists a subspace $W \subset V$ of dimension 2 such that:

(1) $B|_W \simeq c\,\langle 1, a+\epsilon^2\rangle_b$ for some $c \in F^*$ and $\epsilon \in F$.

(2) there exist a nonzero $v' \in V \otimes_F R_{n-1}$ and $b' \in D_F(Q)$ such that $\widetilde{B'}(v') = 0$, where

$$B' = b'(B|_W) \perp (B|_{W^\perp}).$$

Before proving this lemma, let us mention two changes comparing to the original version [32, Lemma] due to Rost: In statement (1), we take the 2-dimensional bilinear forms $\langle 1, a+\epsilon^2\rangle_b$ for $\epsilon \in F$ instead of the form $\langle 1, a\rangle_b$, and in statement (2) we consider the form $b'(B|_W)$ for $b' \in D_F(Q)$ instead of $b(B|_W)$.

Proof of Lemma 10.2. We write

$$v = v_0 + \sum_{i=1}^{n} (v_i y x^{i-1} + w_i x^i); \quad v_i, w_i \in V.$$

Since $\widetilde{B}(v) = 0 \in R$, we get

$$\begin{aligned} 0 &= \widetilde{B}(v_n) y^2 x^{2(n-1)} + \widetilde{B}(w_n) x^{2n} \qquad \pmod{R_{2n-1}} \\ &= (\widetilde{B}(v_n) a + \widetilde{B}(w_n)) x^{2n} \qquad \pmod{R_{2n-1}}. \end{aligned}$$

Thus, $\widetilde{B}(v_n)a = \widetilde{B}(w_n)$ (i.e., $\widetilde{B}$ is isotropic over $F(\sqrt{a})$). Note that $v_n, w_n \neq 0$ since B is anisotropic and $v \notin V \otimes_F R_{n-1}$. Moreover, since $a \notin F^2$, the vectors v_n and w_n are linearly independent.

Let $c = \widetilde{B}(v_n)$, $\alpha = \sqrt{a}$ and W the 2-dimensional vector subspace of V generated by the vectors v_n, w_n. Since $\widetilde{B}|_W \simeq c\langle 1, a\rangle$, it follows from [24, Lemma 3.7] that $B|_W \simeq c\langle 1, a+\epsilon^2\rangle_b$ for a suitable $\epsilon \in F$.

Identifying W with $F(\alpha) = F \oplus F\alpha$ (by $1 \mapsto v_n$ and $\alpha \mapsto w_n$), we get $\widetilde{B}(w) = cw^2$ for all $w \in W$.

Now we write $v = p + q$ with $p \in W \otimes_F R$ and $q \in W^\perp \otimes_F R$. We can express

$$p = (y + x\alpha)x^{n-1} + (y + x\alpha)\mu x^{n-2} + \lambda x^{n-1} + \overline{p}$$

for some $\lambda, \mu \in W$ such that $\overline{p} \in W \otimes_F R_{n-2}$ and $q \in W^\perp \otimes_F R_{n-1}$ (if $n = 1$, we have $\mu = \overline{p} = 0$).

Let $b' = \lambda^2 + b \in F$ and put $v' = (b')^{-1}(\lambda + y + x\alpha)p + q$. Then, $b' \in D_F(Q)$ and v' is a zero of the form $B' = b'(B|_W) \perp (B|_{W^\perp})$ because:

$$\begin{aligned} \widetilde{B'}(v') &= b'\widetilde{B}((b')^{-1}(\lambda + y + x\alpha)p) + \widetilde{B}(q) \\ &= b'c(b')^{-2}(\lambda + y + x\alpha)^2 p^2 + \widetilde{B}(q) \\ &= c(b')^{-1}(\lambda^2 + y^2 + x^2 a)p^2 + \widetilde{B}(q) \\ &= c(b')^{-1}(\lambda^2 + b)p^2 + \widetilde{B}(q) \\ &= \widetilde{B}(p) + \widetilde{B}(q) \\ &= \widetilde{B}(v) \\ &= 0. \end{aligned}$$

Now to show that $v' \in V \otimes_F R_{n-1}$, we just need to verify that $(b')^{-1}(\lambda + y + x\alpha)p \in W \otimes_F R_{n-1}$. We can express $(b')^{-1}(\lambda + y + x\alpha)p$ as:

$$\begin{aligned}(b')^{-1}(\lambda+y+x\alpha)p &= (b')^{-1}(\lambda+y+x\alpha)\Big((\lambda+y+x\alpha)x^{n-1}+(\lambda+y+x\alpha)\mu x^{n-2}+\lambda\mu x^{n-2}+\overline{p}\Big)\\ &= (b')^{-1}(\lambda^2+y^2+x^2a)x^{n-1}+(b^{'})^{-1}(\lambda^2+y^2+x^2a)\mu x^{n-2}\\ &\quad +(b')^{-1}(\lambda+y+x\alpha)(\lambda\mu x^{n-2}+\overline{p})\\ &= x^{n-1}+\mu x^{n-2}+(b')^{-1}(\lambda+y+x\alpha)(\lambda\mu x^{n-2}+\overline{p}).\end{aligned}$$

Therefore, $(b')^{-1}(\lambda + y + x\alpha)p \in W \otimes_F R_{n-1}$ and thus $(b')^{-1}(\lambda + y + x\alpha)p + q = v' \in V \otimes_F R_{n-1}$. $\square$

We would like to stress here that although the main idea of the proof is heavily inspired from [32], certain changes were required for our setting. The important one is the choice of the vector v', we correct it by the factor λ, this is very helpful to show that $v' \in V \otimes_F R_{n-1}$ and then avoid the use of trace form that does not apply in our case. A consequence of this change is the appearance of the scalar $b' \in D_F(Q)$.

A consequence of the previous lemma is the following proposition which is the analogue of [32, Proposition]. Its proof is the same as that done by Rost.

Proposition 10.3. *Let B be a bilinear form over F. Then, there exist an integer $p \geq 0$, bilinear forms B_i, C_i for $0 \leq i \leq p$ and elements $c_i \in F^*, \epsilon_i \in F$ for $0 \leq i \leq p-1$ such that $B = B_0$ and*

(1) $B_i \simeq c_i \langle 1, a+\epsilon_i^2\rangle_b \perp C_i$ for $0 \leq i \leq p-1$.
(2) $B_{i+1} \simeq c_i b_i \langle 1, a+\epsilon_i^2\rangle_b \perp C_i$ and $b_i \in D_F(Q)$ for $0 \leq i \leq p-1$.
(3) $((B_p)_{F(Q)})_{\mathrm{an}} \simeq ((B_p)_{\mathrm{an}})_{F(Q)}$.

Proof. We use induction on the dimension of B_{an}. Thus, we assume that B is anisotropic and $B_{F(Q)}$ is isotropic. Then, there exist $n \geq 0$ and a nonzero $v \in V \otimes_F R_n$ such that $\widetilde{B}(v) = 0$. We proceed by induction on n.

If $n = 0$, then $v \in V$ and B would be isotropic over F. Hence, $n \geq 1$. We may also assume $v \notin V \otimes_F R_{n-1}$ and take $B_1 = B'$, where B' is the bilinear form defined in the Lemma 10.2. If B' is anisotropic, we apply induction hypothesis for $n-1$ and if B' is isotropic we apply the induction hypothesis for $\dim B'_{\mathrm{an}} < \dim B$. In any case we find forms $B' = B'_0, \ldots, B'_p$ as in the proposition and $B = B_0, B_i = B'_{i-1} (i = 1, \ldots, p+1)$ is the required sequence. $\square$

Now we are able to give the proof of Theorem 10.1 including some explanations proper to our setting.

Proof of Theorem 10.1. Let B be an anisotropic bilinear form over F. We keep the same notations as in Proposition 10.3. All the bilinear forms B_i, for $0 \leq i \leq p$, are isometric over $F(Q)$ because $b \in D_{F(Q)}(\langle 1, a\rangle)$ implies $b_i \in D_{F(Q)}(\langle 1, a\rangle) = D_{F(Q)}(\langle 1, a+\epsilon_i^2\rangle)$ for any $b_i \in D_F(Q)$ and $\epsilon_i \in F$, meaning that $(b_i \langle 1, a+\epsilon_i^2\rangle_b)_{F(Q)} \simeq (\langle 1, a+\epsilon_i^2\rangle_b)_{F(Q)}$ by the roundness of a bilinear Pfister form. In particular, $(B_i)_{F(Q)} \simeq (B_{i+1})_{F(Q)}$. Now the bilinear form C needed in the theorem is $(B_p)_{\mathrm{an}}$. $\square$

We finish this section by an example showing the non-excellence of function fields of singular conics for quadratic forms:

Example 10.4. *Let $\mathbb{F}_2$ be the finite field with two elements. Let $\gamma = [t_1, t_2] \perp [t_3, t_4] \perp [1, t_1t_2 + t_3t_4]$ be an Albert quadratic form over the rational function field $F := \mathbb{F}_2(t_1, t_2, t_3, t_4)$*

in the variables t_1, t_2, t_3, t_4 over $\mathbb{F}_2$. Let Q be the totally singular quadratic form $\langle 1, t_1, t_3\rangle$. Then, the form γ is anisotropic over F and becomes isotropic over $F(Q)$, but $(\gamma_{F(Q)})_{\mathrm{an}}$ is not defined over F.

Proof. The form $\gamma_{F(Q)}$ is isotropic because $Q \prec \gamma$. Then, there exists $\tau \in GP_2(F(Q))$ such that $\gamma_{F(Q)} \sim \tau$. Suppose that τ is defined over F and let δ be an F-quadratic form such that $\gamma_{F(Q)} \sim \delta_{F(Q)}$. Note that $\delta \in GP_2(F)$ because if $c \in F$ satisfies $\triangle(\delta) = c + \wp(F) \in F/\wp(F)$, then $[1, c]_{F(Q)} \sim 0$ because $\triangle(\delta)_{F(Q)} = 0$. But then [19, Proposition 1.1] implies $[1, c] \sim 0$, that is, $\triangle(\delta) = 0 \in F/\wp(F)$.

Now the form $\gamma \perp \delta$ belongs to $I_q^2 F$ and becomes hyperbolic over $F(Q)$. Hence, we get $\gamma \perp \delta \in I_q^3 F$ [22, Corollary 4.11]. Passing to $F(\delta)$, we get $\gamma_{F(\delta)} \in I_q^3 F(\delta)$. By the Hauptsatz, $\gamma_{F(\delta)} \sim 0$. Using [20, Theorem 1.2], we get that $\dim \gamma_{\mathrm{an}}$ is divisible by 4, and thus γ is isotropic, a contradiction. $\square$

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