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On composition operators on the Wiener algebra of Dirichlet series

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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Abstract. We show that the symbol of a bounded composition operator on the Wiener algebra of Dirichlet series does not need to belong to this algebra. Our example even gives an absolutely summing (hence compact) composition operator.

MSC 2010 primary: 47B33; secondary: 30B50

Key-words Composition operator; Dirichlet series

1 Introduction

In [3] (see also [5]), composition operators on the Wiener algebra \mathcal{A}^+ of all absolutely convergent Dirichlet series were studied.

Recall that \mathcal{A}^+ is the space of all analytic maps $f: \mathbb{C}_0 \rightarrow \mathbb{C}$ which can be written

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{with} \quad \|f\|_{\mathcal{A}^+} := \sum_{n=1}^{\infty} |a_n| < +\infty,$$

where, for $\theta \in \mathbb{R}$, we note $\mathbb{C}_\theta = \{z \in \mathbb{C}; \Re z > \theta\}$. If $\phi: \mathbb{C}_0 \rightarrow \mathbb{C}_0$ is an analytic function, the composition operator $C_\phi: \mathcal{A}^+ \rightarrow \mathcal{A}^+$ of symbol ϕ on this space is defined as $C_\phi(f) = f \circ \phi$. Gordon and Hedenmalm, for the Hilbert space \mathcal{H}^2 , showed in [6] that such a symbol has necessarily the form

$$(1.1) \quad \phi(s) = c_0 s + \varphi(s),$$

where $c_0 \geq 0$ is an integer and φ is a convergent Dirichlet series with values in \mathbb{C}_0 , that is $\varphi: \mathbb{C}_0 \rightarrow \mathbb{C}_0$ is an analytic function which can be written $\varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ for $\Re s$ large enough. Moreover, this Dirichlet series is uniformly convergent in \mathbb{C}_ε for all $\varepsilon > 0$ ([13, pages 1625–1626 and Theorem 3.1]; see also [12, Theorem 8.4.1, page 245]).

It is shown in [3, Theorem 2.3] that C_ϕ is bounded on \mathcal{A}^+ if and only if $\sup_{N \geq 1} \|N^{-\phi}\|_{\mathcal{A}^+} < +\infty$, and that it is compact if and only if $\|N^{-\phi}\|_{\mathcal{A}^+} \xrightarrow{N \rightarrow \infty} 0$. Note that it is actually proved in [6, Theorem 4] that if $N^{-\phi}$ is a Dirichlet series

for all $N \geq 1$, then ϕ as necessarily the form (1.1). Then $\|N^{-\phi}\|_{\mathcal{A}^+} = \|N^{-\varphi}\|_{\mathcal{A}^+}$, so c_0 plays no role, so we assume in the sequel that $c_0 = 0$.

When X is a Banach space of analytic functions that contains the identity map $u: z \mapsto z$, and $C_\phi: X \rightarrow X$ is a composition operator, then $\phi = C_\phi(u)$ belongs to X . For $X = \mathcal{A}^+$, it is not the case, so it is natural to ask if $\varphi \in \mathcal{A}^+$ when $C_\varphi: \mathcal{A}^+ \rightarrow \mathcal{A}^+$ is a bounded composition operator. The object of this short note is to give a negative answer (Theorem 2.1).

Let us point out that it is proved in [3, Proposition 2.9] that $\varphi \in \mathcal{A}^+$ does not suffice to have a bounded composition operator on \mathcal{A}^+ ; the symbol is even a Dirichlet polynomial

$$\varphi(s) = c_1 + c_r r^{-s} + c_{r^2} r^{-2s}$$

where $r \geq 2$ is an integer and $c_r, c_{r^2} > 0$. For such a Dirichlet polynomial, it is proved that C_φ is not bounded if $\Re c_1 < \frac{(c_r)^2}{8c_{r^2}}$ and $c_r \leq 4c_{r^2}$ (for example, $c_r = 4, c_{r^2} = 1$ and $\Re c_1 < 2$).

2 Main result

Recall that $\varphi: \mathbb{C}_0 \rightarrow \mathbb{C}$ is a convergent Dirichlet series, if φ is analytic on \mathbb{C}_0 and we can write $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ for $\Re s$ large enough.

Theorem 2.1. *There exists a convergent Dirichlet series φ inducing a bounded composition operator $C_\varphi: \mathcal{A}^+ \rightarrow \mathcal{A}^+$, but such that $\varphi \notin \mathcal{A}^+$. Moreover, $\varphi \in \mathcal{H}^p$ for all $p < \infty$ and C_φ is compact and absolutely summing.*

Let us recall the definition of the Hardy space \mathcal{H}^p of Dirichlet series, following [2]. That uses the Bohr representation of Dirichlet series. Let $(p_j)_{j \geq 1}$ be the increasing sequence of all the prime numbers (so $p_1 = 2, p_2 = 3, p_3 = 5$, and so on). If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the decomposition of the integer n in prime factors, to the Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is associated the Taylor series $(\Delta\varphi)(z) = \sum_{\alpha} a_n z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_r^{\alpha_r}$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r, 0, 0, \dots)$. Due to Kronecker's theorem, φ is bounded if and only if $\Delta\varphi$ is bounded, and $\|\varphi\|_{\infty} = \|\Delta\varphi\|_{\infty}$. The Hardy space \mathcal{H}^p is the space of all convergent Dirichlet series φ for which $\Delta\varphi$ belongs to the Hardy space $H^p(\mathbb{T}^{\infty})$, with the norm $\|\varphi\|_{\mathcal{H}^p} = \|\Delta\varphi\|_{H^p}$.

Note that \mathcal{A}^+ is isometrically isomorphic, by this map Δ , to the Wiener algebra $A^+(\mathbb{T}^{\infty})$.

Let us also recall that a bounded linear map $u: X \rightarrow Y$ between two Banach spaces X and Y is r -summing ($1 \leq r < \infty$) if there is a positive constant K such that

$$\left(\sum_{k=1}^n \|u(x_k)\|^r \right)^{1/r} \leq K \sup_{\xi \in B_{X^*}} \left(\sum_{k=1}^n |\xi(x_k)|^r \right)^{1/r}$$

for all $x_1, \dots, x_n \in X, n \geq 1$, and where B_{X^*} is the unit ball of X^* . For $r = 1$, these operators are also said absolutely summing.

Proof of the theorem. We are going to take a symbol φ of the form $\varphi(s) = \sum_{k=1}^{\infty} c_k 2^{-ks} = f(2^{-s})$, where $f: \mathbb{D} \rightarrow \mathbb{C}_0$ is an analytic function such that $\sup_{N \geq 1} \|N^{-f}\|_{A^+(\mathbb{D})} < +\infty$, but $f \notin H^\infty$.

Recall that $A^+ = A^+(\mathbb{D})$ is the space of all analytic functions $u: \mathbb{D} \rightarrow \mathbb{C}$ such that $u(z) = \sum_{n=0}^{\infty} a_n z^n$, with $\|u\|_{A^+(\mathbb{D})} := \sum_{n=0}^{\infty} |a_n| < +\infty$.

We choose for f a conformal map sending the unit disk \mathbb{D} onto the half-strip

$$R = \{z \in \mathbb{C}; \Re z > 1 \text{ and } |\Im z| < \pi\}.$$

Explicitly, we take $f = \tau_1 \circ L \circ h \circ c \circ T$, where

$$\begin{aligned} T(z) &= \frac{1+z}{1-z}; & c(z) &= e^{i\pi/4} \sqrt{z}; \\ h(z) &= \frac{iz+1}{z+i}; & L(z) &= -2 \log z; \end{aligned}$$

and $\tau_1(z) = z + 1$. T maps the unit disk \mathbb{D} onto the right-half plane; then c sends the right-half plane onto the first quadrant; h the first quadrant onto the right-half of \mathbb{D} ; L this right-half of \mathbb{D} onto the half-strip $\{|\Im z| < \pi, \Re z > 0\}$, and finally the translation τ_1 sends this half-strip onto the half-strip R .

This map is clearly not in H^∞ , but, for every $\beta \in (0, \pi/2)$, there is a positive constant C_β such that $R + C_\beta$ is contained in the angular sector of vertex 0 and of opening β ; it follows (see [4, Theorem 3.2]) that $f + C_\beta \in H^p$ for all $p < \pi/\beta$; so $f \in H^p$ for all $p < \infty$. We can also see that

$$f(e^{it}) = \alpha \log |i - e^{it}| + g(t),$$

with $g \in L^\infty$ and α a constant, so that $f \in H^{\Psi_1}$, the Hardy-Orlicz space attached to the Orlicz function $\Psi_1(x) = e^x - 1$, and

$$\|f\|_p = O(p)$$

as p goes to infinity.

Since $f \notin H^\infty$, we a fortiori have $f \notin A^+$, so $\varphi \notin \mathcal{A}^+$. However $\varphi \in \mathcal{H}^p$ for all $p < \infty$ since $f \in H^p$ for these values of p .

We now have to show that $N^{-\varphi} \in \mathcal{A}^+$, i.e. $N^{-f} \in A^+$, for all $N \geq 1$. This is clear for $N = 1$. For $N \geq 2$, we have $N^{-f} = \exp(-f \log N)$, and the range of $f \log N$ is the half-strip

$$R_N = \{z \in \mathbb{C}; \Re z > \log N \text{ and } |\Im z| < \pi \log N\}.$$

Under the exponential map e^{-z} , ∂R_N is transformed as follows:

- 1) the vertical segment $[\log N - i\pi \log N, \log N + i\pi \log N]$ is sent onto the circle of center 0 and radius $1/N$, browsed $\log N$ times;
- 2) the half-line $\{t \log N + i\pi \log N; t \geq 1\}$ is one-to-one mapped onto the radius $(0, e^{-i\pi \log N}/N]$;
- 3) the half-line $\{t \log N - i\pi \log N; t \geq 1\}$ is one-to-one mapped onto the radius $(0, e^{i\pi \log N}/N]$.

Hence, if $F_N = N^{-f}$, we have

$$\int_0^{2\pi} |F'_N(e^{it})| dt = \frac{2}{N} + 2\pi \frac{\log N}{N} < +\infty,$$

so $(N^{-f})' \in H^1$. By Hardy's inequality (see [4, Corollary page 48]), it follows that $N^{-f} \in A^+$, and there exists a positive constant C such that $\|N^{-f}\|_{A^+} \leq C \log N/N$. In particular, $\|N^{-\varphi}\|_{A^+} = \|N^{-f}\|_{A^+} \xrightarrow{N \rightarrow \infty} 0$, so $C_\varphi: \mathcal{A}^+ \rightarrow \mathcal{A}^+$ is compact, by [3, Theorem 2.3].

To end the proof, remark that, writing $u_N(s) = N^{-s}$, we have

$$\sum_{N=2}^{\infty} \|C_\varphi(u_N)\|_{A^+}^2 = \sum_{N=2}^{\infty} \|N^{-\varphi}\|_{A^+}^2 \leq C^2 \sum_{N=2}^{\infty} \frac{(\log N)^2}{N^2} < +\infty.$$

Hence, using the Cauchy-Schwarz inequality, we can define a bounded linear operator $S: \ell_2 \rightarrow \mathcal{A}^+$ by sending the N -th vector e_N of the canonical basis of ℓ_2 to $N^{-\varphi}$, and we have the factorization $C_\varphi = ST$, where $T: \mathcal{A}^+ \rightarrow \ell_2$ is defined by setting $T(u_N) = e_N$. But \mathcal{A}^+ is isometrically isomorphic to ℓ_1 , and the canonical injection from ℓ_1 to ℓ_2 is 1-summing (this was first remarked by Pietsch [10, § 1, Satz 10], and it is a particular case of the Grothendieck theorem). Let us recall why that holds. To each $(\alpha_k)_{k \geq 1} \in \ell_1$, we associate the L^∞ function $\sum_{k=1}^{\infty} \alpha_k r_k$, where $(r_k)_{k \geq 1}$ is the sequence of the Rademacher functions on $[0, 1]$; the canonical injection from $L^\infty(0, 1)$ into $L^1(0, 1)$ is absolutely summing (see [11, top of page 11], or [1, Remark 8.2.9]) and, by Khintchin's inequalities (see [8, Chapitre 0, Théorème IV.1], or [9, Chapter 1, Theorem IV.1]), the L^1 -norm of $\sum_{k=1}^{\infty} \alpha_k r_k$ is equivalent to $(\sum_{k=1}^{\infty} |\alpha_k|^2)^{1/2}$.

It follows that C_φ is 1-summing. \square

Note that, since $\mathcal{A}^+ \cong \ell_1$ has the Schur property, and since every q -summing operator is weakly compact, every q -summing operator into \mathcal{A}^+ is compact.

A slight modification of the proof gives a variant of Theorem 2.1.

Theorem 2.2. *For every $p \in (1, \infty)$, there exists a convergent Dirichlet series φ such that $\varphi \in \mathcal{H}^q$ for all $q < p$, but $\varphi \notin \mathcal{H}^p$, and such that φ induces a bounded composition operator $C_\varphi: \mathcal{A}^+ \rightarrow \mathcal{A}^+$. Moreover, C_φ is compact and is absolutely summing.*

Proof. We replace the conformal map f of Theorem 2.1 by a conformal map f from \mathbb{D} onto the intersection of the angular sector $\{z \in \mathbb{C}_0; |\arg z| < \pi/2p\}$ with the half-plane \mathbb{C}_1 . We have $f \notin H^p$ though $f \in H^q$ for all $q < p$ (see [7, top of page 237]). We set $\varphi(s) = f(2^{-s})$ for $\Re s > 0$. We have $\varphi \in \mathcal{H}^q$ for all $q < p$, but $\varphi \notin \mathcal{H}^p$.

For all $N \geq 1$, we have $N^{-f} \in A^+$. This is clear for $N = 1$. For $N \geq 2$, let $\beta = \pi/2p$ and $\gamma_\pm(t) = \exp(-e^{\pm i\beta}t)$, with $t \geq \log N/\cos \beta$, then the boundary

of the range of $F_N = N^{-f}$ is the union of γ_+ and γ_- , and of the circle of radius $1/N$ browsed $(1/\pi)(\tan \beta) \log N$ times. Since

$$\int_{\log N / \cos \beta}^{+\infty} |\gamma'_{\pm}(t)| dt = \int_{\log N / \cos \beta}^{+\infty} e^{-(\cos \beta)t} dt = \frac{1}{\cos \beta} \frac{1}{N},$$

we get that

$$\int_0^{2\pi} |F'_N(e^{it})| dt = \frac{2}{\cos \beta} \frac{1}{N} + \frac{\tan \beta}{\pi} \frac{\log N}{N} < +\infty,$$

so $F'_N \in H^1$ and $F_N = N^{-f} \in A^+$.

Moreover, $\|N^{-\varphi}\|_{\mathcal{A}^+} = \|N^{-f}\|_{\mathcal{A}^+} \lesssim \log N/N \xrightarrow{N \rightarrow \infty} 0$, so C_φ is compact on \mathcal{A}^+ .

Since $\sum_{N=1}^{\infty} \|N^{-\varphi}\|_{\mathcal{A}^+}^2 < +\infty$, we get, as in the proof of Theorem 2.1, that C_φ is 1-summing. \square

3 Another result

Let us remark that the example of [3, Proposition 2.9] quoted in the Introduction is a Dirichlet polynomial φ such that $N^{-\varphi} \in \mathcal{A}^+$ for all $N \geq 1$, though the associated composition operator C_φ is not bounded from \mathcal{A}^+ into itself.

Theorem 3.1. *For any non-negative number $A \in \mathbb{R}_+$, there exists a convergent Dirichlet series φ such that $\varphi(\mathbb{C}_0) \subseteq \mathbb{C}_A$, but such that, for any $N \geq 2$, we have $N^{-\varphi} \notin \mathcal{A}^+$.*

In particular, the composition operator C_φ is not bounded from \mathcal{A}^+ into itself.

That will follow from the following result.

Lemma 3.2. *Let $N \geq 2$ and let $\varphi: \mathbb{C}_0 \rightarrow \mathbb{C}$ be an analytic function such that $N^{-\varphi} \in \mathcal{A}^+$. Then, for every $a \in \mathbb{R}$, either $\varphi(s)$ has a limit as s tends to ia , or $\Re \varphi(s)$ tends to $+\infty$ as s tends to ia .*

Proof. Since $N^{-\varphi}$ belongs to \mathcal{A}^+ , it is continuous on $\overline{\mathbb{C}_0}$; hence it has limits at every point $ia \in i\mathbb{R}$. If this limit is 0, that means that $\Re \varphi(s) \xrightarrow{s \rightarrow ia} +\infty$. If not, we have $\lim_{s \rightarrow ia} N^{-\varphi(s)} = c \neq 0$. Therefore, if $r < |c|$, there is some open disk V centered at ia such that $N^{-\varphi(s)} \in D(c, r)$ when $s \in V$. Let F be a determination of the logarithm in $D(c, r)$. Then

$$\psi(s) := F[N^{-\varphi(s)}] \xrightarrow{s \rightarrow ia} F(c).$$

Since

$$\exp[-\varphi(s) \log N] = N^{-\varphi(s)} = \exp[\psi(s)],$$

there exists $k = k(s) \in \mathbb{Z}$ such that $\psi(s) = -\varphi(s) \log N + 2k(s)\pi i$. But φ and ψ are continuous on $V \cap \mathbb{C}_0$; it follows that $k(s)$ is constant. Therefore

$$\varphi(s) = -\psi(s) + 2k\pi i \xrightarrow{s \rightarrow ia} -F(c) + 2k\pi i. \quad \square$$

Proof of Theorem 3.1. Let

$$\varphi(s) = A + 1 + \exp\left(-\frac{1 + 2^{-s}}{1 - 2^{-s}}\right).$$

Then φ is a convergent Dirichlet series and maps \mathbb{C}_0 into \mathbb{C}_A . However, $N^{-\varphi} \notin \mathcal{A}^+$ because φ does not have a limit as s goes to 0. \square

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