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On composition operators on the Wiener algebra of Dirichlet series

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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Abstract. We show that the symbol of a bounded composition operator on the Wiener algebra of Dirichlet series does not need to belong to this algebra. Our example even gives an absolutely summing (hence compact) composition operator.

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Key-words Composition operator; Dirichlet series

1 Introduction

In [3] (see also [5]), composition operators on the Wiener algebra \( A^+ \) of all absolutely convergent Dirichlet series were studied.

Recall that \( A^+ \) is the space of all analytic maps \( f: \mathbb{C}_0 \to \mathbb{C} \) which can be written

\[
f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{with} \quad \|f\|_{A^+} := \sum_{n=1}^{\infty} |a_n| < +\infty,
\]

where, for \( \theta \in \mathbb{R} \), we note \( \mathbb{C}_{\theta} = \{ z \in \mathbb{C}; \, \Re z > \theta \} \). If \( \phi: \mathbb{C}_0 \to \mathbb{C}_0 \) is an analytic function, the composition operator \( C_{\phi}: A^+ \to A^+ \) of symbol \( \phi \) on this space is defined as \( C_{\phi}(f) = f \circ \phi \). Gordon and Hedenmalm, for the Hilbert space \( \mathcal{H}^2 \), showed in [6] that such a symbol has necessarily the form

\[
\phi(s) = c_0 s + \varphi(s), \tag{1.1}
\]

where \( c_0 \geq 0 \) is an integer and \( \varphi \) is a convergent Dirichlet series with values in \( \mathbb{C}_0 \), that is \( \varphi: \mathbb{C}_0 \to \mathbb{C}_0 \) is an analytic function which can be written \( \varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s} \) for \( \Re s \) large enough. Moreover, this Dirichlet series is uniformly convergent in \( \mathbb{C}_\varepsilon \) for all \( \varepsilon > 0 \) ([13, pages 1625–1626 and Theorem 3.1]; see also [12, Theorem 8.4.1, page 245]).

It is shown in [3, Theorem 2.3] that \( C_{\phi} \) is bounded on \( A^+ \) if and only if \( \sup_{N\geq 1} \|N^{-\phi}\|_{A^+} < +\infty \), and that it is compact if and only if \( \|N^{-\phi}\|_{A^+} \xrightarrow{N \to \infty} 0 \). Note that it is actually proved in [6, Theorem 4] that if \( N^{-\phi} \) is a Dirichlet series
for all $N \geq 1$, then $\phi$ as necessarily the form (1.1). Then $\|N^{-\phi}\|_{A^+} = \|N^{-\varphi}\|_{A^+}$, so $c_0$ plays no role, so we assume in the sequel that $c_0 = 0$.

When $X$ is a Banach space of analytic functions that contains the identity map $u: z \mapsto z$, and $C_\phi: X \to X$ is a composition operator, then $\phi = C_\phi(u)$ belongs to $X$. For $X = A^+$, it is not the case, so it is natural to ask if $\varphi \in A^+$ when $C_\varphi: A^+ \to A^+$ is a bounded composition operator. The object of this short note is to give a negative answer (Theorem 2.1).

Let us point out that it is proved in [3, Proposition 2.9] that $\varphi \in A^+$ does not suffice to have a bounded composition operator on $A^+$; the symbol is even a Dirichlet polynomial

$$\varphi(s) = c_1 + c_r r^{-s} + c_{r^2} r^{-2s}$$

where $r \geq 2$ is an integer and $c_r, c_{r^2} > 0$. For such a Dirichlet polynomial, it is proved that $C_\varphi$ is not bounded if $\Re c_1 < \frac{(c_r)^2}{8c_{r^2}}$ and $c_r \leq 4 c_{r^2}$ (for example, $c_r = 4, c_{r^2} = 1$ and $\Re c_1 < 2$).

## 2 Main result

Recall that $\varphi: \mathbb{C}_0 \to \mathbb{C}$ is a convergent Dirichlet series, if $\varphi$ is analytic on $\mathbb{C}_0$ and we can write $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ for $\Re s$ large enough.

**Theorem 2.1.** There exists a convergent Dirichlet series $\varphi$ inducing a bounded composition operator $C_\varphi: A^+ \to A^+$, but such that $\varphi \notin A^+$. Moreover, $\varphi \in \mathcal{H}^p$ for all $p < \infty$ and $C_\varphi$ is compact and absolutely summing.

Let us recall the definition of the Hardy space $\mathcal{H}^p$ of Dirichlet series, following [2]. That uses the Bohr representation of Dirichlet series. Let $(p_j)_{j \geq 1}$ be the increasing sequence of all the prime numbers (so $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on). If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is the decomposition of the integer $n$ in prime factors, to the Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is associated the Taylor series $(\Delta \varphi)(z) = \sum_{\alpha} a_\alpha z_1^{a_1} z_2^{a_2} \cdots z_r^{a_r}$, where $\alpha = (a_1, a_2, \ldots, a_r, 0, 0, \ldots)$. Due to Kronecker’s theorem, $\varphi$ is bounded if and only if $\Delta \varphi$ is bounded, and $\|\varphi\|_\infty = \|\Delta \varphi\|_\infty$. The Hardy space $\mathcal{H}^p$ is the space of all convergent Dirichlet series $\varphi$ for which $\Delta \varphi$ belongs to the Hardy space $H^p(\mathbb{T}^\infty)$, with the norm $\|\varphi\|_{\mathcal{H}^p} = \|\Delta \varphi\|_{H^p}$.

Note that $A^+$ is isometrically isomorphic, by this map $\Delta$, to the Wiener algebra $A^+(\mathbb{T}^\infty)$.

Let us also recall that a bounded linear map $u: X \to Y$ between two Banach spaces $X$ and $Y$ is $r$-summing ($1 \leq r < \infty$) if there is a positive constant $K$ such that

$$\left( \sum_{k=1}^{n} \|u(x_k)\|^r \right)^{1/r} \leq K \sup_{\xi \in B_X^*} \left( \sum_{k=1}^{n} |\xi(x_k)|^r \right)^{1/r}$$

for all $x_1, \ldots, x_n \in X$, $n \geq 1$, and where $B_X^*$ is the unit ball of $X^*$. For $r = 1$, these operators are also said absolutely summing.
Proof of the theorem. We are going to take a symbol ϕ of the form ϕ(s) = \sum_{k=1}^{\infty} c_k 2^{-ks} = f(2^{-s})$, where \( f: \mathbb{D} \to \mathbb{C} \) is an analytic function such that 
\[ \sup_{N \geq 1} \| N^{-f} \|_{A^+} < +\infty, \text{ but } f \notin H^\infty. \]

Recall that \( A^+ = A^+ (\mathbb{D}) \) is the space of all analytic functions \( u: \mathbb{D} \to \mathbb{C} \) such that 
\[ u(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ with } \| u \|_{A^+} := \sum_{n=0}^{\infty} |a_n| < +\infty. \]

We choose for \( f \) a conformal map sending the unit disk \( \mathbb{D} \) onto the half-strip 
\[ R = \{ z \in \mathbb{C}; \text{ Re } z > 1 \text{ and } |\text{ Im } z| < \pi \}. \]

Explicitly, we take 
\[ f = \tau_1 \circ L \circ h \circ c \circ T, \]
where
\[ T(z) = \frac{1 + z}{1 - z}; \quad c(z) = e^{i\pi/4} \sqrt{z}; \]
\[ h(z) = \frac{iz + 1}{z + i}; \quad L(z) = -2 \log z; \]

and \( \tau_1(z) = z + 1 \). \( T \) maps the unit disk \( \mathbb{D} \) onto the right-half plane; \( c \) sends the right-half plane onto the first quadrant; \( h \) the first quadrant onto the right-half of \( \mathbb{D} \); \( L \) this right-half of \( \mathbb{D} \) onto the half-strip \( \{ |\text{ Im } z| < \pi, \text{ Re } z > 0 \} \), and finally the translation \( \tau_1 \) sends this half-strip onto the half-strip \( R \).

This map is clearly not in \( H^\infty \), but, for every \( \beta \in (0, \pi/2) \), there is a positive constant \( C_\beta \) such that \( R + C_\beta \) is contained in the angular sector of vertex 0 and of opening \( \beta \); it follows (see [4, Theorem 3.2]) that \( f + C_\beta \in H^p \) for all \( p < \pi/\beta \); so \( f \in H^p \) for all \( p < \infty \). We can also see that
\[ f(e^{i\tau}) = \alpha \log |i - e^{i\tau}| + g(t), \]
with \( g \in L^\infty \) and \( \alpha \) a constant, so that \( f \in H^{\Psi_1} \), the Hardy-Orlicz space attached to the Orlicz function \( \Psi_1(x) = e^x - 1 \), and
\[ \| f \|_p = O(p) \]
as \( p \) goes to infinity.

Since \( f \notin H^\infty \), we a fortiori have \( f \notin A^+ \), so \( \varphi \notin A^+ \). However \( \varphi \in H^p \) for all \( p < \infty \) since \( f \in H^p \) for these values of \( p \).

We now have to show that \( N^{-f} \in A^+ \), i.e. \( N^{-f} \in A^+ \), for all \( N \geq 1 \). This is clear for \( N = 1 \). For \( N \geq 2 \), we have \( N^{-f} = \exp(-f \log N) \), and the range of \( f \log N \) is the half-strip
\[ R_N = \{ z \in \mathbb{C}; \text{ Re } z > \log N \text{ and } |\text{ Im } z| < \pi \log N \}. \]

Under the exponential map \( e^{-z} \), \( \partial R_N \) is transformed as follows:

1) the vertical segment \( [\log N - i\pi \log N, \log N + i\pi \log N] \) is sent onto the circle of center 0 and radius \( 1/N \), browsed \( \log N \) times;

2) the half-line \( \{ t \log N + i\pi \log N; t \geq 1 \} \) is one-to-one mapped onto the radius \( (0, e^{-i\pi \log N}/N) \);

3) the half-line \( \{ t \log N - i\pi \log N; t \geq 1 \} \) is one-to-one mapped onto the radius \( (0, e^{i\pi \log N}/N) \).
Hence, if \( F_N = N^{-f} \), we have

\[
\int_0^{2\pi} |F_N'(e^{it})| \, dt = \frac{2}{N} + 2\pi \frac{\log N}{N} < +\infty,
\]

so \((N^{-f})' \in H^1\). By Hardy’s inequality (see [4, Corollary page 48]), it follows that \(N^{-f} \in A^+\), and there exists a positive constant \(C\) such that \(\|N^{-f}\|_{A^+} \leq C \log N/N\). In particular, \(\|N^{-f}\|_{A^+} = \|N^{-f}\|_{A^+} \rightarrow 0\), so \(C\varphi : A^+ \rightarrow A^+\) is compact, by [3, Theorem 2.3].

To end the proof, remark that, writing \(u_N(s) = N^{-s}\), we have

\[
\sum_{N=2}^{\infty} \|C\varphi(u_N)\|_{A^+}^2 = \sum_{N=2}^{\infty} \|N^{-s}\|_{A^+}^2 \leq C^2 \sum_{N=2}^{\infty} \frac{(\log N)^2}{N^2} < +\infty.
\]

Hence, using the Cauchy-Schwarz inequality, we can define a bounded linear operator \(S : \ell_2 \rightarrow A^+\) by sending the \(N\)-th vector \(e_N\) of the canonical basis of \(\ell_2\) to \(N^{-s}\), and we have the factorization \(C\varphi = ST\), where \(T : A^+ \rightarrow \ell_2\) is defined by setting \(T(u_N) = e_N\). But \(A^+\) is isometrically isomorphic to \(\ell_2\), and the canonical injection from \(\ell_1\) to \(\ell_2\) is 1-summing (this was first remarked by Pietsch [10, § 1, Satz 10], and it is a particular case of the Grothendieck theorem). Let us recall why that holds. To each \((\alpha_k)_{k \geq 1} \in \ell_1\), we associate the \(L^\infty\) function \(\sum_{k=1}^{\infty} \alpha_k r_k\), where \((r_k)_{k \geq 1}\) is the sequence of the Rademacher functions on \([0,1]\); the canonical injection from \(L^\infty(0,1)\) into \(L^1(0,1)\) is absolutely summing (see [11, top of page 11], or [1, Remark 8.2.9]) and, by Khintchin’s inequalities (see [8, Chapitre 0, Théorème IV.1], or [9, Chapter 1, Theorem IV.1]), the \(L^1\)-norm of \(\sum_{k=1}^{\infty} \alpha_k r_k\) is equivalent to \((\sum_{k=1}^{\infty} |\alpha_k|^2)^{1/2}\).

It follows that \(C\varphi\) is 1-summing. \(\square\)

Note that, since \(A^+ \cong \ell_1\) has the Schur property, and since every \(q\)-summing operator is weakly compact, every \(q\)-summing operator into \(A^+\) is compact.

A slight modification of the proof gives a variant of Theorem 2.1.

**Theorem 2.2.** For every \(p \in (1,\infty)\), there exists a convergent Dirichlet series \(\varphi\) such that \(\varphi \in \mathcal{H}^q\) for all \(q < p\), but \(\varphi \notin \mathcal{H}^p\), and such that \(\varphi\) induces a bounded composition operator \(C\varphi : A^+ \rightarrow A^+\). Moreover, \(C\varphi\) is compact and is absolutely summing.

**Proof.** We replace the conformal map \(f\) of Theorem 2.1 by a conformal map \(f\) from \(\mathbb{D}\) onto the intersection of the angular sector \(\{z \in \mathbb{C}_0 : |\arg z| < \pi/2p\}\) with the half-plane \(\mathbb{C}_1\). We have \(f \notin H^p\) though \(f \in H^q\) for all \(q < p\) (see [7, top of page 237]). We set \(\varphi(s) = f(2^{-s})\) for \(\Re s > 0\). We have \(\varphi \in \mathcal{H}^q\) for all \(q < p\), but \(\varphi \notin \mathcal{H}^p\).

For all \(N \geq 1\), we have \(N^{-f} \in A^+\). This is clear for \(N = 1\). For \(N \geq 2\), let \(\beta = \pi/2p\) and \(\gamma_{\pm}(t) = \exp(-e^{\pm i\beta}t)\), with \(t \geq \log N/\cos \beta\), then the boundary
of the range of $F_N = N^{-f}$ is the union of $\gamma_+$ and $\gamma_-$, and of the circle of radius $1/N$ browsed $(1/\pi) (\tan \beta) \log N$ times. Since

$$\int_{\log N/\cos \beta}^{\log N/\cos \beta + \infty} |\gamma'_+(t)| \, dt = \int_{\log N/\cos \beta}^{\log N/\cos \beta + \infty} e^{-(\cos \beta) t} \, dt = \frac{1}{\cos \beta} \frac{1}{N},$$

we get that

$$\int_0^{2\pi} |F_N'(e^{it})| \, dt = \frac{2}{\cos \beta} \frac{1}{N} + \frac{\tan \beta \log N}{\pi} < +\infty,$$

so $F_N' \in H^1$ and $F_N = N^{-f} \in A^+$.

Moreover, $\|N^{-\varphi}\|_{A^+} = \|N^{-f}\|_{A^+} \leq \log N/N \to 0$, so $C_{\varphi}$ is compact on $A^+$.

Since $\sum_{N=1}^{\infty} \|N^{-\varphi}\|^2_{A^+} < +\infty$, we get, as in the proof of Theorem 2.1, that $C_{\varphi}$ is 1-summing. \qed

3 Another result

Let us remark that the example of [3, Proposition 2.9] quoted in the Introduction is a Dirichlet polynomial $\varphi$ such that $N^{-\varphi} \in A^+$ for all $N \geq 1$, though the associated composition operator $C_{\varphi}$ is not bounded from $A^+$ into itself.

**Theorem 3.1.** For any non-negative number $\varphi \in \mathbb{R}_+$, there exists a convergent Dirichlet series $\varphi$ such that $\varphi(\mathbb{C}_0) \subseteq \mathcal{C}_A$, but such that, for any $N \geq 2$, we have $N^{-\varphi} \notin A^+$.

In particular, the composition operator $C_{\varphi}$ is not bounded from $A^+$ into itself.

That will follow from the following result.

**Lemma 3.2.** Let $N \geq 2$ and let $\varphi: \mathbb{C}_0 \to \mathbb{C}$ be an analytic function such that $N^{-\varphi} \in A^+$. Then, for every $a \in \mathbb{R}$, either $\varphi(s)$ has a limit as $s$ tends to $ia$, or $\Re \varphi(s)$ tends to $+\infty$ as $s$ tends to $ia$.

*Proof.* Since $N^{-\varphi}$ belongs to $A^+$, it is continuous on $\overline{\mathbb{C}}_0$; hence it has limits at every point $ia \in i\mathbb{R}$. If this limit is 0, that means that $\Re \varphi(s) \to +\infty$. If not, we have $\lim_{s \to ia} N^{-\varphi(s)} = e \neq 0$. Therefore, if $r < |c|$, there is some open disk $V$ centered at $ia$ such that $N^{-\varphi(s)} \in D(c,r)$ when $s \in V$. Let $F$ be a determination of the logarithm in $D(c,r)$. Then

$$\psi(s) := F[N^{-\varphi(s)}] \to F(c).$$

Since

$$\exp[-\varphi(s) \log N] = N^{-\varphi(s)} = \exp[\psi(s)],$$

there exists $k = k(s) \in \mathbb{Z}$ such that $\psi(s) = -\varphi(s) \log N + 2k(s)\pi i$. But $\varphi$ and $\psi$ are continuous on $V \cap \mathbb{C}_0$; it follows that $k(s)$ is constant. Therefore

$$\varphi(s) = -\psi(s) + 2k\pi i \to -F(c) + 2k\pi i. \quad \square$$
Proof of Theorem 3.1. Let

\[ \varphi(s) = A + 1 + \exp \left( \frac{1 + 2^{-s}}{1 - 2^{-s}} \right). \]

Then \( \varphi \) is a convergent Dirichlet series and maps \( C_0 \) into \( C_A \). However, \( N^{-\varphi} \notin \mathcal{A}^+ \) because \( \varphi \) does not have a limit as \( s \) goes to 0. \( \square \)

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References


Daniel Li
Univ. Artois, UR 2462, Laboratoire de Mathématiques de Lens (LML), F-62 300 LENS, FRANCE
daniel.li@univ-artois.fr

Hervé Queffélec
Univ. Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524, F-59 655 VILLENEUVE D’ASCQ Cedex, FRANCE
Herve.Queffelec@univ-lille.fr

Luis Rodríguez-Piazza
Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Calle Tarfia s/n 41 012 SEVILLA, SPAIN
piazza@us.es