HAL open science

Variations on Poincaré duality for intersection homology

Martintxo Saralegi-Aranguren, Daniel Tanré

▶ To cite this version:

Martintxo Saralegi-Aranguren, Daniel Tanré. Variations on Poincaré duality for intersection homology. L'Enseignement Mathématique , 2020, 65 (1-2), pp.117-154. 10.4171/LEM/65-1/2-4 . hal-03660402

HAL Id: hal-03660402

https://univ-artois.hal.science/hal-03660402v1

Submitted on 9 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:1812.03072v2 [math.AT] 24 Sep 2019

VARIATIONS ON POINCARÉ DUALITY FOR INTERSECTION HOMOLOGY

MARTINTXO SARALEGI-ARANGUREN AND DANIEL TANRÉ

Abstract. Intersection homology with coefficients in a field restores Poincaré duality for some spaces with singularities, as stratified pseudomanifolds. But, with coefficients in a ring, the behaviours of manifolds and stratified pseudomanifolds are different. This work is an overview, with proofs and explicit examples, of various possible situations with their properties.

We first set up a duality, defined from a cap product, between two intersection cohomologies: the first one arises from a linear dual and the second one from a simplicial blow up. Moreover, from this property, Poincaré duality in intersection homology looks like the Poincaré-Lefschetz duality of a manifold with boundary. Besides that, an investigation of the coincidence of the two previous cohomologies reveals that the only obstruction to the existence of a Poincaré duality is the homology of a well defined complex. This recovers the case of the peripheral sheaf introduced by Goresky and Siegel for compact PL-pseudomanifolds. We also list a series of explicit computations of peripheral intersection cohomology. In particular, we observe that Poincaré duality can exist in the presence of torsion in the "critical degree" of the intersection homology of the links of a stratified pseudomanifold.

Contents

Introduction 2
1. Background 6
2. Verdier dual of intersection cochain complexes 13
3. Poincaré duality with pairings 16
4. A relative complex 20
5. Components of the peripheral complex 22
6. Examples 25
References 29

Date: September 25, 2019.
2010 *Mathematics Subject Classification.* 55N33, 57P10, 57N80, 55U30.
Key words and phrases. Intersection homology; Peripheral complex; Poincaré duality.
The second author was supported by the MINECO and FEDER research project MTM2016-78647-P. and the ANR-11-LABX-0007-01 "CEMPI".

Introduction

In this introduction, for sake of simplicity, we restrict the coefficients to $\mathbb{Z}$ and $\mathbb{Q}$. We consider also Goresky and MacPherson perversities, depending only on the codimensions of strata. A more general situation is handled in the text and specified in the various statements. Recollections of definitions and main properties can be found in Section 1.

Let M be a compact, n-dimensional, oriented manifold. The famous Poincaré duality gives a non-singular pairing

$$H_k(M;\mathbb{Q}) \otimes H_{n-k}(M;\mathbb{Q}) \to \mathbb{Q},$$

defined by the intersection product. This feature has been extended to the existence of singularities by M. Goresky and R. MacPherson. In [16], they introduce the *intersection homology* associated to a perversity $\overline{p}$ and prove the existence of a non-singular pairing in intersection homology,

$$H_k^{\overline{p}}(X;\mathbb{Q}) \otimes H_{n-k}^{D\overline{p}}(X;\mathbb{Q}) \to \mathbb{Q},$$

when $\overline{p}$ and $D\overline{p}$ are complementary perversities and X is a compact, oriented, n-dimensional PL-pseudomanifold. If we replace the field of rational numbers by the ring of integers, the situation becomes more complicated. In the case of a compact oriented manifold, we still have non-singular pairings,

$$\mathrm{F}\, H_k(M;\mathbb{Z}) \otimes \mathrm{F}\, H_{n-k}(M;\mathbb{Z}) \to \mathbb{Z}, \tag{0.1}$$

between the torsion free parts of homology groups, and

$$\mathrm{T} H_k(M;\mathbb{Z}) \otimes \mathrm{T} H_{n-k-1}(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \tag{0.2}$$

between the torsion parts. In contrast, these two properties can disappear in intersection homology as it has been discovered and studied by M. Goresky and P. Siegel in [18]. In their work, they define a class of compact PL-pseudomanifolds called *locally $\overline{p}$-torsion free* (see Definition 3.6) for which there exist non-singular pairings in intersection homology,

$$\mathrm{F}\, H_k^{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{F}\, H_{n-k}^{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z} \tag{0.3}$$

and

$$\mathrm{T}\, H_k^{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{T}\, H_{n-k-1}^{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}. \tag{0.4}$$

But there are examples of PL-pseudomanifolds for which the previous pairings are singular, as for example the Thom space associated to the tangent space of the 2-sphere for (0.3) and the suspension of the real projective space $\mathbb{R}P^3$ for (0.4), see Examples 6.4 and 6.3.

Let us come back to recollections on Poincaré duality. For an oriented, n-dimensional manifold, M, the cap product with the fundamental class is an isomorphism,

$$- \frown [M] \colon H_c^k(M;\mathbb{Z}) \xrightarrow{\cong} H_{n-k}(M;\mathbb{Z}), \tag{0.5}$$

between cohomology with compact supports and homology. The existence of the non-singular pairings (0.1) and (0.2) are then consequences of (0.5) and the universal coefficient formula. In intersection homology, this method was investigated by G. Friedman and J.E. McClure in [15] and taken over in [11, Section 8.2]. As cohomology groups, the

authors consider the homology of the dual $C^*_{\overline{p}}(X;\mathbb{Z}) = \mathrm{Hom}(C^{\overline{p}}_*(X;\mathbb{Z}),\mathbb{Z})$ of the complex of $\overline{p}$-intersection chains and, under the same restriction as in Goresky and Siegel's paper, they prove the existence of an isomorphism induced by a cap product with a fundamental class,

$$H^k_{\overline{p},c}(X;\mathbb{Z}) \xrightarrow{\cong} H^{D\overline{p}}_{n-k}(X;\mathbb{Z}), \tag{0.6}$$

for a locally $\overline{p}$-torsion free, oriented, paracompact, n-dimensional stratified pseudomanifold X. With this restriction, the pairings (0.3) and (0.4) are then deduced from (0.6) and a formula of universal coefficients, as in the case of a manifold.

In [8], we take over the approach (0.5) using blown-up cochains with compact supports, $\widetilde{N}^*_{\overline{\bullet},c}(-)$, that we have introduced and studied in previous papers [10], [4], [5], [9], [6], [7] (also called Thom-Whitney cochains in some of these works). One of their features is the existence of cup and cap products (see [6] or Section 1) for any ring of coefficients and without any restriction on the stratified pseudomanifold. Indeed we prove in [8, Theorem B] that, for any oriented, paracompact, n-dimensional stratified pseudomanifold, X, and any perversity $\overline{p}$, the cap product with a fundamental class is an isomorphism,

$$- \frown [X] \colon \mathscr{H}^k_{\overline{p},c}(X;\mathbb{Z}) \xrightarrow{\cong} H^{\overline{p}}_{n-k}(X;\mathbb{Z}), \tag{0.7}$$

between the blown-up cohomology with compact supports $\mathscr{H}^k_{\overline{p},c}(-)$ and the intersection homology. The blown-up cohomology is not defined from the dual complex of intersection chains but proceeds from a simplicial blow up process recalled in Section 1. Thus, there is no universal coefficients formula between $\mathscr{H}^*_{\overline{p}}(-)$ and $H^{\overline{p}}_*(-)$ and we cannot deduce from (0.7) a non-singular bilinear form as in the classical case of a manifold. In [8, Theorem C], for a compact oriented stratified pseudomanifold X, we prove the non-degeneracy of the bilinear form

$$\Phi_{\overline{p}} \colon \mathrm{F}\mathscr{H}^k_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{F}\mathscr{H}^{n-k}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z} \tag{0.8}$$

built from the cup product. (There are examples of stratified pseudomanifolds where this bilinear form is singular, see [8, Example 4.10] or Example 6.4.) In contrast, there are examples (see Example 6.3) of the degeneracy of the associated bilinear form

$$L_{\overline{p}} \colon \mathrm{T}\mathscr{H}^k_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{T}\mathscr{H}^{n+1-k}_{D\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}. \tag{0.9}$$

The existence of such examples is not surprising: as the blown-up cohomology is isomorphic through (0.7) to the intersection homology, the defect of duality detected by Goresky and Siegel is also present in (0.8) and (0.9). In sum, we have two intersection cohomologies, $H^*_{\overline{\bullet}}(-)$ and $\mathscr{H}^*_{\overline{\bullet}}(-)$: the first one has a universal coefficient formula and the second one satisfies the isomorphism (0.7) through a cap product with a fundamental class. But, as the quoted examples show, neither satisfies a Poincaré duality with cup products and coefficients in $\mathbb{Z}$, in all generality. (However, the blown-up cohomology satisfies (0.7) over $\mathbb{Z}$ without restriction on the torsion of links.)

This work is also concerned with not necessarily compact stratified pseudomanifolds and, for having a complete record, let us also mention the existence of an isomorphism,

$$- \frown [X] \colon \mathscr{H}^k_{\overline{p}}(X;\mathbb{Z}) \xrightarrow{\cong} H^{\infty,\overline{p}}_{n-k}(X;\mathbb{Z}), \tag{0.10}$$

between the blown-up intersection cohomology and the Borel-Moore intersection homology, (see [24] or [9] in the PL case) for any paracompact, separable and oriented stratified pseudomanifold of dimension n.

After this not so brief "state of the art", we present the results of this work. *The starting point is the existence of a duality between the two intersection cohomologies,* developed in Section 2. To express it, we use the injective resolution, $I^*_{\mathbb{Z}}\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, and the Verdier dual, DA^*, defined as the Hom functor of a cochain complex A^* with value in $I^*_{\mathbb{Z}}$, see (1.14).

Theorem A. [Theorem 2.2] *Let X be a paracompact, separable and oriented stratified pseudomanifold of dimension n and $\overline{p}$ be a perversity. Then, there exist two quasi-isomorphisms, defined from the cap product with a cycle representing the fundamental class $[X] \in H_n^{\infty,\overline{0}}(X;\mathbb{Z})$,*

$$\mathscr{C}_{\overline{p}}\colon C^\star_{\overline{p}}(X;\mathbb{Z}) \to (D\widetilde{N}^*_{\overline{p},c}(X;\mathbb{Z}))_{n-\star} \text{ and } \mathscr{N}_{\overline{p}}\colon \widetilde{N}^\star_{\overline{p}}(X;\mathbb{Z}) \to (DC^*_{\overline{p},c}(X;\mathbb{Z})_{n-\star}.$$

As a consequence, in the compact case, we deduce two non-singular pairings between the two intersection cohomologies,

$$\mathrm{F}H^k_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{F}\mathscr{H}^{n-k}_{\overline{p}}(X;\mathbb{Z}) \to \mathbb{Z} \text{ and } \mathrm{T}H^k_{\overline{p}}(X;\mathbb{Z}) \otimes \mathrm{T}\mathscr{H}^{n-k+1}_{\overline{p}}(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}. \quad (0.11)$$

In a second step, we are looking for a quasi-isomorphism between $\widetilde{N}^*_{\overline{p}}(X;\mathbb{Z})$ and $D\widetilde{N}^*_{D\overline{p},c}(X;\mathbb{Z})$. This can be deduced from Theorem A and the existence of a quasi-isomorphism between $\widetilde{N}^*_{\overline{p}}(X;\mathbb{Z})$ and $C^*_{D\overline{p}}(X;\mathbb{Z})$. For investigating that, we use the existence (see Proposition 3.1) of a cochain map, $\chi_{\overline{p}}\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \to C^*_{D\overline{p}}(X;\mathbb{Z})$, and its version with compact supports, $\chi_{\overline{p},c}\colon \widetilde{N}^*_{\overline{p},c}(X;\mathbb{Z}) \to C^*_{D\overline{p},c}(X;\mathbb{Z})$. So, by setting $\mathscr{D}_{\overline{p}} = \mathscr{C}_{D\overline{p}} \circ \chi_{\overline{p}}$, we get a cochain map,

$$\mathscr{D}_{\overline{p}}\colon \widetilde{N}^*_{\overline{p}}(X;\mathbb{Z}) \to D\widetilde{N}^*_{D\overline{p},c}(X;\mathbb{Z}), \quad (0.12)$$

which is a quasi-isomorphism if, and only if, the map $\chi_{\overline{p}}$ is a quasi-isomorphism. Hence, the homotopy cofiber of $\chi_{\overline{p}}$ in the category of cochain complexes plays a fundamental role in Poincaré duality. We study it in Section 3. We call it the *peripheral complex* and denote it and its homology by $R^*_{\overline{p}}$ and $\mathscr{R}^*_{\overline{p}}$, respectively. (A brief analysis shows that it corresponds effectively to the global sections of the peripheral sheaf of [18], in the PL compact case.) This complex, which personifies the non-duality, owns itself a duality in the compact case. To write it in our framework, we introduce the compact supports analogues, $R^*_{\overline{p},c}$ and $\mathscr{R}^*_{\overline{p},c}$, of $R^*_{\overline{p}}$ and $\mathscr{R}^*_{\overline{p}}$.

Theorem B. [Theorem 3.4] *Let X be a paracompact, separable and oriented stratified pseudomanifold of dimension n and $\overline{p}$ be a perversity. Then, there exists a quasi-isomorphism,*

$$R^\star_{\overline{p}}(X;\mathbb{Z}) \to (DR^*_{D\overline{p},c}(X;\mathbb{Z}))_{n-\star-1}.$$

We also describe some properties of this complex, established in [18] for PL compact stratified pseudomanifolds. For instance, as $\chi_{\overline{p}}$ induces a quasi-isomorphism when the ring of coefficients is a field, the homology $\mathscr{R}^*_{\overline{p}}(X;\mathbb{Z})$ is entirely torsion. As the nullity of $\mathscr{R}^*_{\overline{p}}(X;\mathbb{Z})$ is a sufficient and necessary condition for having the quasi-isomorphism $\mathscr{D}_{\overline{p}}$, we may enquire what means the "locally $\overline{p}$-torsion free" requirement appearing in [18] and [15]. In Proposition 3.9, we show that it is equivalent to the nullity of the peripheral cohomology $\mathscr{R}^*_{\overline{p}}(U;\mathbb{Z})$ for *any open subset of X*. Example 6.7 shows that this last property is not necessary for getting the quasi-isomorphism $\mathscr{D}_{\overline{p}}$.

Suppose $\overline{p} \leq D\overline{p}$. We denote by $\widetilde{N}_{D\overline{p}/\overline{p}}(X;R)$ the homotopy cofiber of the inclusion of cochain complexes, $\widetilde{N}^*_{\overline{p}}(X;R) \to \widetilde{N}^*_{D\overline{p}}(X;R)$ and by $\widetilde{N}_{D\overline{p}/\overline{p},c}(X;R)$ the compact support version of it. In [14, Lemma 3.7], G. Friedman and E. Hunsicker prove that the homology analogue of this relative complex owns a self-duality for compact PL-pseudomanifolds and intersection homology with rational coefficients. In Section 4, we extend this result to paracompact, separable and oriented stratified pseudomanifolds of dimension n, X. Denote by $C^*_{\overline{p}/D\overline{p},c}(X;\mathbb{Z})$ the cofiber of the inclusion $C^*_{D\overline{p},c}(X;\mathbb{Z}) \to C^*_{\overline{p},c}(X;\mathbb{Z})$. In Proposition 4.1, we get a quasi-isomorphism, similar to the one of Theorem 2.2,

$$\widetilde{N}^\star_{D\overline{p}/\overline{p}}(X;\mathbb{Z}) \to (DC^*_{\overline{p}/D\overline{p},c})_{n-\star}.$$

Next, if $\chi_{\overline{p}}$ and $\chi_{\overline{p},c}$ are quasi-isomorphisms, we prove the existence of a quasi-isomorphism,

$$\widetilde{N}^\star_{D\overline{p}/\overline{p}}(X;\mathbb{Z}) \to (D\widetilde{N}^*_{D\overline{p}/\overline{p},c}(X;\mathbb{Z}))_{n-\star-1},$$

which gives back the self-duality of [14], see Corollary 4.2.

In Section 5, we study some components of the peripheral cohomology for compact oriented stratified pseudomanifolds. The pairings deduced from (0.11) are investigated separately for the existence of non-singular pairings in the torsion or in the torsion free parts. In Section 6, examples of the different possibilities are described. In particular, Example 6.7 is a not locally $\overline{p}$-torsion free stratified pseudomanifold with Poincaré duality over $\mathbb{Z}$. Finally, let us emphasize that most of the duality results in Sections 2, 3, 4 do not require an hypothesis of finitely generated homology.

Notations and conventions. In this work, homology and cohomology are considered with coefficients in a principal ideal domain, R, or in its field of fractions QR and, if there is no ambiguity, we do not mention the coefficient explicitly in the proofs. For any R-module, A, we denote by $\mathrm{T}A$ the *R-torsion submodule* of A and by $\mathrm{F}A = A/\mathrm{T}A$ the R-torsion free quotient of A. Recall that a pairing $A \otimes B \to R$ is *non-degenerate* if the two adjunction maps, $A \to \mathrm{Hom}(B,R)$ and $B \to \mathrm{Hom}(A,R)$, are injective. The pairing is *non-singular* if they are both isomorphisms.

For any topological space X, we denote by $\mathtt{c}X = X \times [0,1]/X \times \{0\}$ the cone on X and by $\mathring{\mathtt{c}}X = X \times [0,1[/X \times \{0\}$ the open cone on X. Elements of the cones are denoted $[x,t]$ and the apex is $\mathtt{v} = [-,0]$.

In the previous introduction, $H^{\overline{p}}_i(-)$ denotes the intersection homology of [16] or [19]. It can be obtained from the chain complex of filtered simplices of Definition 1.7, see [7, Proposition A.29]. The perversities used in this work are completely general: they are defined on the set of strata and do not only depend on the codimension. Moreover, we lift any restriction on the values taken by a perversity. An issue of that freedom is that an allowable simplex in the sense of [16] or [19] may have a support totally included in the singular subset. This has bad consequences, as the breakdown of Poincaré duality. To overcome this failure, we use a complex built from filtered simplices that are not totally included in the singular set, see Remark 1.9. As it differs from the complex of [16] or [19], we denote it by $\mathfrak{C}^{\overline{p}}_*(-)$ and its homology by $\mathfrak{H}^{\overline{p}}_*(-)$. We emphasize that for the original perversities of the loc. cit. references, we have $C^{\overline{p}}_*(-) = \mathfrak{C}^{\overline{p}}_*(-)$ and $H^{\overline{p}}_*(-) = \mathfrak{H}^{\overline{p}}_*(-)$. Simply, our approach allows an extension of the original historical definition that leaves it unchanged. Thus we call it intersection homology without ambiguity.

The dual complex $\mathfrak{C}^*_{\overline{p}}(-) = \mathrm{Hom}(\mathfrak{C}^{\overline{p}}_*(-), R)$ gives birth to a cohomology $\mathfrak{H}^*_{\overline{p}}(-)$. As explained before in the introduction, this cohomology does not satisfy a Poincaré duality, through a cap product, with intersection homology for any coefficients. For having this property, we use a cohomology constructed from a simplicial blow up. For a clear distinction with the previous cohomology obtained with a linear dual, we denote $\mathscr{H}^*_{\overline{p}}(-)$ the blown-up cohomology and $\widetilde{N}^*_{\overline{p}}(-)$ its corresponding cochain complex .

1. Background

We recall the basics we need, sending the reader to [7], [6], [11] or [16], for more details.

`Pseudomanifolds.` First come the geometrical objects, the stratified pseudomanifolds. In this work, we authorize them to have strata of codimension 1.

Definition 1.1. A *topological stratified pseudomanifold of dimension n* (or a stratified pseudomanifold) is a Hausdorff space together with a filtration by closed subsets,

$$X_{-1} = \emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-2} \subseteq X_{n-1} \subsetneqq X_n = X,$$

such that, for each $i \in \{0, \ldots, n\}$, $X_i \backslash X_{i-1}$ is a topological manifold of dimension i or the empty set. The subspace X_{n-1} is called *the singular set* and each point $x \in X_i \backslash X_{i-1}$ with $i \neq n$ admits

(i) an open neighborhood V of x in X, endowed with the induced filtration,
(ii) an open neighborhood U of x in $X_i \backslash X_{i-1}$,
(iii) a compact stratified pseudomanifold L of dimension $n - i - 1$, whose cone $\mathring{c}L$ is endowed with the conic filtration, $(\mathring{c}L)_i = \mathring{c}L_{i-1}$,
(iv) a homeomorphism, $\varphi\colon U \times \mathring{c}L \to V$, such that
  (a) $\varphi(u, \mathtt{v}) = u$, for any $u \in U$, where $\mathtt{v}$ is the apex of $\mathring{c}L$,
  (b) $\varphi(U \times \mathring{c}L_j) = V \cap X_{i+j+1}$, for any $j \in \{0, \ldots, n - i - 1\}$.

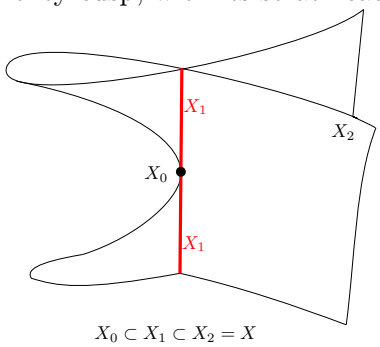
A topological stratified pseudomanifold of dimension 0 is a discrete set of points.

The stratified pseudomanifold L is called the *link* of x. The connected components S of $X_i \backslash X_{i-1}$ are the *strata* of X of dimension i. The strata of dimension n are said to be *regular* and we denote by $\mathcal{S}_X$ (or $\mathcal{S}$ if there is no ambiguity) the set of non-empty strata. We have proven in [7, Proposition A.22] that $S \preceq S'$ if, and only if, $S \subset \overline{S'}$, defines an order relation. We also denote $S \prec S'$ if $S \preceq S'$ and $S \neq S'$.

Definition 1.1 of stratified pseudomanifold is slightly more general than the one in [17] where it is supposed $X_{n-1} = X_{n-2}$. In this work, we are concerned with Poincaré duality and general perversities, for which the previous restriction is not necessary. On the other hand, the hypothesis $X_n \neq X_{n-1}$ implies that the links of the singular strata are always non-empty sets, therefore $X_n \backslash X_{n-1}$ is dense in X. This infers a "good" notion of dimension on X which is the relevant point in [17, Page 82] and motivates us for keeping the appellation of pseudomanifold in this case.

Example 1.2. Among stratified pseudomanifolds, let us quote the manifolds, the open subsets of a stratified pseudomanifold (with the induced structure), the cones on compact manifolds with the singular set reduced to the apex, the Thom spaces filtered by the compactification point. As relevant examples of spaces admitting a structure of stratified

pseudomanifolds, we may also take over the list of [17]: complex algebraic varieties, complex analytic varieties, real analytic varieties, Whitney stratified sets, Thom-Mather stratified spaces. For instance, the following picture represents the real part of the hypersurface of $\mathbb{C}^3$ called Whitney cusp, with its stratification,



$X_0 \subset X_1 \subset X_2 = X$

Definition 1.3. The *depth* of a topological stratified pseudomanifold X is the largest integer ℓ for which there exists a chain of strata, $S_0 \prec S_1 \prec \cdots \prec S_\ell$. It is denoted by $\operatorname{depth} X$.

In particular, $\operatorname{depth} X = 0$ if, and only if, all the strata of X are regular.

`Perversity.` The second concept in intersection homology is that of perversity. We consider the perversities of [21] defined on each stratum. They are already used in [22], [23], [12], [13], [15].

Definition 1.4. A *perversity on a stratified pseudomanifold,* X, is a map, $\overline{p}\colon \mathcal{S}_X \to \mathbb{Z}$, defined on the set of strata of X and taking the value 0 on the regular strata. The pair $(X, \overline{p})$ is called a *perverse pseudomanilfold.* If $\overline{p}$ and $\overline{q}$ are two perversities on X, we set $\overline{p} \leq \overline{q}$ if we have $\overline{p}(S) \leq \overline{q}(S)$, for all $S \in \mathcal{S}_X$.

Among perversities, there are those considered in [16] and whose values depend only on the codimension of the strata.

Definition 1.5. A GM-*perversity* is a map $\overline{p}\colon \mathbb{N} \to \mathbb{Z}$ such that $\overline{p}(0) = \overline{p}(1) = \overline{p}(2) = 0$ and $\overline{p}(i) \leq \overline{p}(i+1) \leq \overline{p}(i) + 1$, for all $i \geq 2$. As particular case, we have the null perversity $\overline{0}$ constant with value 0 and the *top perversity* defined by $\overline{t}(i) = i - 2$ if $i \geq 2$. For any perversity, $\overline{p}$, the perversity $D\overline{p} := \overline{t} - \overline{p}$ is called the *complementary perversity* of $\overline{p}$. A GM-perversity induces a perversity on X by $\overline{p}(S) = \overline{p}(\operatorname{codim} S)$.

Example 1.6. Let us mention the lower-middle and the upper-middle perversities, respectively defined on the singular strata by

$$\overline{m}(S) = \left\lfloor \frac{(\operatorname{codim} S) - 2}{2} \right\rfloor \text{ and } \overline{n}(S) = D\overline{m}(S) = \left\lceil \frac{(\operatorname{codim} S) - 2}{2} \right\rceil,$$

which play an important role in intersection homology. They coincide for Witt spaces ([18, Definition 11.1]) and, for them, a non-singular pairing exists in intersection homology with rational coefficients, see [18]. For instance, this is the case for the Thom space of the tangent bundle of the 2-sphere and there is a pairing induced by the cup product, $\mathscr{H}^k_{\overline{m}}(X;\mathbb{Q}) \otimes \mathscr{H}^{n-k}_{\overline{m}}(X;\mathbb{Q}) \to \mathbb{Q}$. Its behaviour with integer coefficients is analyzed in [8, Example 4.10].

`Intersection Homology.` We specify the chain complex used for the determination of intersection homology of a stratified pseudomanifold X equipped with a perversity $\overline{p}$, cf. [10].

Definition 1.7. A *filtered simplex* is a continuous map $\sigma\colon \Delta \to X$, from a Euclidean simplex endowed with a decomposition $\Delta = \Delta_0 * \cdots * \Delta_n$, called *$\sigma$-decomposition of* Δ, such that $\sigma^{-1}X_i = \Delta_0 * \cdots * \Delta_i$, for all $i \in \{0, \dots, n\}$, where $*$ denotes the join. The sets Δ_i may be empty, with the convention $\emptyset * Y = Y$, for any space Y. The simplex σ is *regular* if $\Delta_n \neq \emptyset$. A chain is *regular* if it is a linear combination of regular simplices.

Given a Euclidean regular simplex $\Delta = \Delta_0 * \cdots * \Delta_n$, we consider as "boundary" of Δ the regular part $\mathfrak{d}\Delta$ of the chain $\partial\Delta$. That is $\mathfrak{d}\Delta = \partial(\Delta_0 * \cdots * \Delta_{n-1}) * \Delta_n$, if $|\Delta_n| = 0$, or $\mathfrak{d}\Delta = \partial\Delta$, if $|\Delta_n| \geq 1$. For any regular simplex $\sigma\colon \Delta \to X$, we set $\mathfrak{d}\sigma = \sigma_* \circ \mathfrak{d}$ and denote by $\mathfrak{C}_*(X; R)$ the complex of linear combinations of regular simplices (called finite chains) with the differential $\mathfrak{d}$.

Definition 1.8. The *perverse degree of* a filtered simplex $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ is the $(n+1)$-uple, $\|\sigma\| = (\|\sigma\|_0, \dots, \|\sigma\|_n)$, with $\|\sigma\|_i = \dim(\Delta_0 * \cdots * \Delta_{n-i})$ and the convention $\dim \emptyset = -\infty$. The *perverse degree of σ along a stratum S* is defined by

$$\|\sigma\|_S = \begin{cases} -\infty, & \text{if } S \cap \sigma(\Delta) = \emptyset, \\ \|\sigma\|_{\operatorname{codim} S}, & \text{otherwise.} \end{cases}$$

A *regular simplex is $\overline{p}$-allowable* if

$$\|\sigma\|_S \leq \dim \Delta - \operatorname{codim} S + \overline{p}(S) = \dim \Delta - D\overline{p}(S) - 2, \tag{1.1}$$

for each stratum S of X. A chain ξ is *$\overline{p}$-allowable* if it is a linear combination of $\overline{p}$-allowable simplices, and of *$\overline{p}$-intersection* if ξ and its boundary $\mathfrak{d}\xi$ are $\overline{p}$-allowable. Let $\mathfrak{C}^{\overline{p}}_*(X; R)$ be the complex of $\overline{p}$-intersection chains and $\mathfrak{H}^{\overline{p}}_*(X; R)$ its homology, called *$\overline{p}$-intersection homology.*

Remark 1.9. This homology is called tame intersection homology in [10] and non-GM intersection homology in [11], see [10, Theorem B]. It coincides with the intersection homology for the original perversities of [16], see [10, Remark 3.9].

We introduce also the complex $\mathfrak{C}^{\infty,\overline{p}}_*(X; R)$ of locally finite chains of $\overline{p}$-intersection with the differential $\mathfrak{d}$. If X is locally compact, metrizable and separable, this complex is isomorphic (see [9, Proposition 3.4]) to the inverse limit,

$$\mathfrak{C}^{\infty,\overline{p}}_*(X; R) \cong \varprojlim_{K \subset X} \mathfrak{C}^{\overline{p}}_*(X, X \backslash K; R),$$

where K runs over the compact subsets of X. Its homology, $\mathfrak{H}^{\infty,\overline{p}}_*(X; R)$, is called *the locally finite (or Borel-Moore) $\overline{p}$-intersection homology.*

`Blown-up cohomology.` Let $N^*(\Delta)$ be the simplicial cochain complex of a Euclidean simplex Δ, with coefficients in R. Given a face F of Δ, we write $\mathbf{1}_F$ for the element of $N^*(\Delta)$ taking the value 1 on F and 0 otherwise. We denote also by $(F, 0)$ the same face viewed as face of the cone $\mathtt{c}\Delta = [\mathrm{v}] * \Delta$ and by $(F, 1)$ the face $\mathtt{c}F$ of $\mathtt{c}\Delta$. The apex is

denoted $(\emptyset, 1) = \mathtt{c}\emptyset = [\mathtt{v}]$. Cochains on the cone $\mathtt{c}\Delta$ are denoted $\mathbf{1}_{(F,\varepsilon)}$ with $\varepsilon = 0$ or 1. If $\Delta = \Delta_0 * \cdots * \Delta_n$, let us set

$$\widetilde{N}^*(\Delta) = N^*(\mathtt{c}\Delta_0) \otimes \cdots \otimes N^*(\mathtt{c}\Delta_{n-1}) \otimes N^*(\Delta_n).$$

A basis of $\widetilde{N}^*(\Delta)$ is formed of the elements $\mathbf{1}_{(F,\varepsilon)} = \mathbf{1}_{(F_0,\varepsilon_0)} \otimes \cdots \otimes \mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \otimes \mathbf{1}_{F_n}$, where $\varepsilon_i \in \{0,1\}$ and F_i is a face of Δ_i for $i \in \{0,\dots,n\}$ or the empty set with $\varepsilon_i = 1$ if $i < n$. We set $|\mathbf{1}_{(F,\varepsilon)}|_{>s} = \sum_{i>s}(\dim F_i + \varepsilon_i)$.

Definition 1.10. Let $\ell \in \{1,\dots,n\}$. The *ℓ-perverse degree* of $\mathbf{1}_{(F,\varepsilon)} \in \widetilde{N}^*(\Delta)$ is

$$\|\mathbf{1}_{(F,\varepsilon)}\|_\ell = \begin{cases} -\infty & \text{if} \quad \varepsilon_{n-\ell} = 1, \\ |\mathbf{1}_{(F,\varepsilon)}|_{>n-\ell} & \text{if} \quad \varepsilon_{n-\ell} = 0. \end{cases}$$

For a cochain $\omega = \sum_b \lambda_b\ \mathbf{1}_{(F,\varepsilon)_b} \in \widetilde{N}^*(\Delta)$ with $\lambda_b \neq 0$ for all b, the *ℓ-perverse degree* is

$$\|\omega\|_\ell = \max_b \|\mathbf{1}_{(F,\varepsilon)_b}\|_\ell.$$

By convention, we set $\|0\|_\ell = -\infty$.

Let $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ be a filtered simplex. We set $\widetilde{N}^*_\sigma = \widetilde{N}^*(\Delta)$. If $\delta_\ell\colon \Delta' \to \Delta$ is an inclusion of a face of codimension 1, we have $\partial_\ell\sigma = \sigma \circ \delta_\ell\colon \Delta' \to X$. If $\Delta = \Delta_0 * \cdots * \Delta_n$ is filtered, the induced filtration on Δ' is denoted $\Delta' = \Delta'_0 * \cdots * \Delta'_n$ and $\partial_\ell\sigma$ is a filtered simplex. The *blown-up intersection complex* of X is the cochain complex $\widetilde{N}^*(X)$ composed of the elements ω associating to each regular filtered simplex $\sigma\colon \Delta_0 * \cdots * \Delta_n \to X$ an element $\omega_\sigma \in \widetilde{N}^*_\sigma$ such that $\delta_\ell^*(\omega_\sigma) = \omega_{\partial_\ell\sigma}$, for any face operator $\delta_\ell\colon \Delta' \to \Delta$ with $\Delta'_n \neq \emptyset$. The differential $d\omega$ is defined by $(d\omega)_\sigma = d(\omega_\sigma)$. The *perverse degree of ω along a singular stratum S* equals

$$\|\omega\|_S = \sup\left\{\|\omega_\sigma\|_{\operatorname{codim} S} \mid \sigma\colon \Delta \to X \text{ regular such that } \sigma(\Delta) \cap S \neq \emptyset\right\}.$$

We denote $\|\omega\|$ the map which associates $\|\omega\|_S$ to any singular stratum S and 0 to any regular one. A *cochain $\omega \in \widetilde{N}^*(X;R)$ is $\overline{p}$-allowable* if $\|\omega\| \leq \overline{p}$ and of *$\overline{p}$-intersection* if ω and $d\omega$ are $\overline{p}$-allowable. Let $\widetilde{N}^*_{\overline{p}}(X;R)$ be the complex of $\overline{p}$-intersection cochains and $\mathscr{H}^*_{\overline{p}}(X;R)$ its homology, called *blown-up $\overline{p}$-intersection cohomology* of X.

Finally, we mention the existence of a version with compact supports, $\widetilde{N}^*_{\overline{p},c}(X;R)$ and $\mathscr{H}^*_{\overline{p},c}(X;R)$, whose properties have been established in [8].

`Products.` Let X be a stratified pseudomanifold equipped with two perversities, $\overline{p}$ and $\overline{q}$. In [6, Proposition 4.2], we prove the existence of a map

$$- \smile - \colon \widetilde{N}^i_{\overline{p}}(X;R) \otimes \widetilde{N}^j_{\overline{q}}(X;R) \to \widetilde{N}^{i+j}_{\overline{p}+\overline{q}}(X;R), \tag{1.2}$$

inducing an associative and commutative graded product, called *intersection cup product,*

$$- \smile - \colon \mathscr{H}^i_{\overline{p}}(X;R) \otimes \mathscr{H}^j_{\overline{q}}(X;R) \to \mathscr{H}^{i+j}_{\overline{p}+\overline{q}}(X;R). \tag{1.3}$$

We mention also from [6, Propositions 6.6 and 6.7] the existence of cap products,

$$- \frown - \colon \widetilde{N}^i_{\overline{q}}(X;R) \otimes \mathfrak{C}^{\overline{p}}_j(X;R) \to \mathfrak{C}^{\overline{p}+\overline{q}}_{j-i}(X;R), \tag{1.4}$$

such that $(\eta \smile \omega) \frown \xi = \eta \frown (\omega \frown \xi)$. (By definition, we say that the collection $\{\mathfrak{C}^{\overline{p}}_*(X;R)\}_{\overline{p}\in\mathcal{P}}$ is a left perverse module over the perverse algebra $\left\{\widetilde{N}^*_{\overline{q}}(X;R)\right\}_{\overline{q}\in\mathcal{P}}$.) Moreover, we have

$$\mathfrak{d}(\omega \frown \xi) = d\omega \frown \xi + (-1)^{|\omega|}\omega \frown \mathfrak{d}\xi \tag{1.5}$$

and the cap product induces a map in homology,

$$- \frown -\colon \mathscr{H}^i_{\overline{q}}(X;R) \otimes \mathfrak{H}^{\overline{p}}_j(X;R) \to \mathfrak{H}^{\overline{p}+\overline{q}}_{j-i}(X;R). \tag{1.6}$$

The map (1.4) can be extended to maps,

$$- \frown - \quad : \quad \widetilde{N}^i_{\overline{p}}(X;R) \otimes \mathfrak{C}^{\infty,\overline{q}}_j(X;R) \to \mathfrak{C}^{\infty,\overline{p}+\overline{q}}_{j-i}(X;R), \tag{1.7}$$

$$- \frown - \quad : \quad \widetilde{N}^i_{\overline{p},c}(X;R) \otimes \mathfrak{C}^{\infty,\overline{q}}_j(X;R) \to \mathfrak{C}^{\overline{p}+\overline{q}}_{j-i}(X;R), \tag{1.8}$$

which induce,

$$- \frown - \quad : \quad \mathscr{H}^i_{\overline{p}}(X;R) \otimes \mathfrak{H}^{\infty,\overline{q}}_j(X;R) \to \mathfrak{H}^{\infty,\overline{p}+\overline{q}}_{j-i}(X;R), \tag{1.9}$$

$$- \frown - \quad : \quad \mathscr{H}^i_{\overline{p},c}(X;R) \otimes \mathfrak{H}^{\infty,\overline{q}}_j(X;R) \to \mathfrak{H}^{\overline{p}+\overline{q}}_{j-i}(X;R). \tag{1.10}$$

`A second cohomology coming from a linear dual.` Let X be a stratified pseudo-manifold with a perversity $\overline{p}$. We set

$$\mathfrak{C}^*_{\overline{p}}(X;R) = \mathrm{Hom}_R(\mathfrak{C}^{\overline{p}}_*(X;R), R)$$

with the differential $\mathfrak{d}c(\xi) = -(-1)^{|c|}c(\mathfrak{d}\xi)$. The homology of $\mathfrak{C}^*_{\overline{p}}(X;R)$ is denoted $\mathfrak{H}^*_{\overline{p}}(X;R)$ (or $\mathfrak{H}^*_{\overline{p}}(X)$ if there is no ambiguity) and called *$\overline{p}$-intersection cohomology*. From Remark 1.9 and the Universal Coefficients Theorem [11, Theorem 7.1.4], we deduce that this cohomology coincides with the non-GM cohomology of [11].

The cap product (1.4) defines a star map

$$\star\colon \mathfrak{C}^j_{\overline{p}}(X;R) \otimes \widetilde{N}^i_{\overline{q}}(X;R) \longrightarrow \mathfrak{C}^{j+i}_{\overline{p}-\overline{q}}(X;R) \tag{1.11}$$

by

$$(c \star \omega)(\xi) = c(\omega \frown \xi).$$

We check easily $c \star (\omega \smile \eta) = (c \star \omega) \star \eta$. Hence, the collection $\{\mathfrak{C}^*_{\overline{p}}(X;R)\}_{\overline{p}\in\mathcal{P}}$ is a right perverse module over the perverse algebra $\left\{\widetilde{N}^*_{\overline{q}}(X;R)\right\}_{\overline{q}\in\mathcal{P}}$. Moreover, we have

$$\mathfrak{d}(c \star \omega) = \mathfrak{d}c \star \omega + (-1)^{|c|}c \star d\omega \tag{1.12}$$

and the star product induces

$$- \star -\colon \mathfrak{H}^j_{\overline{p}}(X;R) \otimes \mathscr{H}^i_{\overline{q}}(X;R) \to \mathfrak{H}^{j+i}_{\overline{p}-\overline{q}}(X;R). \tag{1.13}$$

The module structures (1.11) and (1.13) have also variants with compact supports. We do not describe them in detail.

`Background on Poincaré duality.` This notion has been described in the introduction, for compact oriented manifolds and stratified pseudomanifolds. We recall the main results of [8] and [24] which represent a first step for a duality over a ring.

Proposition 1.11. [8, Theorem B], [24, Theorem B] *Let $(X, \overline{p})$ be an oriented paracompact, perverse stratified pseudomanifold of dimension n. The cap product with the fundamental class $[X] \in \mathfrak{H}_n^{\infty,\overline{0}}(X;R)$ induces an isomorphism*

$$\mathscr{H}^k_{\overline{p},c}(X;R) \xrightarrow{\cong} \mathfrak{H}^{\overline{p}}_{n-k}(X;R).$$

Moreover, if X is second countable, this cap product also induces an isomorphism,

$$\mathscr{H}^k_{\overline{p}}(X;R) \xrightarrow{\cong} \mathfrak{H}^{\infty,\overline{p}}_{n-k}(X;R).$$

`Dual of a complex.` Let $0 \to R \to QR \xrightarrow{\rho} QR/R \to 0$ be an injective resolution of the principal ideal domain R.

We denote by I^*_R the cochain complex $I^0_R = QR \overset{\rho}{\twoheadrightarrow} I^1_R = QR/R$ and define the *dual complex of a cochain complex*, A^*, as the chain complex

$$(DA^*)_k = (\mathrm{Hom}(A^*, I^*_R))_k = \mathrm{Hom}_R(A^k, QR) \oplus \mathrm{Hom}_R(A^{k+1}, QR/R) \tag{1.14}$$

with the differential $\partial(\varphi_0, \varphi_1) = (-(-1)^k\varphi_0 \circ d, -(-1)^k\varphi_1 \circ d - \rho \circ \varphi_0)$. This dual complex verifies a universal coefficient formula, see [20, Lemma 1.2] for instance,

$$0 \longrightarrow \mathrm{Ext}_R(H^{k+1}(A^*), R) \longrightarrow H_k(DA^*) \xrightarrow{\kappa} \mathrm{Hom}_R(H^k(A^*), R) \longrightarrow 0, \tag{1.15}$$

where κ is the canonical map defined by $(\kappa[\varphi_0, \varphi_1])([a]) = \varphi_0(a)$. The complex DA^* plays the same role as the Verdier dual in sheaf theory. A *self-dual cochain complex of dimension n* is a complex, A^*, together with a quasi-isomorphism

$$A^\star \to (DA^*)_{n-\star}. \tag{1.16}$$

Similarly, we define the dual of a chain complex, A_*, as the cochain complex,

$$(DA_*)^k = (\mathrm{Hom}(A_*, I^*_R))^k = \mathrm{Hom}_R(A_k, QR) \oplus \mathrm{Hom}_R(A_{k-1}, QR/R), \tag{1.17}$$

with the differential $d(\psi_0, \psi_1) = (-(-1)^k\psi_0 \circ \partial, -(-1)^k\psi_1 \circ \partial - \rho \circ \psi_0)$. This dual complex also verifies a universal coefficient formula,

$$0 \longrightarrow \mathrm{Ext}_R(H_{k-1}(A_*), R) \longrightarrow H^k(DA_*) \longrightarrow \mathrm{Hom}_R(H_k(A_*), R) \longrightarrow 0. \tag{1.18}$$

`Torsion and torsion free pairings.` We recall how the existence of a duality gives pairings between the torsion and torsion free parts, see [11, Section 8.4] for a similar treatment.

Proposition 1.12. *Let A^* and B^* be two cochain complexes with finitely generated cohomology. To any cochain map, $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1) \colon B^* \to DA^*$, sending B^k to $(DA^*)_{n-k}$, we can associate two pairings,*

$$\mathcal{P}_F \colon FH^k(B) \to \mathrm{Hom}(H^{n-k}(A), R) \text{ and } \mathcal{P}_T \colon \mathrm{T}H^k(B) \to \mathrm{Hom}(\mathrm{T}H^{n-k+1}(A), QR/R).$$

The first one is defined by

$$\mathcal{P}_F([b]([a]) = \mathcal{P}_0(b)(a) \in R.$$

For the second one, let $[b] \in \mathrm{T}H^k(B)$. *There exists* $b' \in B^{k-1}$ *and* $\ell \in R$ *such that* $db' = \ell b$ *and we set*

$$\mathcal{P}_T([b]) = \rho\left(\frac{\mathcal{P}_0(b')}{\ell}\right) + \mathcal{P}_1(b).$$

Moreover, the pairings $\mathcal{P}_F$ *and* $\mathcal{P}_T$ *are non-singular if, and only if,* $\mathcal{P}$ *is a quasi-isomorphism.*

Proof. We first construct the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{T}H^k(B) & \xrightarrow{j_1} & H^k(B) & \longrightarrow & \mathrm{F}H^k(B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathcal{P}_T} & & \downarrow{\scriptstyle \mathcal{P}} & & \downarrow{\scriptstyle \mathcal{P}_F} & & \\ 0 & \longrightarrow & \mathrm{Hom}(\mathrm{T}H^{n-k+1}(A), QR/R) & \xrightarrow{j_2} & H_{n-k}(DA^*) & \xrightarrow{\kappa} & \mathrm{Hom}(H^k(A), R) & \longrightarrow & 0. \end{array} \tag{1.19}$$

The upper line is the decomposition of a module in torsion and torsion free parts. The lower one is a universal coefficient formula. Recall that the short exact sequence,

$$0 \longrightarrow \mathrm{Hom}(A^{*+1}, QR/R) \hookrightarrow DA^* \longrightarrow \mathrm{Hom}(A^*, QR) \longrightarrow 0,$$

gives a long exact sequence with connecting map denoted δ,

$$\ldots \longrightarrow \mathrm{Hom}(H^{k+1}(A), QR) \xrightarrow{\delta} \mathrm{Hom}(H^{k+1}(A), QR/R) \longrightarrow H_{n-k}(DA^*) \xrightarrow{\kappa} \ldots$$

As QR is injective and $H(A^*)$ finitely generated, there are isomorphisms $\mathrm{Ker}\,\delta \cong \mathrm{Hom}(H^{*+1}(A;R), R)$ and $\mathrm{Coker}\,\delta \cong \mathrm{Ext}(H^{*+1}(A), R) \cong \mathrm{Hom}(\mathrm{T}H^{*+1}(A), QR/R)$. Hence, the map j_2 is induced by the canonical inclusion $\mathrm{Hom}(A^{*+1}, QR/R) \hookrightarrow DA^*$.

As $\mathrm{Hom}(H^k(A), R)$ is torsion free, the composite $\kappa \circ \mathcal{P} \circ j_1$ is zero and there exists a lifting $\mathcal{P}_T$ such that $j_2 \circ \mathcal{P}_T = \mathcal{P} \circ j_1$. This map induces $\mathcal{P}_F$ making commutative the diagram. The map $\mathcal{P}_F$ is easily determined as in the statement.

We now determine the map $\mathcal{P}_T$. With the notations of the statement, we analyze the compatibility of $\mathcal{P}$ with the differentials. We first have:

- $(\mathcal{P} \circ d)(b') = (\mathcal{P}_0(db'), \mathcal{P}_1(db')) = (\ell\mathcal{P}_0(b), \ell\mathcal{P}_1(b))$,
- $\begin{aligned} \partial\mathcal{P}(b') &= \partial(\mathcal{P}_0(b'), \mathcal{P}_1(b')) \\ &= (-(-1)^{n-k+1}\mathcal{P}_0(b') \circ d, -(-1)^{n-k+1}\mathcal{P}_1(b') \circ d - \rho \circ \mathcal{P}_0(b')). \end{aligned}$

The equality $\partial \circ \mathcal{P} = \mathcal{P} \circ d$ implies

$$\left\{ \begin{array}{lcl} \ell\mathcal{P}_0(b) &=& -(-1)^{n-k+1}\mathcal{P}_0(b') \circ d, \\ \ell\mathcal{P}_1(b) &=& -(-1)^{n-k+1}\mathcal{P}_1(b') \circ d - \rho \circ \mathcal{P}_0(b'). \end{array} \right. \tag{1.20}$$

We now show the commutation $j_2 \circ \mathcal{P}_T = \mathcal{P} \circ j_1$ by proving that the difference is zero in homology:

$$\begin{aligned} \partial\left(\frac{\mathcal{P}_0(b')}{\ell}, 0\right) &= \left(-(-1)^{n-k+1}\frac{\mathcal{P}_0(b') \circ d}{\ell}, -\rho\left(\frac{\mathcal{P}_0(b')}{\ell}\right)\right) \\ &= (\mathcal{P}_0(b), -\mathcal{P}_T([b]) + \mathcal{P}_1(b)) = \mathcal{P}(b) - (0, \mathcal{P}_T([b])). \end{aligned}$$

The last equality comes from the definition of $\mathcal{P}_T$ and (1.20).

If $\mathcal{P}_F$ and $\mathcal{P}_T$ are isomorphisms, $\mathcal{P}$ is one also, from the five lemma. Conversely, suppose that $\mathcal{P}$ is an isomorphism. The Ker-Coker exact sequence associated to (1.19)

implies $\operatorname{Ker}\mathcal{P}_T = \operatorname{Coker}\mathcal{P}_F = 0$ and $\operatorname{Ker}\mathcal{P}_F \cong \operatorname{Coker}\mathcal{P}_T$. But $\operatorname{Ker}\mathcal{P}_F$ is free and $\operatorname{Coker}\mathcal{P}_T$ is torsion, thus $\operatorname{Ker}\mathcal{P}_F = \operatorname{Coker}\mathcal{P}_T = 0$. □

2. Verdier dual of intersection cochain complexes

In Theorem 2.2, we prove Theorem A for any principal ideal domain as ring of coefficients. The main feature is the use of a cap product which gives the duality map between the two intersection cohomologies. We continue with a necessary and sufficient condition for the existence of a duality at the level of the blown-up intersection cohomology (or intersection homology) itself.

Proposition 2.1. *Let $(X,\overline{p})$ be an oriented perverse stratified pseudomanifold of dimension n and γ_X a representing cycle of the fundamental class $[X]\in\mathfrak{H}_n^{\infty,\overline{0}}(X;R)$. The two following maps,*

$$\mathscr{N}_{\overline{p}}\colon \widetilde{N}^{\star}_{\overline{p}}(X;R)\to (D\mathfrak{C}^*_{\overline{p},c}(X;R))_{n-\star},\ \text{and}\ \mathscr{C}_{\overline{p}}\colon \mathfrak{C}^{\star}_{\overline{p}}(X;R)\to (D\widetilde{N}^*_{\overline{p},c}(X;R))_{n-\star},$$

defined by

- $\mathscr{N}_{\overline{p}}(\omega)=(\varphi(\omega),0)$ *with* $\varphi(\omega)(c)=(-1)^{|\omega|\,|c|}(c\star\omega)(\gamma_X)$,
- $\mathscr{C}_{\overline{p}}(c)=(\psi(c),0)$ *with* $\psi(c)(\omega)=(c\star\omega)(\gamma_X)$,

are cochain maps.

Proof. Let $\rho\colon QR\to QR/R$ be the quotient map. We first observe that $\rho\varphi(\omega)(c)=0\in QR/R$ since $\varphi(\omega)(c)\in R$. Also, as γ_X is a cocycle, we have $\mathfrak{d}(c\star\omega)(\gamma_X)=0$. With (1.12), we deduce

$$(\mathfrak{d}c\star\omega)(\gamma_X)+(-1)^{|c|}(c\star d\omega)(\gamma_X)=0.$$

Thus, we have $(-1)^{|\omega|(|c|+1)}\varphi(\omega)(\mathfrak{d}c)+(-1)^{|\omega|\,|c|}\varphi(d\omega)(c)=0$ which implies $\varphi(d\omega)=-(-1)^{|\omega|}\varphi(\omega)\circ\mathfrak{d}$. From these observations, we get

$$\begin{aligned}\partial\mathscr{N}_{\overline{p}}(\omega) &= (\partial\varphi(\omega),0)=(-(-1)^{|\omega|}\varphi(\omega)\circ\mathfrak{d},-\rho\varphi(\omega))=(\varphi(d\omega),0)\\ &= \mathscr{N}_{\overline{p}}(d\omega).\end{aligned}$$

The proof is similar for $\mathscr{C}_{\overline{p}}$. □

Theorem 2.2. *Let $(X,\overline{p})$ be a paracompact, separable and oriented perverse stratified pseudomanifold of dimension n. Then, the two maps, $\mathscr{N}_{\overline{p}}\colon \widetilde{N}^{\star}_{\overline{p}}(X;R)\to (D\mathfrak{C}^*_{\overline{p},c}(X;R))_{n-\star}$ and $\mathscr{C}_{\overline{p}}\colon \mathfrak{C}^{\star}_{\overline{p}}(X;R)\to (D\widetilde{N}^*_{\overline{p},c}(X;R))_{n-\star}$, of Proposition 2.1, are quasi-isomorphisms.*

We need some lemmas before giving the proof. The first one is proven in [6].

Lemma 2.3. [6, Lemma 13.3] *Let X be a locally compact topological space, metrizable and separable. We are given an open basis of X, $\mathfrak{U}=\{U_\alpha\}$, closed by finite intersections, and a statement $P(U)$ on open subsets of X satisfying the following three properties.*

a) The property $P(U_\alpha)$ is true for all α.
b) If U, V are open subsets of X for which properties $P(U)$, $P(V)$ and $P(U\cap V)$ are true, then $P(U\cup V)$ is true.
c) If $(U_i)_{i\in I}$ is a family of open subsets of X, pairwise disjoint, verifying the property $P(U_i)$ for all $i\in I$, then $P(\bigsqcup_i U_i)$ is true.

Then the property $P(X)$ is true.

Lemma 2.4. *Suppose given a cochain map, $\psi_X\colon A^*(X) \to B^*(X)$, for any paracompact, separable perverse stratified pseudomanifold X, satisfying the following three properties.*

i) The map ψ_X is a quasi-isomorphism for any $X = \mathbb{R}^a \times \mathring{c}L$, with L a compact stratified pseudomanifold or the emptyset.
ii) The two complexes, $A^*(X)$ and $B^*(X)$, verify the Mayer-Vietoris property and ψ_X induces a morphism of exact sequences (up to sign).
iii) If ψ_{U_i} is a quasi-isomorphism for a family of disjoint stratified pseudomanifolds, then $\psi_{\sqcup_i U_i}$ is a quasi-isomorphism.

Then ψ_X is a quasi-isomorphism for any X.

Proof. As X is metrizable (cf. [10, Proposition 1.11]) we may use Lemma 2.3. We denote by $P(X)$ the property "ψ_X is a quasi-isomorphism". We consider the family $\mathcal{U} = \{U_\alpha\}$ formed of the open subsets of charts of the topological stratified pseudomanifold X together with the open subsets of the conical charts of the topological manifold $X\backslash X_{n-1}$.

Observe that Property b)$_{\mathcal{U}}$ is a direct consequence of the existence of a morphism between the Mayer-Vietoris sequences in the domain and codomain. Also, Property c)$_{\mathcal{U}}$ coincides with the hypothesis iii). We are reduced to establish a)$_{\mathcal{U}}$.

For that, we proceed by induction on the depth of X. If $\operatorname{depth} X = 0$, the stratified pseudomanifold X is a manifold and we have that U_α is an open subset of $\mathbb{R}^n$. We now consider the basis $\mathcal{V}$ formed of the open n-cubes of $\mathbb{R}^n$ included in U_α. This family is closed by finite intersections and verifies the hypotheses of Lemma 2.3, Property a)$_{\mathcal{V}}$ being given by the hypothesis i). This proves $P(U_\alpha)$.

To carry out the inductive step, we first observe that $P(U)$ is already established for each open subset U of $X\backslash X_{n-1}$ since $\operatorname{depth} U = 0$. We consider an open subset U_α of a conical chart $Y = \mathbb{R}^a \times \mathring{c}L$ of X, with L a compact stratified pseudomanifold. We choose the basis $\mathcal{W}$ of open subsets of U_α, formed of the open subsets $W \subset U_\alpha$ with $W \cap (\mathbb{R}^a \times \{\mathtt{v}\}) = \emptyset$, which are stratified pseudomanifolds with $\operatorname{depth} W \leq \operatorname{depth}(\mathbb{R}^a \times (\mathring{c}L\backslash\{\mathtt{v}\}) < \operatorname{depth} Y \leq \operatorname{depth} X$, together with the open subsets $W = B \times \mathring{c}_r L \subset U_\alpha$, where B is an open a-cube, $r > 0$ and $\mathring{c}_r L = (L \times [0, r[)/(L \times \{0\})$. The family $\mathcal{W}$ is closed by finite intersections and verifies the hypotheses of Lemma 2.3, the property a)$_{\mathcal{W}}$ being given by induction and the hypothesis i). This proves $P(U_\alpha)$. □

The third lemma is the proof of Theorem 2.2 in a particular generic case.

Lemma 2.5. *The conclusion of Theorem 2.2 is true if $X = \mathbb{R}^a \times \mathring{c}L$, where L is a compact oriented perverse stratified pseudomanifold of dimension $m-1$.*

Proof. We begin by checking the finite generation of the various homologies and cohomologies. First, we know that the intersection homology of a compact stratified pseudomanifold is finitely generated, see [11, Corollary 6.3.40] for instance. From Poincaré duality, universal coefficients formula or direct computations, this infers the finite generation of $\mathscr{H}^*_{\overline{p}}(X)$, $\mathfrak{H}^*_{\overline{p}}(X)$ and $\mathfrak{H}^{\infty,\overline{p}}_*(X)$. For the blown-up cohomology with compact supports, $\mathscr{H}^*_{\overline{p},c}(X)$, this is a consequence of [8, Propositions 2.18 and 2.19]. As we do not find an explicit reference for the last one, $\mathfrak{H}^*_{\overline{p},c}(X)$, we supply a short direct proof.

Set $K_n = [-n,n]^a \times \mathring{c}_n L$ with $\mathring{c}_n L = L \times [0,(n-1)/n[/L \times \{0\}$. The family $(K_n)_n$ being cofinal among the compact subsets of $\mathbb{R}^a \times \mathring{c}L$, we have

$$\mathfrak{H}^*_{\overline{p},c}(\mathbb{R}^a \times \mathring{c}L) = \varinjlim_n \mathfrak{H}^*_{\overline{p}}(\mathbb{R}^a \times \mathring{c}L, (\mathbb{R}^a \times \mathring{c}L)\backslash K_n).$$

As all the open subsets $(\mathbb{R}^a \times \mathring{c}L)\backslash K_n$ are stratified homeomorphic, it suffices to consider $n = 0$ and

$$\mathfrak{H}^*_{\overline{p},c}(\mathbb{R}^a \times \mathring{c}L) = \mathfrak{H}^*_{\overline{p}}(\mathbb{R}^a \times \mathring{c}L, (\mathbb{R}^a \times \mathring{c}L)\backslash\{(0,\mathtt{v})\}). \tag{2.1}$$

As we observed before, the cohomology $\mathfrak{H}^*_{\overline{p}}(\mathbb{R}^a \times \mathring{c}L)$ is finitely generated. For the second one, we know that $\mathbb{R}^a \times \mathring{c}L\backslash\{(0,\mathtt{v})\}$ is stratified homeomorphic to $\mathring{c}(S^{a-1} * L)\backslash\{\mathtt{u}\}$, cf. [2, 5.7.4] and proof of [9, Proposition 3.7]. As $\mathfrak{H}^*_{\overline{p}}(\mathring{c}(S^{a-1} * L)\backslash\{\mathtt{u}\})$ is also finitely generated, so is the relative homology of (2.1).

For the rest of this proof, we set $X = \mathbb{R}^a \times \mathring{c}L$. Let us observe that the two following maps are quasi-isomorphisms,

$$\widetilde{N}^\star_{\overline{p},c}(X) \to \mathfrak{C}^{\overline{p}}_{n-\star}(X) \to (D\mathfrak{C}^*_{\overline{p}}(X))_{n-\star}.$$

The left-hand map is the Poincaré duality of [8, Theorem B]. For the right hand one, this comes from the fact that $\mathfrak{C}^{\overline{p}}_*(X)$ is a free module with finitely generated homology, see [20, Proof of Proposition 1.3] for instance. By applying the dual functor to this composition, we get the map

$$\mathscr{C}_{\overline{p}}\colon \mathfrak{C}^\star_{\overline{p}}(X) \to (DD\mathfrak{C}^*_{\overline{p}}(X))^\star \to (D\widetilde{N}^*_{\overline{p},c}(X))_{n-\star},$$

which is a quasi-isomorphism, since the homologies are finitely generated, see [20, Proposition 1.3].

For the second quasi-isomorphism, $\mathscr{N}_{\overline{p}}$, we decompose it as

$$\widetilde{N}^\star_{\overline{p}}(X) \to \mathfrak{C}^{\infty,\overline{p}}_\star(X) \to (D\mathfrak{C}^*_{\overline{p},c}(X))_{n-\star},$$

where the left-hand map is the duality of [24] (recalled in Proposition 1.11). Thus the proof is reduced to the study of the right-hand map. First, recall from [15] and [24, Proposition 2.2], that $\mathfrak{C}^*_{\overline{p},c}(X) = \varinjlim_K \mathfrak{C}^*_{\overline{p}}(X, X\backslash K) = \mathfrak{C}^*_{\overline{p}}(X, X\backslash\{(0,\mathtt{v})\})$ and $\mathfrak{C}^{\infty,\overline{p}}_*(X) = \varprojlim_K \mathfrak{C}^{\overline{p}}_*(X, X\backslash K) = \mathfrak{C}^*_{\overline{p}}(X, X\backslash\{(0,\mathtt{v})\})$. Thus, it is sufficient to prove the existence of a quasi-isomorphism,

$$\mathfrak{C}^{\overline{p}}_\star(X, X\backslash\{(0,\mathtt{v})\}) \to (D\mathfrak{C}^*_{\overline{p}}(X, X\backslash\{(0,\mathtt{v})\}))_{n-\star}.$$

As the complex $\mathfrak{C}^{\overline{p}}_k(X)$ is free with finitely generated homology, the evaluation map $\mathfrak{C}^{\overline{p}}_\star(X) \to (D\mathfrak{C}^*_{\overline{p}}(X))_{n-\star}$ is a quasi-isomorphism.

Replacing the subspace $X\backslash\{(0,\mathtt{v})\}$ by $\mathring{c}(S^{a-1} * L)\backslash\{\mathtt{u}\}$ as we do above, we also get a quasi-isomorphism $\mathfrak{C}^{\overline{p}}_k(X\backslash\{(0,\mathtt{v})\}) \to (D\mathfrak{C}^*_{\overline{p}}(X\backslash\{(0,\mathtt{v})\}))_{n-k}$. From a five lemma argument, we get that the map $\mathfrak{C}^{\infty,\overline{p}}_\star(X) \to (D\mathfrak{C}^*_{\overline{p},c}(X))_{n-\star}$ is a quasi-isomorphism. □

Proof of Theorem 2.2. We check the hypotheses of Lemma 2.4.

i) This is Lemma 2.5.

ii) We already know that each complex has a Mayer-Vietoris sequence. The fact that any of the maps under consideration induces a morphism of exact sequence comes from the natutality of the choice of the fundamental classes: for an open subset $U \subset X$, we may choose the restriction of a fixed cycle $\gamma_X \in \mathfrak{C}_n^{\infty,\overline{0}}(X)$ representing the fundamental class of X to define the fundamental class of U.

iii) This is a consequence of the fact that the duality D sends inductive limits to projective limits and of the following properties:
$\widetilde{N}^*_{\overline{p}}(\sqcup_i U_i) = \prod_i \widetilde{N}^*_{\overline{p}}(U_i)$, $\widetilde{N}^*_{\overline{p},c}(\sqcup_i U_i) = \oplus_i \widetilde{N}^*_{\overline{p},c}(U_i)$, $\mathfrak{C}^*_{\overline{p}}(\sqcup_i U_i) = \prod_i \mathfrak{C}^*_{\overline{p}}(U_i)$, and $\mathfrak{C}^*_{\overline{p},c}(\sqcup_i U_i) = \oplus_i \mathfrak{C}^*_{\overline{p},c}(U_i)$.

iv) This is immediate.

□

Remark 2.6. The two complexes, $\widetilde{N}^*_{\bullet}(-)$ and $\mathfrak{C}^*_{\bullet}(-)$, have elements of the same nature (they associate a number to chains) but have a different behaviour.

- In $\widetilde{N}^*_{\overline{p}}(-)$ a blown-up cochain is defined on each filtered simplex.
- The cochains in $\mathfrak{C}^*_{\overline{p}}(-)$ are defined only on the chains of $\overline{p}$-intersection. We can view them as relative cochains taking the value 0 on chains which are not of $\overline{p}$-intersection.

Viewing $\mathscr{H}^*_{\overline{p}}(-)$ as an absolute cohomology and $\mathfrak{H}^*_{\overline{p}}(-)$ as a relative one, the pairings coming from Theorem 2.2 look like the Poincaré-Lefschetz non-singular pairings of a compact oriented manifold with boundary, that is, by example for the torsion free part,

$$\mathrm{F}H^*(M, \partial M; R) \otimes \mathrm{F}H^{n-*}(M; R) \to R.$$

Remark 2.7. For sake of simplicity, we suppose that X is an oriented compact stratified pseudomanifold. In [8, Theorem B], we prove that the chain map defined by the cap product with a cycle γ_X representing the fundamental class, is a quasi-isomorphism,

$$- \cap \gamma_X \colon \widetilde{N}^\star_{\overline{p}}(X, R) \to \mathfrak{C}^{\overline{p}}_{n-\star}(X, R).$$

Theorem 2.2 shows that the composition with a certain dualization of the chain complex, in fact the Verdier dual of the linear dual, is a quasi-isomorphism as well,

$$\widetilde{N}^\star_{\overline{p}}(X; R) \to (D\mathfrak{C}^*_{\overline{p}}(X; R))_{n-\star},$$

making of the blown-up cochain complex a Verdier dual of $\mathfrak{C}^*_{\overline{p}}(X; R)$. In the next section, we are now looking for a duality involving only the blown-up cochains.

3. Poincaré duality with pairings

After defining the peripheral complex, we prove two main properties of it (see [18] in the case of compact PL-pseudomanifolds): its link with the occurrence of a duality in intersection homology and the existence of a duality on itself. In Proposition 3.9, we show that the locally torsion free condition, required by Goresky and Siegel (see Definition 3.6) is equivalent to a local acyclicity of the peripheral complex. Finally, we give an example of a stratified pseudomanifold which is not locally torsion free and has an acyclic peripheral complex, thus satisfies Poincaré duality.

A Poincaré duality on an oriented stratified pseudomanifold, X, similar to the duality on manifolds, should be the existence of a quasi-isomorphism

$$\mathscr{D}_{\overline{p}}\colon \widetilde{N}^{\star}_{\overline{p}}(X;R) \to D(\widetilde{N}^{*}_{D\overline{p},c}(X;R))_{n-\star}. \tag{3.1}$$

With Theorem 2.2, such map $\mathscr{D}_{\overline{p}}$ can be obtained from the composition of $\mathscr{C}_{D\overline{p}}$ with a quasi-isomorphism $\widetilde{N}^{*}_{\overline{p}}(X;R) \xrightarrow{\simeq} \mathfrak{C}^{*}_{D\overline{p}}(X;R)$. We introduce now such crucial map, already present in [7] and [6, Section 13].

Proposition 3.1. *Let $(X,\overline{p})$ be a perverse stratified pseudomanifold and $\varepsilon\colon (\mathfrak{C}^{\overline{t}}_{*}(X;R),\mathfrak{d}) \to (R,0)$ an augmentation. Then there is a cochain map,*

$$\chi_{\overline{p}}\colon \widetilde{N}^{*}_{\overline{p}}(X;R) \to \mathfrak{C}^{*}_{D\overline{p}}(X;R), \tag{3.2}$$

defined by $\chi_{\overline{p}}(\omega) = \varepsilon \star \omega$. We denote by $\chi_{\overline{p},c}\colon \widetilde{N}^{*}_{\overline{p},c}(X;R) \to \mathfrak{C}^{*}_{D\overline{p},c}(X;R)$ the restriction of $\chi_{\overline{p}}$ to the cochains with compact supports.

Proof. Let $\omega \in \widetilde{N}^{k}_{\overline{p}}(X;R)$ and $\xi \in \mathfrak{C}^{D\overline{p}}_{k}(X;R)$. With the notation of the statement, we observe from (1.4) that $\omega \frown \xi \in \mathfrak{C}^{\overline{t}}_{0}(X;R)$ and thus $\varepsilon(\omega \frown \xi)$ is well defined, To check the compatibility with the differentials, we apply ε at the two sides of (1.5). First, we have $\varepsilon(\mathfrak{d}(\omega \frown \xi)) = 0$ which implies,

$$\begin{aligned} 0 &= \varepsilon((d\omega) \frown \xi) + (-1)^{|\omega|}\varepsilon(\omega \frown (\mathfrak{d}\xi)) = \chi_{\overline{p}}(d\omega)(\xi) + (-1)^{|\omega|}\chi_{\overline{p}}(\omega)(\mathfrak{d}\xi) \\ &= \chi_{\overline{p}}(d\omega)(\xi) - \mathfrak{d}\chi_{\overline{p}}(\omega)(\xi)) \end{aligned}$$

and $\mathfrak{d}\chi_{\overline{p}}(\omega) = \chi_{\overline{p}}(d\omega)$. □

Corollary 3.2. *Let $(X,\overline{p})$ be a paracompact, separable and oriented perverse stratified pseudomanifold of dimension n and γ_X a representing cycle of the fundamental class $[X] \in H^{\infty,\overline{0}}_{n}(X;R)$. Then, the map*

$$\mathscr{D}_{\overline{p}}\colon \widetilde{N}^{\star}_{\overline{p}}(X;R) \to (D\widetilde{N}^{*}_{D\overline{p},c}(X;R))_{n-\star} \tag{3.3}$$

defined by $\mathscr{D}_{\overline{p}}(\omega)(\omega',\omega'') = ((\varepsilon \star \omega \star \omega')(\gamma_X),0)$ is a quasi-isomorphism if, and only if, the map $D\chi_{D\overline{p},c}$ is one also.

The torsion and torsion free pairings arising from $\mathscr{D}_{\overline{p}}$ are studied in Section 5.

Proof. With the notation of Proposition 2.1, the map $\mathscr{D}_{\overline{p}}$ is equal to the following composition,

$$\widetilde{N}^{\star}_{\overline{p}}(X) \xrightarrow{\mathscr{N}_{\overline{p}}} (D\mathfrak{C}^{*}_{\overline{p},c}(X))_{n-\star} \xrightarrow{D\chi_{D\overline{p},c}} (D\widetilde{N}^{*}_{D\overline{p},c}(X))_{n-\star}.$$

Thus the result is a consequence of Theorem 2.2. Let us also notice that $\mathscr{D}_{\overline{p}} = \mathscr{C}_{D\overline{p}} \circ \chi_{\overline{p}}$. □

In view of Corollary 3.2, the cofibers of $\chi_{\overline{p}}$ and $\chi_{\overline{p},c}$ in the category of cochain complexes play a fundamental role in Poincaré duality. We call them the *peripheral complexes.* (A brief analysis shows that they correspond to the global sections of the peripheral sheaf of [18].)

Definition 3.3. Let $(X,\overline{p})$ be a perverse stratified pseudomanifold. The *$\overline{p}$-peripheral complex* of X is the mapping cone of $\chi_{\overline{p}}\colon \widetilde{N}^*_{\overline{p}}(X;R) \to \mathfrak{C}^*_{D\overline{p}}(X;R)$; i.e.,

$$R^*_{\overline{p}}(X;R) = (\mathfrak{C}^*_{D\overline{p}}(X;R) \oplus \widetilde{N}^{*+1}_{\overline{p}}(X;R), D), \text{ with } D(c,\omega) = (dc + \chi_{\overline{p}}(\omega), -d\omega).$$

We denote by $\mathscr{R}^*_{\overline{p}}(X;R)$ the homology of $R^*_{\overline{p}}(X;R)$ and call it the *peripheral $\overline{p}$-intersection cohomology of X*. Similarly, we define $R^*_{\overline{p},c}(X;R)$ and $\mathscr{R}^*_{\overline{p},c}(X;R)$ from $\chi_{\overline{p},c}$.

If R is a field, the maps $\chi_{\overline{p}}$ and $\chi_{\overline{p},c}$ are quasi-isomorphisms, see [6, Theorem F] and [8, Proposition 2.23]. Therefore, the *peripheral cohomologies* $\mathscr{R}^*_{\overline{p}}(X;R)$ and $\mathscr{R}^*_{\overline{p},c}(X;R)$ *are R-torsion.* Also, from a classical argument, as $\widetilde{N}^*_{\overline{p}}(-;R)$ and $\mathfrak{C}^*_{\overline{p}}(-;R)$ have Mayer-Vietoris exact sequences, so does the peripheral complex.

The next result concerns the existence of a duality on the peripheral cohomology, $\mathscr{R}^*_\bullet(-;R)$, we follow the same way as in [18, Proposition 9.3]. (Let us also notice that this technique works in the general framework of a triangulated category, see [1, Theorem 1.6].)

Theorem 3.4 ([18])**.** *Let $(X,\overline{p})$ be a paracompact, separable and oriented perverse stratified pseudomanifold of dimension n and γ_X a representing cycle of the fundamental class $[X] \in H^{\infty,\overline{0}}_n(X;R)$. Then, there is a cochain map,*

$$\varphi_{\overline{p}}\colon R^\star_{\overline{p}}(X;R) \to (DR^*_{D\overline{p},c}(X;R))_{n-1-\star},$$

inducing an isomorphism in homology.

Proof. The various arrows of the following diagram are specified below.

$$\begin{tikzcd}
 & R^k_{\overline{p}}(X) \arrow[dl, "{[1]}"'] \arrow[dr, "\chi_{\overline{p}}"'] & \\
\widetilde{N}^k_{\overline{p}}(X) \arrow[rr] \arrow[d, "\mathscr{N}_{\overline{p}}"'] & & \mathfrak{C}^k_{D\overline{p}}(X) \arrow[dl, "\varphi_{\overline{p}}"] \arrow[d, "\mathscr{C}_{D\overline{p}}"] \\
(D\mathfrak{C}^{n-k}_{\overline{p},c}(X))_{n-k} \arrow[rr, "\chi^\sharp_{D\overline{p},c}"'] & (DR^*_{D\overline{p},c}(X))_{n-k-1} \arrow[ul] & (D\widetilde{N}^*_{D\overline{p},c}(X))_{n-k} \arrow[ul, "{-[1]}"']
\end{tikzcd} \tag{3.4}$$

The map $\chi_{\overline{p}}$ is recalled in (3.2) and $\chi^\sharp_{D\overline{p},c} = D\chi_{D\overline{p},c}$ is defined by duality. The two vertical maps of the front square are defined in Proposition 2.1. By construction, the front square commutes and induces the cochain map $\varphi_{\overline{p}}$. From Theorem 2.2 and the 5-lemma, we get that $\varphi_{\overline{p}}$ induces an isomorphism. □

Corollary 3.5. *Let $(X,\overline{p})$ be a paracompact, separable and oriented perverse stratified pseudomanifold. Then, the following conditions are equivalent.*

(1) The stratified pseudomanifold $(X,\overline{p})$ verifies Poincaré duality; i.e., the map $\mathscr{D}_{\overline{p}}$ is a quasi-isomorphism.
(2) The map $D\chi_{D\overline{p},c}$ is a quasi-isomorphism.
(3) The map $\chi_{\overline{p}}$ is a quasi-isomorphism

Proof. The equivalence of (1) and (2) is done in Corollary 3.2 and the equivalence of (2) and (3) comes from the commutativity of the front face of (3.4) and Theorem 2.2. □

This corollary means that $\mathscr{D}_{\overline{p}}$ is a quasi-isomorphism if, and only if, the peripheral complex $R^k_{\overline{p}}(X;R)$ is acyclic. In [18], Goresky and Siegel give a sufficient condition of acyclicity for the peripheral complex that we describe now.

First, let us observe that the two complexes, $\widetilde{N}^*_{\overline{p}}(X;R)$ and $\mathfrak{C}^*_{D\overline{p}}(X;R)$, are connected by a cochain map, have Mayer-Vietoris sequences, coincide on Euclidean spaces and have the same behaviour for disjoint union of open subsets. Therefore (see Lemma 2.4), the map $\chi_{\overline{p}}$ induces an isomorphism if it does on the products $\mathbb{R}^n \times \mathring{c}L$ where L is a compact stratified pseudomanifold. To exemplify this point, we first reduce to the particular case of a cone over a compact manifold, $X = \mathring{c}M$. Already known computations (see Example 6.1) show that in this case, the difference between the two cohomology groups is concentrated in one degree, where we have

$$\mathscr{H}^{\overline{p}(\mathrm{v})+1}_{\overline{p}}(\mathring{c}M;R) = 0 \text{ and } \mathfrak{H}^{\overline{p}(\mathrm{v})+1}_{D\overline{p}}(\mathring{c}M;R) = \mathrm{T}\mathfrak{H}_{\overline{p}(\mathrm{v})}(M;R). \tag{3.5}$$

Thus the lack of torsion in the homology of the manifold M, in this critical degree, is a necessary and sufficient condition for having an isomorphism between the two cohomologies $\mathscr{H}^*_{\overline{p}}(\mathring{c}M;R)$ and $\mathfrak{H}^*_{D\overline{p}}(\mathring{c}M;R)$. We examine now the general case.

First, observe that "the" link of a stratum is not uniquely determined but all the links of points lying in the same stratum have isomorphic intersection homology groups, see [11, Corollary 5.3.14]. Thus, for sake of simplicity, we use the expression *the link L_S of a stratum S* if only the intersection homology groups of the links appear, as in the following definition.

Definition 3.6 ([18])**.** A stratified pseudomanifold X is *locally $(D\overline{p},R)$-torsion free* if

$$\mathrm{T}\,\mathfrak{H}^{\overline{p}}_{D\overline{p}(S)}(L_S;R) = 0, \tag{3.6}$$

for each stratum S with associated link L_S.

As $\dim X = \dim L_S + \dim S + 1$, we have $D\overline{p}(S) = \overline{t}(S) - \overline{p}(S) = \dim L_S - \overline{p}(S) - 1$. From Poincaré duality, one can deduce (see for instance [11, Corollary 8.2.5]) that X is locally $(\overline{p},R)$-torsion free if, and only if, it is locally $(D\overline{p},R)$-torsion free. We therefore use them indifferently. Let us also notice that any open subset of a locally $(\overline{p},R)$-torsion free stratified pseudomanifold is a locally $(\overline{p},R)$-torsion free stratified pseudomanifold.

Proposition 3.7. *Let $(X,\overline{p})$ be a paracompact, separable, perverse, stratified pseudomanifold. If X is locally $(\overline{p},R)$-torsion free, then the maps $\chi_{\overline{p}}$ and $\chi_{\overline{p},c}$ induce isomorphisms,*

$$\chi^*_{\overline{p}}\colon \mathscr{H}^*_{\overline{p}}(X;R) \xrightarrow{\cong} \mathfrak{H}^*_{D\overline{p}}(X;R) \text{ and } \chi^*_{\overline{p},c}\colon \mathscr{H}^*_{\overline{p},c}(X;R) \xrightarrow{\cong} \mathfrak{H}^*_{D\overline{p},c}(X;R). \tag{3.7}$$

Proof. The assertion for $\chi^*_{\overline{p}}$ is proven in [6, Theorem F] and in [8, Proposition 2.23] for $\chi^*_{\overline{p},c}$. □

By using that the dual of a quasi-isomorphism is a quasi-isomorphism and Corollary 3.5, we deduce that a locally $(\overline{p},R)$-torsion free stratified pseudomanifold satisfies Poincaré duality and we recover [18, Theorem 4.4]. The reverse way is not true in general, as Example 6.7 shows.

Proposition 3.8. *There are examples of compact oriented stratified pseudomanifolds with a perversity $\overline{p}$, which are not locally $(\overline{p},R)$-torsion free and whose $\overline{p}$-intersection homology satisfies Poincaré duality.*

We complete this section with a characterization of the property "$(\overline{p}, R)$-torsion free" in terms of local acyclicity of the peripheral complex, which is equivalent to the nullity of the associated sheaf, considered in [18].

Proposition 3.9. *Let X be a compact oriented stratified pseudomanifold of dimension n and $\overline{p}$ a perversity. Then, the stratified pseudomanifold X is locally $(\overline{p}, R)$-torsion free if, and only if, $\mathscr{R}^*_{\overline{p}}(U; R) = 0$ for any open subset $U \subset X$.*

Proof. Suppose $\mathscr{R}^*_{\overline{p}}(U) = 0$ for any open subset U of X. We choose a conical chart $U = \mathbb{R}^{n-k} \times \mathring{c}L$. From Example 6.1, we observe that the condition $\mathscr{R}^*_{\overline{p}}(U) = 0$ implies

$$\mathscr{R}^*(\mathring{c}L) = \mathrm{T}\mathfrak{H}^{D\overline{p}}_{\overline{p}(\mathrm{v})}(L) = 0.$$

Therefore, the stratified pseudomanifold X is locally $(\overline{p}, R)$-torsion free.

We establish now the reverse way and suppose that the stratified pseudomanifold X is locally $(\overline{p}, R)$-torsion free. We apply Lemma 2.3 taking for $P(U)$ the property

"for any open subset V of U, we have $\mathscr{R}^*_{\overline{p}}(V) = 0$."

We proceed by induction on the depth of the stratified pseudomanifold, starting easily with the case of a manifold with empty singular set. The induction uses two steps.

• First, we prove $P(U)$ for any open subset U of a fixed conical chart $Y = \mathbb{R}^m \times \mathring{c}L$. This is obvious if $L = \emptyset$ therefore, we suppose $L \neq \emptyset$. We consider the following basis, $\mathcal{V}$, of open subsets V of U composed of subsets of two kinds:

- The open subsets V of U that do not contain the apex of $\mathring{c}L$. They are stratified pseudomanifolds of depth less than depth X and the induction hypothesis can be used.
- The open subsets $V = B \times \mathring{c}_\varepsilon L$, where $B \subset \mathbb{R}^m$ is an open cube, $\varepsilon > 0$ and $\mathring{c}_\varepsilon L = (L \times [0, \varepsilon[)/(L \times \{0\})$. The acyclicity of $R^*_{\overline{p}}(V)$ comes from the local $(D\overline{p}, R)$-torsion freeness of X, as at the beginning of this proof.

This family $\mathcal{V}$ is closed for finite intersections and satisfies the hypotheses of Lemma 2.3. We have just proved condition a). Property b) is a consequence of the existence of Mayer-Vietoris sequences and c) is direct. Thus, $P(U)$ is true.

• Finally, for establishing the property $P(X)$, we choose the open basis composed of open subsets of conical charts or regular open subsets and apply Lemma 2.3. Note that condition a) is proved in the first step. For b) and c), the arguments used for a conical chart apply also. □

Note that Example 6.7 is in accordance with Proposition 3.9. Here, conical charts are products, $]0, 1[\times \mathring{c}(S^1 \times S^1 \times \mathbb{R}P^3)$, that are not locally torsion free.

4. A relative complex

In this section, we take over the relative complex introduced by Friedman and Hunsicker [14], for locally torsion free compact PL-pseudomanifolds. We extend the properties given in loc. cit. to the case of an acyclic peripheral complex, with coefficients in R.

Let $(X, \overline{p})$ be a perverse space such that $\overline{p} \leq D\overline{p}$. We consider the homotopy cofiber sequence,

$$\widetilde{N}^*_{\overline{p}}(X; R) \to \widetilde{N}^*_{D\overline{p}}(X; R) \to \widetilde{N}_{D\overline{p}/\overline{p}}(X; R).$$

We call it the $(D\overline{p},\overline{p})$-relative complex (or relative complex if there is no ambiguity) and denote its homology by $\mathscr{H}^*_{D\overline{p}/\overline{p}}(X;R)$. Similarly, we consider the homotopy cofiber sequence

$$\mathfrak{C}^*_{D\overline{p},c}(X;R) \to \mathfrak{C}^*_{\overline{p},c}(X;R) \to \mathfrak{C}^*_{\overline{p}/D\overline{p},c}(X;R).$$

In [14, Lemma 3.7], G. Friedman and E. Hunsicker also introduce relative complexes for intersection homology with rational coefficients of compact PL-pseudomanifolds. Their general purpose is the extension of Novikov additivity and Wall non-additivity in the case $R=\mathbb{Q}$, for $4n$-dimensional PL-pseudomanifolds.

Proposition 4.1. *Let $(X,\overline{p})$ be a paracompact, separable and oriented perverse stratified pseudomanifold of dimension n with $\overline{p}\leq D\overline{p}$. Then there is a quasi-isomorphism*

$$\psi_{\overline{p}}\colon \widetilde{N}^{\star}_{D\overline{p}/\overline{p}}(X;R) \to (D\mathfrak{C}^*_{\overline{p}/D\overline{p},c}(X;R))_{n-\star-1}.$$

Proof. We introduce a diagram, as in the proof of Theorem 3.4,

$$\begin{tikzcd}
 & \widetilde{N}^k_{D\overline{p}/\overline{p}}(X) \arrow[dl, "{[1]}"'] \arrow[dr, "\psi_{\overline{p}}"] & \\
\widetilde{N}^k_{\overline{p}}(X) \arrow[rr] \arrow[d, "\mathscr{N}_{\overline{p}}"'] & (D\mathfrak{C}^*_{\overline{p}/D\overline{p},c}(X))_{n-k-1} \arrow[dl] & \widetilde{N}^k_{D\overline{p}}(X) \arrow[ul] \arrow[d, "\mathscr{N}_{D\overline{p}}"] \\
(D\mathfrak{C}^*_{\overline{p},c}(X))_{n-k} \arrow[rr] & & (D\mathfrak{C}^*_{D\overline{p},c}(X))_{n-k}. \arrow[ul, "-[1]"']
\end{tikzcd} \tag{4.1}$$

The two vertical maps of the front face are quasi-isomorphisms. They induce, the back vertical arrow, $\psi_{\overline{p}}$, which is also a quasi-isomorphism. □

By construction, we have $\psi_{\overline{p}}(\omega)(c,c') = ((dc\star\omega + (-1)^{|c|}c\star d\omega)(\gamma_X),0)$, where γ_X is a cycle representing the fundamental class.

Corollary 4.2. *Let $(X,\overline{p})$ be a paracompact, separable and oriented perverse stratified pseudomanifold of dimension n such that $\chi_{\overline{p}}$ and $\chi_{\overline{p},c}$ are quasi-isomorphisms. Denote by $\widetilde{N}^*_{D\overline{p}/\overline{p},c}(X;R)$ the cofiber of $\widetilde{N}^*_{\overline{p},c}(X;R)\to\widetilde{N}^*_{D\overline{p},c}(X;R)$. Then there is a quasi-isomorphism*

$$\Psi_{\overline{p}}\colon \widetilde{N}^{\star}_{D\overline{p}/\overline{p}}(X;R) \to (D\widetilde{N}^*_{D\overline{p}/\overline{p},c}(X;R))_{n-\star-1}.$$

As in Proposition 1.12, such quasi-isomorphism induces non-singular pairings for the torsion and the torsion free parts of the homology of $\widetilde{N}^*_{D\overline{p}/\overline{p}}(X;R)$. In the compact PL-case, the previous statement corresponds to the duality obtained in [14]. In contrast with the peripheral complex of the previous section, the homology of this relative complex is not entirely torsion, see Example 6.8.

Proof. Let us observe that the maps $\chi_{\overline{p},c}$ and $\chi_{D\overline{p},c}$ induce a map

$$\chi_{D\overline{p}/\overline{p},c}\colon \widetilde{N}^*_{D\overline{p}/\overline{p},c}(X;R) \to \mathfrak{C}^*_{\overline{p}/D\overline{p},c}(X;R).$$

As $\chi_{\overline{p},c}$ is a quasi-isomorphism, its dual $D\chi_{\overline{p},c}$ is one also. On the other hand, with Corollary 3.5, as $\chi_{\overline{p}}$ is a quasi-isomorphism, then $D\chi_{D\overline{p},c}$ is one also. Therefore, with the five lemma, we deduce that $D\chi_{D\overline{p}/\overline{p},c}$ is a quasi-isomorphism. In conclusion, the composition $D\chi_{D\overline{p}/\overline{p},c}\circ\psi_{\overline{p}}$ is the quasi-isomorphism $\Psi_{\overline{p}}$. □

5. Components of the peripheral complex

In this section, we study the peripheral complex in the compact case. It is constituted of "three components" coming from the torsion and torsion free parts of the two cohomologies defining it. They correspond to failures of the existence of non-singular torsion or torsion free Poincaré pairings for intersection homology and blown-up cohomology.

If X is compact, the map $\mathscr{D}_{\overline{p}}\colon \widetilde{N}^k_{\overline{p}}(X;R) \to (D\widetilde{N}^*_{D\overline{p}}(X;R))_{n-k}$ generates two pairings,

$$\Phi_{\overline{p}}\colon \mathrm{F}\mathscr{H}^k_{\overline{p}}(X;R) \otimes \mathrm{F}\mathscr{H}^{n-k}_{D\overline{p}}(X;R) \to R \tag{5.1}$$

and

$$L_{\overline{p}}\colon \mathrm{T}\mathscr{H}^k_{\overline{p}}(X;R) \otimes \mathrm{T}\mathscr{H}^{n+1-k}_{D\overline{p}}(X;R) \to QR/R. \tag{5.2}$$

For sake of simplicity, we call $\Phi_{\overline{p}}$ the *Poincaré torsion free pairing* and $L_{\overline{p}}$ the *Poincaré torsion pairing.* Let us also observe that any of the isomorphisms of Proposition 1.11 allows the replacement of $\mathscr{H}^k_{\overline{p}}(-)$ by $\mathfrak{H}^{\overline{p}}_{n-k}(-)$, giving pairings of the intersection homology itself. If $\chi_{\overline{p}}$ is a quasi-isomorphism, these two pairings are non-singular. In this section, we are looking for sufficient conditions suitable for one of them to be non-singular.

`Components of the peripheral complex.` From $\chi^*_{\overline{p}}\colon \mathscr{H}^*_{\overline{p}}(X;R) \to \mathfrak{H}^*_{D\overline{p}}(X;R)$, we construct, by restriction and projection, a morphism of exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{T}\mathscr{H}^*_{\overline{p}}(X;R) & \longrightarrow & \mathscr{H}^*_{\overline{p}}(X;R) & \longrightarrow & F\mathscr{H}^*_{\overline{p}}(X;R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \chi^*_{\overline{p},T}} & & \downarrow{\scriptstyle \chi^*_{\overline{p}}} & & \downarrow{\scriptstyle \chi^*_{\overline{p},F}} & & \\ 0 & \longrightarrow & \mathrm{T}\mathfrak{H}^*_{D\overline{p}}(X;R) & \longrightarrow & \mathfrak{H}^*_{D\overline{p}}(X;R) & \longrightarrow & F\mathfrak{H}^*_{D\overline{p}}(X;R) & \longrightarrow & 0. \end{array} \tag{5.3}$$

As $\chi^*_{\overline{p}} \otimes QR$ is an isomorphism, the map $\chi^*_{\overline{p},F}$ is injective and $\operatorname{Coker}\chi^*_{\overline{p},F}$ is entirely torsion. Therefore, we can define.

$$\begin{aligned} \mathscr{F}^*_{\overline{p}}(X;R) &= \operatorname{Coker}\chi^*_{\overline{p},F} \\ \mathscr{T}^*_{\overline{p},\mathrm{C}}(X;R) &= \operatorname{Coker}\chi^*_{\overline{p},\mathrm{T}} \text{ and } \mathscr{T}^*_{\overline{p},\mathrm{K}}(X;R) = \operatorname{Ker}\chi^*_{\overline{p},\mathrm{T}} \cong \operatorname{Ker}\chi^*_{\overline{p}}. \end{aligned}$$

As first observation, we deduce from the Ker-Coker Lemma applied to (5.3) the short exact sequences,

$$0 \longrightarrow \mathscr{T}^*_{\overline{p},\mathrm{C}}(X;R) \longrightarrow \operatorname{Coker}\chi^*_{\overline{p}} \longrightarrow \mathscr{F}^*_{\overline{p}}(X;R) \longrightarrow 0. \tag{5.4}$$

By definition of the peripheral complex, we also have short exact sequences,

$$0 \to \operatorname{Coker}\chi^*_{\overline{p}} \to \mathscr{R}^*_{\overline{p}}(X;R) \to \operatorname{Ker}\chi^{*+1}_{\overline{p}} \to 0. \tag{5.5}$$

Observe from these two series of sequences that $\mathscr{T}^*_{\overline{p},\mathrm{C}}(X;R)$ is a submodule of $\mathscr{R}^*_{\overline{p}}(X;R)$.

Proposition 5.1. *Let $(X,\overline{p})$ be a compact perverse stratified pseudomanifold. Then, there exists an exact sequence:*

$$0 \to \mathscr{F}^*_{\overline{p}}(X;R) \to \mathscr{R}^*_{\overline{p}}(X;R)/\mathscr{T}^*_{\overline{p},\mathrm{C}}(X;R) \to \mathscr{T}^{*+1}_{\overline{p},\mathrm{K}}(X;R) \to 0. \tag{5.6}$$

Proof. The proof follows directly from the commutative diagram of exact sequences,

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{F}^*_{\overline{p}}(X;R) \arrow[r] \arrow[u] & \mathscr{R}^*_{\overline{p}}(X;R)/\mathscr{T}^*_{\overline{p},\mathrm{C}}(X;R) \arrow[r] \arrow[u] & \mathscr{T}^{*+1}_{\overline{p},\mathrm{K}}(X;R) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \operatorname{Coker}\chi_{\overline{p}} \arrow[r] \arrow[u] & \mathscr{R}^*_{\overline{p}}(X;R) \arrow[r] \arrow[u] & \mathscr{T}^{*+1}_{\overline{p},\mathrm{K}}(X;R) \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & \mathscr{T}^*_{\overline{p},\mathrm{C}}(X;R) \arrow[r, equal] \arrow[u] & \mathscr{T}^*_{\overline{p},\mathrm{C}}(X;R) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{5.7}$$

where the first column is (5.4) and the middle row (5.5). □

We continue by establishing the existence of a non-singular pairing between the two components coming from the restriction of $\chi^*_{\overline{p}}$ to the torsion submodules, $\mathscr{T}^*_{\overline{p},\mathrm{C}}(X;R)$ and $\mathscr{T}^*_{\overline{p},\mathrm{K}}(X;R)$.

Proposition 5.2. *Let $(X,\overline{p})$ be an oriented compact perverse stratified pseudomanifold. Then, there is a non-singular pairing,*

$$\mathscr{K}_{\overline{p}}\colon \mathscr{T}^k_{\overline{p},\mathrm{K}}(X;R)\otimes \mathscr{T}^{n+1-k}_{D\overline{p},\mathrm{C}}(X;R)\to QR/R.$$

Proof. Consider the following commutative diagram, whose columns are exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathscr{T}^k_{\overline{p},\mathrm{K}}(X;R) \arrow[r] \arrow[d] & \operatorname{Hom}(\mathscr{T}^{n+1-k}_{D\overline{p},\mathrm{C}}(X;R),QR/R) \arrow[d] \\
\mathrm{T}\mathscr{H}^k_{\overline{p}}(X;R) \arrow[r, "\mathfrak{D}'_{\overline{p},\mathrm{T}}", "\cong"'] \arrow[d, "\chi^*_{\overline{p},T}"'] & \operatorname{Hom}(\mathrm{T}\mathfrak{H}^{n+1-k}_{\overline{p}}(X;R),QR/R) \arrow[d, "(\chi^*_{D\overline{p},T})^\sharp"] \\
\mathrm{T}\mathfrak{H}^k_{D\overline{p}}(X;R) \arrow[r, "\mathfrak{D}''_{\overline{p},\mathrm{T}}", "\cong"'] \arrow[d] & \operatorname{Hom}(\mathrm{T}\mathscr{H}^{n+1-k}_{D\overline{p}}(X;R),QR/R) \arrow[d] \\
\mathscr{T}^k_{\overline{p},\mathrm{C}}(X;R) \arrow[r] \arrow[d] & \operatorname{Hom}(\mathscr{T}^{n+1-k}_{D\overline{p},\mathrm{K}}(X;R),QR/R) \arrow[d] \\
0 & 0.
\end{tikzcd}$$

Above, the maps $\mathfrak{D}'_{\overline{p},\mathrm{T}}$ and $\mathfrak{D}''_{\overline{p},\mathrm{T}}$ are the isomorphisms of the torsion pairing associated via Proposition 1.12 to the dualities $\mathscr{N}_{\overline{p}}$ and $\mathscr{C}_{D\overline{p}}$ of Theorem 2.2. The left-hand column is exact by construction and the right-hand one also, since QR/R is injective. As $\mathfrak{D}'_{\overline{p},\mathrm{T}}$ and $\mathfrak{D}''_{\overline{p},\mathrm{T}}$ are isomorphisms, the result follows. □

`Poincaré torsion and torsion free pairings.` As observed before, if the peripheral intersection cohomology vanishes, we have two non-singular pairings (5.1) and (5.2). We study now the existence of one of these two dualities, independently of the other one. Proposition 1.12, Corollary 3.2 and Corollary 3.5 give directly the following observations.

Proposition 5.3. *Let X be a compact oriented stratified pseudomanifold of dimension n and $\overline{p}$ be a perversity.*

1) The non-degenerate torsion free pairing

$$\Phi_{\overline{p}}\colon F\mathscr{H}_k^{\overline{p}}(X;R)\otimes F\mathscr{H}_{n-k}^{D\overline{p}}(X;R)\to R$$

is non-singular if, and only if, $\mathscr{F}_{\overline{p}}^*(X;R)=\operatorname{Coker}\chi_{\overline{p},F}^*=0$.

2) The torsion pairing

$$L_{\overline{p}}\colon \mathrm{T}\mathscr{H}_k^{\overline{p}}(X;R)\otimes \mathrm{T}\mathscr{H}_{n+1-k}^{D\overline{p}}(X;R)\to QR/R$$

can be degenerate and is non-singular if, and only if, $\mathscr{T}_{\overline{p},\mathrm{C}}^*(X;R)=\mathscr{T}_{\overline{p},\mathrm{K}}^*(X;R)=0$, which is also equivalent to $\mathscr{R}_{\overline{p}}^*=\mathscr{F}_{\overline{p}}^*(X;R)=\operatorname{Coker}\chi_{\overline{p},F}^*$.

Remark 5.4. In the previous statement, the different possibilities can occur.

- In Example 6.3, the torsion free pairing is non-singular and the torsion pairing is degenerate.
- In Example 6.4, the torsion free pairing is singular and the torsion pairing is non-singular.
- In Example 6.5, the torsion free pairing is singular and the torsion pairing is degenerate.

`Poincaré duality for intersection homology.` Let X be a compact oriented PL-pseudomanifold of dimension n and $\overline{p}$ be a GM-perversity. As recalled in the introduction, Goresky and MacPherson ([16]) proved that the intersection pairing defined on the $\overline{p}$-intersection homology,

$$\pitchfork\colon H_*^{\overline{p}}(X;\mathbb{Q})\otimes H_{n-*}^{D\overline{p}}(X;\mathbb{Q})\to\mathbb{Q},$$

is non-singular. This duality has been extended over $\mathbb{Z}$ in (0.3) and (0.4) by Goresky and Siegel ([18]) in the case locally $\overline{p}$-torsion free.

The existence ([8]) of the isomorphism $-\frown[X]\colon \mathscr{H}_{\overline{p}}^k(X;R)\xrightarrow{\cong}H_{n-k}^{\overline{p}}(X;R)$ allows the definition of an "intersection product" defined on the intersection homology of a topological stratified pseudomanifold from the commutativity of the following diagram.

$$\begin{array}{ccc}
\mathscr{H}_{\overline{p}}^k(X;R)\otimes\mathscr{H}_{\overline{q}}^\ell(X;R) & \xrightarrow{-\cup-} & \mathscr{H}_{\overline{p}+\overline{q}}^{k+\ell}(X;R)\\
{\scriptstyle -\frown[X]\otimes-\frown[X]}\downarrow\cong & & \cong\downarrow{\scriptstyle -\frown[X]}\\
H_{n-k}^{\overline{p}}(X;R)\otimes H_{n-\ell}^{\overline{q}}(X;R) & \xrightarrow{-\pitchfork-} & H_{n-k-\ell}^{\overline{p}+\overline{q}}(X;R).
\end{array}$$

With this structure, the map $\mathscr{D}_{\overline{p}}$ of (3.3) and Proposition 1.12 give two pairings:

$$\mathscr{G}_{\overline{p},\mathrm{F}}\colon \mathrm{F}H_k^{\overline{p}}(X;R)\otimes\mathrm{F}H_{n-k}^{D\overline{p}}(X;R)\to R \tag{5.8}$$

and

$$\mathscr{G}_{\overline{p},T}\colon \mathrm{T}H_k^{\overline{p}}(X;R)\otimes\mathrm{T}H_{n-k-1}^{D\overline{p}}(X;R)\to QR/R, \tag{5.9}$$

which are non-singular if, and only if, the peripheral complex is acyclic. The previous results on components of the peripheral cohomology can also be translated here through the duality map $- \frown [X]$. In particular, the torsion pairing may be non-singular even if the stratified pseudomanifold X is not $(D\overline{p}, R)$-locally torsion free (see Example 6.7) or even if the peripheral term $\mathscr{R}^*_{\overline{p}}(X;R)$ is not trivial (see Example 6.4).

6. Examples

This section contains references and details on the examples appearing in the text. The most significant example is Example 6.7 which presents a compact stratified pseudomanifold which is not locally $\overline{p}$-torsion free but whose intersection homology satisfies Poincaré duality.

In the case of isolated singularities on an n-dimensional stratified pseudomanifold, $n \geq 2$, a GM-perversity $\overline{p}$ is defined by the natural number $\overline{p}(n) = k$; we denote it by $\overline{k}$.

Example 6.1 (*Cone on a pseudomanifold*)**.** Let L be an $(n-1)$-dimensional compact stratified pseudomanifold. Recall the computations [6, Theorem E] and [11, Proposition 7.1.5],

$$\mathscr{H}^j_{\overline{p}}(\mathring{c}L;R) = \begin{cases} \mathscr{H}^j_{\overline{p}}(L;R) & \text{if} \quad j \leq \overline{p}(\mathtt{v}), \\ 0 & \text{if} \quad j > \overline{p}(\mathtt{v}), \end{cases} \tag{6.1}$$

$$\mathfrak{H}^j_{D\overline{p}}(\mathring{c}L;R) = \begin{cases} \mathfrak{H}^j_{D\overline{p}}(L;R) & \text{if} \quad j \leq \overline{p}(\mathtt{v}), \\ \mathrm{Ext}(\mathfrak{H}^{D\overline{p}}_{j-1}(L;R),R) & \text{if} \quad j = \overline{p}(\mathtt{v})+1, \\ 0 & \text{if} \quad j > \overline{p}(\mathtt{v})+1. \end{cases} \tag{6.2}$$

Moreover, we have also (see [8, Proposition 2.18]),

$$\mathscr{H}^j_{\overline{p},c}(\mathring{c}L;R) = \begin{cases} \mathscr{H}^{j-1}_{\overline{p}}(L;R) & \text{if} \quad j \geq \overline{p}(\mathtt{v})+2, \\ 0 & \text{if} \quad j < \overline{p}(\mathtt{v})+2. \end{cases} \tag{6.3}$$

From (2.1), (6.2) and the exact sequence associated to the pair $(\mathring{c}L, \mathring{c}L\backslash\{\mathtt{v}\})$, we deduce,

$$\mathfrak{H}^j_{D\overline{p},c}(\mathring{c}L;R) = \begin{cases} \mathfrak{H}^{j-1}_{D\overline{p}}(L;R) & \text{if} \quad j \geq \overline{p}(\mathtt{v})+3, \\ \mathrm{F}\mathfrak{H}^{j-1}_{D\overline{p}}(L;R) & \text{if} \quad j = \overline{p}(\mathtt{v})+2, \\ 0 & \text{if} \quad j < \overline{p}(\mathtt{v})+2. \end{cases} \tag{6.4}$$

From (6.3) and (6.4), we get,

$$\mathscr{R}^j_{\overline{p}}(\mathring{c}L;R) = \begin{cases} \mathscr{R}^j_{\overline{p}}(L;R) & \text{if} \quad j \leq \overline{p}(\mathtt{v})-1, \\ \mathrm{Coker}\left\{\chi_{\overline{p}}\colon \mathscr{H}^j_{\overline{p}}(L;R) \to \mathfrak{H}^j_{D\overline{p}}(L;R)\right\} & \text{if} \quad j = \overline{p}(\mathtt{v}), \\ \mathrm{T}\mathfrak{H}^j_{D\overline{p}}(L;R) & \text{if} \quad j = \overline{p}(\mathtt{v})+1, \\ 0 & \text{if} \quad j \geq \overline{p}(\mathtt{v})+2. \end{cases} \tag{6.5}$$

In the particular case of an oriented, compact *manifold* M, we get the peripheral complexes,

- $\mathscr{R}^*_{\overline{p}}(\mathring{c}M;R) = \mathscr{R}^{\overline{p}(\mathtt{v})+1}_{\overline{p}}(\mathring{c}M;R) = \mathrm{Ext}(H_{\overline{p}(\mathtt{v})}(M;R),R) = \mathrm{T}H^{\overline{p}(\mathtt{v})+1}(M;R)$,
- $\mathscr{R}^*_{\overline{p},c}(\mathring{c}M;R) = \mathscr{R}^{\overline{p}(\mathtt{v})+1}_{\overline{p},c}(\mathring{c}M;R) = \mathrm{Ext}(H_{\overline{p}(\mathtt{v})}(M;R),R) = \mathrm{T}H^{\overline{p}(\mathtt{v})+1}(M;R)$.

Let M be $(n-1)$-dimensional. Observe that $(\overline{p}(\mathtt{v})+1)+(D\overline{p}(\mathtt{v})+1)=n$. Thus, the non-singular pairing of the torsion part of the peripheral cohomology (see Theorem 3.4) corresponds to the classical Poincaré duality of the manifold M,

$$\mathrm{T}H^{\overline{p}(\mathtt{v})+1}(M;R)\otimes \mathrm{T}H^{D\overline{p}(\mathtt{v})+1}(M;R)\to QR/R.$$

Moreover, the condition "locally $(\overline{p},R)$-torsion free" of Definition 3.6 for $\mathring{\mathtt{c}}M$ is exactly what we need for having an acyclic peripheral complex and thus a non-singular pairing in blown-up intersection cohomology, since

$$\mathrm{T}\mathfrak{H}^{\overline{p}}_{D\overline{p}(\mathtt{v})}(M;R)=\mathrm{T}H_{n-2-\overline{p}(\mathtt{v})}(M;R)\cong \mathrm{T}H_{\overline{p}(\mathtt{v})}(M;R)=\mathrm{T}H^{\overline{p}(\mathtt{v})+1}(M;R)=\mathscr{R}^*_{\overline{p}}(\mathring{\mathtt{c}}M;R).$$

Example 6.2 (*Isolated singularities*)**.** Let X be a stratified pseudomanifold of dimension n with isolated singularities Σ. Let $\mathbb{R}^m\times\mathring{\mathtt{c}}L^a$ be a conical chart for any singularity $a\in\Sigma$ and set $U=\cup_{a\in\Sigma}\mathbb{R}^m\times\mathring{\mathtt{c}}L^a$. As there is no singularity on $V=X\backslash\Sigma$, the peripheral and compact peripheral cohomologies of V and $U\cap V$ are reduced to 0. From the Mayer-Vietoris sequences, we get

$$\mathscr{R}^*_{\overline{p}}(X;R)=\mathscr{R}^{\overline{p}(a)+1}_{\overline{p}}(X;R)=\oplus_{a\in\Sigma}\mathscr{R}^{\overline{p}(a)+1}_{\overline{p}}(\mathring{\mathtt{c}}L^a;R)=\oplus_{a\in\Sigma}\mathrm{T}H^{\overline{p}(a)+1}(L^a;R)$$

and

$$\mathscr{R}^*_{\overline{p},c}(X;R)=\mathscr{R}^{\overline{p}(a)+1}_{\overline{p},c}(X;R)=\oplus_{a\in\Sigma}\mathscr{R}^{\overline{p}(a)+1}_{\overline{p},c}(\mathring{\mathtt{c}}L^a;R)=\oplus_{a\in\Sigma}\mathrm{T}H^{\overline{p}(a)+1}(L^a;R).$$

Example 6.3 (*Non-singular torsion free pairing with degenerate torsion pairing*)**.** Let M be an oriented compact manifold. From the previous example, we deduce:

$$\mathscr{R}^*_{\overline{p}}(\Sigma M;R)=\mathscr{R}^*_{\overline{p}}(\mathring{\mathtt{c}}M;R)\oplus\mathscr{R}^*_{\overline{p}}(\mathring{\mathtt{c}}M;R)=\mathrm{T}H^{\overline{p}(\mathtt{v})+1}(M;R)\oplus\mathrm{T}H^{\overline{p}(\mathtt{v})+1}(M;R).$$

We also have

$$\mathscr{T}^*_{\overline{p},C}(\Sigma M;R)=\mathscr{T}^{\overline{p}(\mathtt{v})+1}_{\overline{p},C}(\Sigma M;R)=\mathrm{T}H^{\overline{p}(\mathtt{v})+1}(M;R)$$

and

$$\mathscr{T}^*_{\overline{p},K}(\Sigma M;R)=\mathscr{T}^{\overline{p}(\mathtt{v})+2}_{\overline{p},K}(\Sigma M;R)=\mathrm{T}H^{\overline{p}(\mathtt{v})+1}(M;R).$$

Thus, here, the duality of Proposition 5.2 is given by the Poincaré duality on the manifold M. Moreover, as $\mathscr{F}^*_{\overline{p}}(X;R)=0$, *the torsion free pairing* $\Phi_{\overline{p}}$ *of (5.1) is non-singular for any perversity.* Let us consider two examples where the torsion pairing $L_{\overline{p}}$ of (5.2) is degenerate.

a) Consider $X=\Sigma\mathbb{R}P^3$, $R=\mathbb{Z}$ and $\overline{p}=\overline{1}=D\overline{p}$ (the middle perversity), we have

$$\mathscr{R}^*_{\overline{1}}(X;\mathbb{Z})=\mathscr{T}^3_{\overline{1},K}(X;\mathbb{Z})\oplus\mathscr{T}^2_{\overline{1},C}(X;\mathbb{Z})=\mathbb{Z}_2\oplus\mathbb{Z}_2.$$

b) Consider $X=\Sigma(S^1\times S^1\times\mathbb{R}P^3)$, $R=\mathbb{Z}$ and $\overline{p}=\overline{2}=D\overline{p}$ (the middle perversity), we have

$$\mathscr{R}^*_{\overline{2}}(X;\mathbb{Z})=\mathscr{T}^4_{\overline{2},K}(X;\mathbb{Z})\oplus\mathscr{T}^3_{\overline{2},C}(X;\mathbb{Z})=\mathbb{Z}_2^2\oplus\mathbb{Z}_2^2.$$

Here, in contrast with a), the torsion free pairing is non-trivial.

The next example is an illustration of a peripheral cohomology which comes from the torsion free part of the map $\chi_{\overline{p}}$.

Example 6.4 (*Singular torsion free pairing with non-singular torsion pairing*)**.** We present two examples of Thom space built from the circle space of a manifold B relatively to an Euler class e.

a) We choose $B = S^2$, $R = \mathbb{Z}$, $\overline{p} = D\overline{p} = \overline{1}$ and $e = 2w$ where $w \in H^2(S^2;\mathbb{Z})$ is a generator. This example has been described in [8, Example 4.10] by using the Thom isomorphism and the Gysin sequence. We deduce from this reference that the map $\chi_{\overline{1}}\colon \mathscr{H}^2_{\overline{1}}(X;\mathbb{Z}) = \mathbb{Z} \to \mathfrak{H}^2_{\overline{1}}(X;\mathbb{Z}) = \mathbb{Z}$ is the multiplication by 2. Since both cohomologies are abelian free groups, we have

$$\mathscr{R}^*_{\overline{1}}(X;\mathbb{Z}) = \mathscr{R}^2_{\overline{1}}(X;\mathbb{Z}) = \mathscr{F}^2_{\overline{1}}(X;\mathbb{Z}) = \mathbb{Z}_2.$$

Thus, *the torsion pairing $L_{\overline{1}}$ of (5.2) is non-singular and the torsion free pairing $\Phi_{\overline{1}}$ of (5.1) is singular.*

b) We choose $B = \mathbb{R}P^3 \times \mathbb{C}P^2 \times S^1$, $R = \mathbb{Z}$, $\overline{p} = D\overline{p} = \overline{4}$ and $e = (\alpha, 3\omega, 0)$, where $\alpha \in H^2(\mathbb{R}P^3;\mathbb{Z})$ and $\omega \in H^2(\mathbb{C}P^2;\mathbb{Z})$ are generators. We have,

$$\mathscr{R}^*_{\overline{4}}(X;\mathbb{Z}) = \mathscr{R}^5_{\overline{4}}(X;\mathbb{Z}) = \mathscr{F}^5_{\overline{4}}(X;\mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_3.$$

We leave the details to the reader. Here, in contrast with a), we have a non-trivial torsion pairing.

For the suspension of $\mathbb{R}P^3$, the peripheral cohomology comes entirely from the torsion in cohomology; in other words, this is a case where the sequence in [18, Remark 9.2.(3)] is exact. This is not the case in the following example.

Example 6.5 (*Singular torsion free pairing with degenerate torsion pairing*)**.** We consider the Thom space built from the circle bundle over $S^2 \times \mathbb{R}P^3 \times S^3$, relatively to the Euler class $e = (3\omega, a, 0)$, where $\omega \in H^2(S^2;\mathbb{Z})$ and $a \in H^2(\mathbb{R}P^3;\mathbb{Z})$ are generators. We choose $R = \mathbb{Z}$, $\overline{p} = D\overline{p} = \overline{4}$. With the same process than in Example 6.4 a), we prove that $\chi^*_{\overline{4}}\colon \mathscr{H}^k_{\overline{4}}(X;\mathbb{Z}) \cong \mathfrak{H}^k_{\overline{4}}(X;\mathbb{Z})$, if $k \neq 5$, 6, and that

$$\chi^*_{\overline{4}}\colon \mathscr{H}^5_{\overline{4}}(X;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \to \mathfrak{H}^5_{\overline{4}}(X;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$$

is defined by $\chi^*_{\overline{4}}(a,b) = (3a, 3b, \overline{b})$. Finally, we have $\chi^*_{\overline{4}}\colon \mathscr{H}^6_{\overline{4}}(X;\mathbb{Z}) = \mathbb{Z}_2 \to \mathfrak{H}^6_{\overline{4}}(X;\mathbb{Z}) = 0$. We compute

$$\mathscr{R}^5_{\overline{4}}(X;\mathbb{Z}) = \mathbb{Z}_6 \oplus \mathbb{Z}_6.$$

If we go deeper in the torsion and the torsion free parts of $\chi^*_{\overline{4}}$, we get

$$\begin{array}{rclcl}
\mathscr{F}^*_{\overline{4}}(X;\mathbb{Z}) & = & \mathscr{F}^5_{\overline{4}}(X;\mathbb{Z}) & = & \mathbb{Z}_3 \oplus \mathbb{Z}_3, \\
\mathscr{T}^*_{\overline{4},K}(X;\mathbb{Z}) & = & \mathscr{T}^6_{\overline{4},K}(X;\mathbb{Z}) & = & \mathbb{Z}_2, \\
\mathscr{T}^*_{\overline{4},C}(X;\mathbb{Z}) & = & \mathscr{T}^5_{\overline{4},C}(X;\mathbb{Z}) & = & \mathbb{Z}_2.
\end{array}$$

Here, *the torsion free pairing $\Phi_{\overline{4}}$ is singular and the torsion pairing $L_{\overline{4}}$ is degenerate. Moreover, the exact sequence (5.6) is non-trivial:* $0 \to \mathbb{Z}_3 \oplus \mathbb{Z}_3 \to (\mathbb{Z}_6 \oplus \mathbb{Z}_6)/\mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$.

In the previous examples, the peripheral cohomology is non-trivial and the stratified pseudomanifold is not locally $\overline{p}$-torsion free. The general situation can be more elaborate. We first study the peripheral cohomology of the suspension of a stratified homeomorphism of a stratified pseudomanifold, see (6.6). Next, we give a specific example of a non-locally $\overline{p}$-torsion free stratified pseudomanifold with trivial peripheral cohomology.

Example 6.6. [*Suspension of a stratum-preserving homeomorphism.*] Let $(L,\overline{p})$ be a stratified pseudomanifold and $f\colon L\to L$ a stratified homeomorphism, cf. [6, Definition 1.5]. It induces homomorphisms, $f^*\colon \mathscr{H}^*_{\overline{p}}(L;R)\to\mathscr{H}^*_{\overline{p}}(L;R)$ and $f^*\colon \mathfrak{H}^*_{D\overline{p}}(L;R)\to\mathfrak{H}^*_{D\overline{p}}(L;R)$ (see [6, Proposition 3.5] and [10, Proposition 3.11]) and therefore $f^*\colon \mathscr{R}^*_{\overline{p}}(L;R)\to\mathscr{R}^*_{\overline{p}}(L;R)$. The *suspension of* f is the quotient

$$X=L\times[0,1]/\sim, \tag{6.6}$$

with $(x,0)\sim(f(x),1)$ for any $x\in L$. We obtain a stratified pseudomanifold relatively to the filtration $X_k=L_k\times[0,1]/\sim$. Locally, this stratified pseudomanifold is stratified homeomorphic to $L\times J$, where $J\subset\mathbb{R}$ is an interval. So, the perversity $\overline{p}$ on L extends naturally to a perversity on X, also denoted $\overline{p}$. We cover X with two open subsets, $\{U,V\}$, where

$$U=(L\times([0,1[\backslash\{1/2\})/\sim\ \text{ and } V=(L\times]0,1[)/\sim=L\times]0,1[.$$

We have $U\cap V=L\times(]0,1[\backslash\{1/2\})$ and the restriction map in the Mayer-Vietoris sequence, $\mathscr{R}^k_{\overline{p}}(U)\oplus\mathscr{R}^k_{\overline{p}}(V)\to\mathscr{R}^k_{\overline{p}}(U\cap V)$, becomes

$$\nu\colon \mathscr{R}^k_{\overline{p}}(L)\oplus\mathscr{R}^k_{\overline{p}}(L)\to\mathscr{R}^k_{\overline{p}}(L)\oplus\mathscr{R}^k_{\overline{p}}(L), \tag{6.7}$$

with $\nu(x,y)=(x-y,x-f^*(y))$. The correspondences $(x,y)\mapsto x$ and $(x,y)\mapsto y-x$ giving isomorphisms, $\operatorname{Ker}\nu\cong\operatorname{Ker}(f-\mathrm{id})^*$ and $\operatorname{Coker}\nu\cong\operatorname{Coker}(f-\mathrm{id})^*$, the Mayer-Vietoris sequence reduces to short exact sequences

$$0\longrightarrow\operatorname{Coker}(f^*-\mathrm{id})^k\longrightarrow\mathscr{R}^k_{\overline{p}}(X)\longrightarrow\operatorname{Ker}(f^*-\mathrm{id})^{k+1}\longrightarrow 0. \tag{6.8}$$

Example 6.7 (*Pseudomanifold which is not locally $\overline{p}$-torsion free and whose $\overline{p}$-intersection homology has a Poincaré duality*)**.** With the notation of Example 6.6, we choose the stratified pseudomanifold $L=\Sigma(S^1\times S^1\times\mathbb{R}P^3)$, $R=\mathbb{Z}$ and $\overline{p}=D\overline{p}=\overline{2}$. The corresponding peripheral cohomology can be determined from Example 6.3 as

$$\begin{aligned}\mathscr{R}^*_{\overline{2}}(L)=\mathscr{R}^3_{\overline{2}}(L) &= \mathrm{T}H^3(S^1\times S^1\times\mathbb{R}P^3)\oplus\mathrm{T}H^3(S^1\times S^1\times\mathbb{R}P^3) \\ &= H^1(S^1\times S^1)\otimes H^2(\mathbb{R}P^3)\oplus H^1(S^1\times S^1)\otimes H^2(\mathbb{R}P^3)=\mathbb{Z}_2^2\oplus\mathbb{Z}_2^2.\end{aligned} \tag{6.9}$$

Let $H^1(S^1\times S^1)=\mathbb{Z}[a,b]$, $H^2(\mathbb{R}P^3)=\mathbb{Z}_2[u]$ be described by generators. Then, using the cross product, we have

$$\mathscr{R}^3_{\overline{2}}(L)=\mathbb{Z}_2[a\times u]\oplus\mathbb{Z}_2[b\times u]\oplus\mathbb{Z}_2[a'\times u]\oplus\mathbb{Z}_2[b'\times u]. \tag{6.10}$$

For the stratified homeomorphism $f\colon L\to L$, we choose the suspension of the map $g\colon S^1\times S^1\times\mathbb{R}P^3\to S^1\times S^1\times\mathbb{R}P^3$, defined by $g(x,y,z)=(x+y,-x,z)$. Using the notations of (6.9), the induced endomorphism of the peripheral cohomology group $\mathscr{R}^3_{\overline{2}}(L)$ satisfies

$$\begin{aligned} f^*(a\times u)&=(a+b)\times u, & f^*(b\times u)&=-a\times u,\\ f^*(a'\times u)&=(a'+b')\times u, & f^*(b'\times u)&=-a'\times u.\end{aligned} \tag{6.11}$$

We can now prove the two required properties on the stratified pseudomanifold X obtained from the suspension of f.

(i) The link of singular points of X is the product $(S^1 \times S^1 \times \mathbb{R}P^3)$ which verifies

$$\mathrm{T}\mathfrak{H}_2^{\overline{2}}(S^1 \times S^1 \times \mathbb{R}P^3) = \mathrm{T}H_2(S^1 \times S^1 \times \mathbb{R}P^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \neq 0.$$

Thus X is not $(\overline{2}, \mathbb{Z})$-locally torsion free.

(ii) From (6.11) and (6.8), we deduce $\mathrm{Ker}\,(f^* - \mathrm{id}) = \mathrm{Coker}\,(f^* - \mathrm{id}) = 0$ and the triviality of the peripheral cohomology of X. Therefore, the $\overline{p}$-intersection homology of X satisfies Poincaré duality.

This example is inspired by an example of H. King ([19, §1]). The devotees of Riemannian foliations can observe a similar idea in an example of Y. Carrière ([3]).

Example 6.8 (*Relative complex of a suspension*)**.** This example shows that the homology of the relative complex of Section 4 is not entirely torsion, in contrast to the peripheral cohomology. Let $M = \mathbb{C}P^2 \times S^1$ with the perversities $\overline{p} = 1$, $D\overline{p} = 3$. Using the Mayer-Vietoris sequence and the classical conical calculation, one gets

$$\mathscr{H}^j_{D\overline{p}/\overline{p}}(\Sigma M; \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } j = 2,\, 3, \\ 0 & \text{if not.} \end{cases}$$

References

[1] Paul Balmer, *Triangular Witt groups. I. The 12-term localization exact sequence*, K-Theory **19** (2000), no. 4, 311–363. MR 1763933

[2] Ronald Brown, *Topology and groupoids*, BookSurge, LLC, Charleston, SC, 2006, Third edition of ıt Elements of modern topology [McGraw-Hill, New York, 1968; MR0227979], With 1 CD-ROM (Windows, Macintosh and UNIX). MR 2273730

[3] Yves Carrière, *Flots riemanniens*, Astérisque (1984), no. 116, 31–52, Transversal structure of foliations (Toulouse, 1982). MR 755161 (86m:58125a)

[4] David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré, *Steenrod squares on intersection cohomology and a conjecture of M Goresky and W Pardon*, Algebr. Geom. Topol. **16** (2016), no. 4, 1851–1904. MR 3546453

[5] ———, *Singular decompositions of a cap product*, Proc. Amer. Math. Soc. **145** (2017), no. 8, 3645–3656. MR 3652815

[6] ———, *Blown-up intersection cohomology*, An alpine bouquet of algebraic topology, Contemp. Math., vol. 708, Amer. Math. Soc., Providence, RI, 2018, pp. 45–102. MR 3807751

[7] ———, *Intersection cohomology, simplicial blow-up and rational homotopy*, Mem. Amer. Math. Soc. **254** (2018), no. 1214, viii+108. MR 3796432

[8] ———, *Poincaré duality with cap products in intersection homology*, Adv. Math. **326** (2018), 314–351. MR 3758431

[9] ———, *Blown-up intersection cochains and Deligne's sheaves*, Geometriae Dedicata. https://doi.org/10.1007/s10711-019-00458-w (2019).

[10] ———, *Intersection homology: General perversities and topological invariance*, Illinois J. Math. **63** (2019), no. 1, 127–163. MR 3959870

[11] Greg Friedman, *Singular intersection homology*, Available at `http://faculty.tcu.edu/gfriedman/index.html`.

[12] ———, *Intersection homology with general perversities*, Geom. Dedicata **148** (2010), 103–135. MR 2721621

[13] ———, *An introduction to intersection homology with general perversity functions*, Topology of stratified spaces, Math. Sci. Res. Inst. Publ., vol. 58, Cambridge Univ. Press, Cambridge, 2011, pp. 177–222. MR 2796412 (2012h:55007)

[14] Greg Friedman and Eugénie Hunsicker, *Additivity and non-additivity for perverse signatures*, J. Reine Angew. Math. **676** (2013), 51–95. MR 3028755

[15] Greg Friedman and James E. McClure, *Cup and cap products in intersection (co)homology*, Adv. Math. **240** (2013), 383–426. MR 3046315

[16] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162. MR 572580 (82b:57010)

[17] ———, *Intersection homology. II*, Invent. Math. **72** (1983), no. 1, 77–129. MR 696691 (84i:57012)

[18] Mark Goresky and Paul Siegel, *Linking pairings on singular spaces*, Comment. Math. Helv. **58** (1983), no. 1, 96–110. MR 699009 (84h:55004)

[19] Henry C. King, *Intersection homology and homology manifolds*, Topology **21** (1982), no. 2, 229–234. MR 642001 (83j:57010)

[20] Erkki Laitinen, *End homology and duality*, Forum Math. **8** (1996), no. 1, 121–133. MR 1366538

[21] Robert MacPherson, *Intersection homology and perverse sheaves*, Unpublished AMS Colloquium Lectures, San Francisco, 1991.

[22] Martin Saralegi, *Homological properties of stratified spaces*, Illinois J. Math. **38** (1994), no. 1, 47–70. MR 1245833 (95a:55011)

[23] Martintxo Saralegi-Aranguren, *de Rham intersection cohomology for general perversities*, Illinois J. Math. **49** (2005), no. 3, 737–758 (electronic). MR 2210257 (2006k:55013)

[24] Martintxo Saralegi-Aranguren and Daniel Tanré, *Poincaré duality, cap products and Borel-Moore intersection Homology*, arXiv e-prints (2018).

Laboratoire de Mathématiques de Lens, EA 2462, Université d'Artois, SP18, rue Jean Souvraz, 62307 Lens Cedex, France

E-mail address: martin.saraleguiaranguren@univ-artois.fr

Département de Mathématiques, Faculté des Sciences et Technologies, UMR 8524, Université de Lille, 59655 Villeneuve d'Ascq Cedex, France

E-mail address: Daniel.Tanre@univ-lille.fr