Accurate Projection Method of Source Quantities in Coupled Finite-Element Problems

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Projecting discrete quantities from one mesh to another one is a critical operation which is usually required in multi-physical problems. The efficiency of a Galerkin projection method is studied for magneto-thermal and magneto-elasticity couplings. As examples of application, two electromagnetic devices are studied: a three-phase busbar system as a magneto-thermal problem and an electro-magnet as a magneto-mechanical problem.

***Index Terms*—Coupled Problems, Finite Element Method, Projection Method.**

I. Introduction

In many studies, such as multi-physical coupled problems, one can need to use different meshes to take into account some regions or not according to their influence on the computed field. In addition, each problem can ask for a particular fineness of mesh. Hence, the main issue about coupled problems using dedicated meshes is to properly transfer each discrete field obtained on one mesh onto another one.

The finite element (FE) method allows evaluating a field everywhere in a studied domain via the direct interpolation of basis functions. This method can be used to perform mesh to mesh interpolation. Another approach for mesh to mesh interpolation consists in defining the discrete expression of a field on the two considered meshes and then minimize the gap between them through a weighted residual method to build an easy to solve linear matrix system. In a previous study [1], the theory of the method has been presented for the projection of magnetic, thermal and mechanical quantities. In this paper, this theory is recalled and the method is used to perform the transfer of source quantities in the cases of magneto-thermal and magneto-mechanical problems using dedicated meshes for each physical phenomenon. As examples of applications, two electromagnetic devices are studied: a three-phase busbar system as a magneto-thermal problem and an electro-magnet as a magneto-mechanical problem. The influence of the projected source quantities on the temperature – in the first case – and the displacement – in the second case – is studied.

II. Fields Computation

A. Magnetic Field Computation

Maxwell's equations and material constitutive laws are written in a studied domain Ω:

$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{j} \quad , \quad \operatorname{curl} \boldsymbol{e} = -\partial_t \boldsymbol{b} \tag{1}$$

$$\boldsymbol{j} = \sigma\, \boldsymbol{e} \quad , \quad \boldsymbol{b} = \mu\, \boldsymbol{h} \tag{2}$$

where $\boldsymbol{h}$ is the magnetic field, $\boldsymbol{b}$ is the magnetic flux density, $\boldsymbol{e}$ is the electric field, $\boldsymbol{j}$ is the current density, μ is the magnetic permeability and σ is the electric conductivity. Boundary conditions are defined on two complementary parts Γ_b and Γ_h of the boundary of the studied domain:

$$\boldsymbol{b}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_b \text{ and } \boldsymbol{h}\times\boldsymbol{n} = 0 \text{ on } \Gamma_h \tag{3}$$

with $\boldsymbol{n}$ the unit normal exterior to Ω.

By combining equations (1) with (2) one can obtain either the magnetodynamic $\boldsymbol{h}$-formulation (4) or $\boldsymbol{a}$-v formulation (5).

$$\operatorname{curl}(\sigma^{-1}\operatorname{curl}\boldsymbol{h}) + \partial_t(\mu\,\boldsymbol{h}) = 0 \tag{4}$$

$$\operatorname{curl}(\mu^{-1}\operatorname{curl}\boldsymbol{a}) + \sigma(\partial_t\boldsymbol{a} + \operatorname{grad} v) = 0 \tag{5}$$

where $\boldsymbol{a}$ is the magnetic vector potential and v is the scalar electric potential, with $\boldsymbol{b} = \operatorname{curl}\boldsymbol{a}$ and $\boldsymbol{e} = -\partial_t\boldsymbol{a} - \operatorname{grad} v$.

B. Thermal Field Computation

The heat transfer in a studied domain is described by the Fourier's law:

$$\rho\, C_p\, \partial_t T - \operatorname{div}\left(\lambda \operatorname{grad} T\right) = p \tag{6}$$

where ρ is the mass density, C_p is the specific heat and λ is the thermal conductivity. In addition to (6) boundary conditions – ideal insulation, isotherm wall, radiation and convection – also have to be taken into account.

C. Elastic Field Computation

In the case of small displacements $\boldsymbol{u}$, the equilibrium equation is given by:

$$\nabla\boldsymbol{\sigma} + \boldsymbol{F}_B = 0 \tag{7}$$

where $\boldsymbol{F}_B$ represents body forces and $\boldsymbol{\sigma}$ is the stress. The latter is linked to the strain $\boldsymbol{\varepsilon}$ via Hook's law $\boldsymbol{\sigma} = \boldsymbol{E}\cdot\boldsymbol{\varepsilon}$ where $\boldsymbol{E}$ is Young modulus.

III. Coupling Strategies and Quantities

For coupled problems, two different coupling strategies, either direct (strong) or indirect (weak), can be used. The most natural and obvious approach from a mathematical point of view is the direct solution since it allows taking into account every single field interaction between each other. Nevertheless, such a strategy can be computationally

expensive and is only justified in particular configurations.

On the other hand, under certain conditions, one can use indirect coupling strategy. For magneto-thermal coupled problems, using such an approach is justified by the fact that each field has its own specific time scale. For magneto-mechanical coupled problems, indirect coupling can be used in the case of small deformations.

With such an approach, the interaction between the different fields is taken into account through coupling quantities, which are the heat sources and the temperature (which influences the electrical conductivity) in a magneto-thermal study and the magnetic force in a magneto-mechanical study.

In our case, using dedicated grids for each physical phenomenon requires projecting those coupling quantities onto different meshes (Fig. 1).

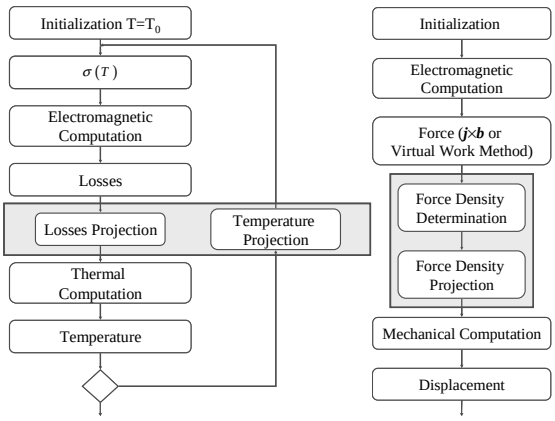

Fig. 1. Magneto-thermal (left) and magneto-mechanical (right) indirect coupled problems using dedicated meshes

IV. Projection Method

A. Galerkin Projection Method General Formulation

One considers a studied domain Ω and two function spaces $U \in L^2(\Omega)$ and $V \in L^2(\Omega)$ with a known field $u \in U$. The aim of the method is to find $v \in V$ by using a weighted residual method, with test functions ψ, so that:

$$\int_{\Omega} \psi (v-u)\, d\tau = 0 \quad , \quad \forall \psi \in V \tag{8}$$

In the discrete domain, a field can be interpolated via the basis functions defined on the mesh used to discretize Ω. Hence, if the field u is discretized on a *source* mesh and v on a *target* mesh, writing (8) in the discrete domain leads to:

$$\int_{\Omega} \psi \sum_{j \in \varepsilon_{t_v}} v_{t_j} w_{t_j} d\tau = \int_{\Omega} \psi \sum_{i \in \varepsilon_{s_u}} u_{s_i} w_{s_i} d\tau \quad , \quad \forall \psi \in V \tag{9}$$

where ε_{s_u} and ε_{t_v} are sets of geometrical entities belonging to the source and the target mesh respectively. Moreover, w_{s_i} and w_{t_j} are the basis functions associated to the entity i of the source mesh and j of the target mesh respectively. Hence, the Galerkin method is applied with each basis function w_{t_j} as a test function ψ so (9) leads to a linear matrix system where the discrete coefficients v_{t_j} are the unknown entities. The numerical integration has to be done using an appropriate number of Gauss points to accurately sample to source field variations.

B. Projection of Magneto-Thermal Coupling Quantities

As we mentioned it earlier, coupling quantities in a magneto-thermal problem are heat sources and temperature. According to Tonti diagram (Fig. 2) those quantities are discretized in the volume FE space E^3 and the nodal FE space E^0 respectively [2].

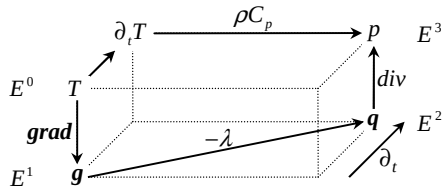

Fig. 2. Tonti diagram applied to thermal problem

Hence, the projection of the heat source and of the temperature is performed by replacing w_{t_j}, and so the test function ψ, by a volume basis function and a nodal basis function respectively. However, the projection of the heat sources can be tricky. Indeed, the fact that it is discretized in the volume FE space means that the heat sources are constant per FE. Hence, in some cases, such as devices with skin effects for example, a step effect can appear on the projected quantity. Moreover, most of thermal codes historically use nodal element space to discretize the heat sources. Hence, for both reasons, one can use nodal basis functions for both the projected heat source and its associated test function ψ in the projection formulation.

C. Projection of the Force

The proposed projection method (9) is made for the projection of discrete quantities in the sense of FE. That means that one has to be careful about the space in which quantities to be projected take place, especially about the force since there are various methods to determine it. For example, in the case of Laplace's force $\boldsymbol{j} \times \boldsymbol{b}$, the right function to use to replace the test function ψ is nodal basis function [4].

In the case of nodal forces given by the virtual work method, the projection cannot be done directly by using (9). Indeed, those forces do not define a field [5], but rather a set of discrete force components acting on each node. Hence, once the nodal forces $\boldsymbol{F}_n$ given by the virtual work method are computed for each node n of the mesh supporting the magnetic computation, one needs to determine the vectorial values $\boldsymbol{f}_n$ of the corresponding force field. Those $\boldsymbol{f}_n$ then define a force density and are, by definition, related to $\boldsymbol{F}_n$ via the expression

$$\int_{\Omega} w_{n_s} \sum_{i \in N_s} w_{s_i} \boldsymbol{f}_{s_i} d\tau = \boldsymbol{F}_{n_s} \quad , \quad \forall n_s \in N_s \tag{10}$$

with N_s the set of nodes belonging to the source mesh.

Hence, once the force density is evaluated on the source mesh (supporting the magnetic computation), it can be

projected onto the target one (supporting the mechanical computation) by replacing the test function ψ by a nodal basis function.

V. Applications

A. Magneto-Thermal Coupled Problem

A three-phase busbar system is taken as an example for the application of the coupling strategy presented in Fig. 1 (left). It is composed of five copper bars included in a concrete bloc (Fig. 3). Dimensions of each bar and of the concrete bloc are 100×8 mm² and 104×200 mm² respectively.

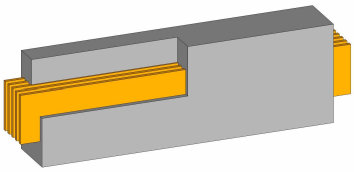

Fig. 3. Three-phase busbar system

Three different meshes are generated for the study. M_{mag} – supporting the electromagnetic computation – is fine enough to take into account the skin effect that appears in the conducting parts with good accuracy, whereas M_{ther1} and M_{ther2} are coarser. The number of elements of these three meshes are 28791, 3763 and 1144, respectively.

The study is performed in two steps. M_{mag} is first used to perform a magneto-thermal computation via the formulations mentioned in section II. Due to the high number of elements – and so high number of degrees of freedom – belonging to M_{mag}, the so-computed FE solution is considered as a reference one for both magnetic and thermal problems. On a second step, M_{mag} is used for the magnetodynamic computation only. The Joule losses are then projected successively onto M_{ther1} and M_{ther2} using (3). For that computation, both nodal and volume basis functions are used. Results obtained for that operation are compared to the reference solution in Fig. 4. Moreover, total Joule losses evaluated on target meshes are compared in table I.

The good accuracy between the projected Joule losses evaluated on thermal meshes and the reference one is pointed out in Fig. 4 and table I. Moreover, even if local error due to "step effect" appear when using volume basis functions for the projection, the global Joule losses do not suffer from any major loss of accuracy, which is an advantage of the proposed method in comparison with the classical interpolation technique.

Then, the so-computed heat sources are used to perform a thermal computation via (6) on M_{ther1} and M_{ther2}. Results of that computation are shown in Fig. 5. The results obtained with projected heat sources are in very good agreement with those obtained by using one single mesh.

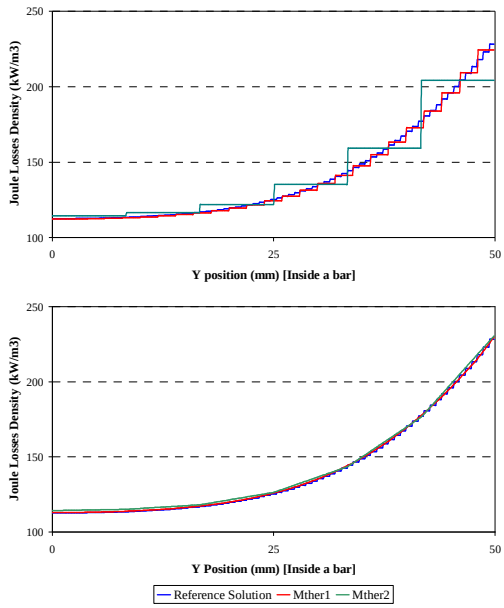

Fig. 4 Joule losses densities versus y-position in a bar; projected quantities with volume (*top*) or nodal (*bottom*) basis functions

TABLE I

Mesh Used for Thermal Computation	Value of the total losses in the device after projection		Value of the relative error compared with the reference solution (%)	
	w_v	w_n	w_v	w_n
M_{ther1}	347.76	347.76	0.028	0.028
M_{ther2}	346.98	346.98	0.252	0.252

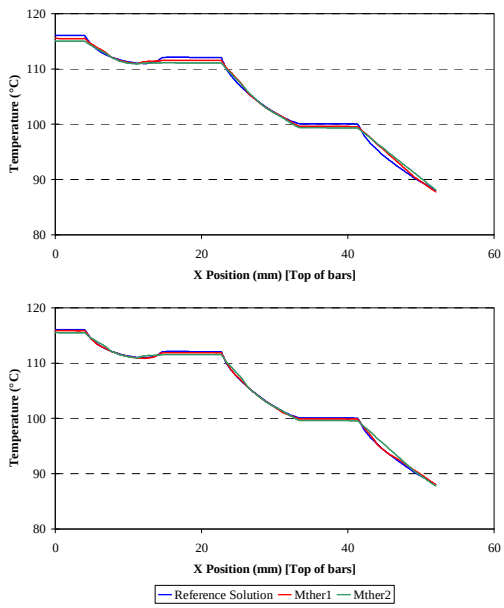

Fig. 5. Evolution of the temperature versus x-position on top of bars; projected Joule losses with volume (*top*) or nodal (*bottom*) basis functions

B. Magneto-Elastic Coupled Problem

An electro-magnet above a plate is taken as an example for

the application of the coupling strategy presented in Fig. 1 (right). For the mechanical computation, the left side of the plate is clamped and the studied quantity is the displacement of the right side of the plate.

Since the mechanical studied domain is the plate, we generate three different meshes – M_{mag}, M_{mech1} and M_{mech2} – differing from each other on the number of elements inside that region, which are 2000, 1200 and 800 respectively. As in the previous section, the study is performed in two steps. A magneto-mechanical computation is first performed on M_{mag} to get a reference solution in term of force densities and displacement at the free side of the plate. Hence, in a second step, the electromagnetic and mechanical problems are solved on dedicated meshes, requiring the projection of the force densities from M_{mag} onto M_{mech1} and M_{mech2}.

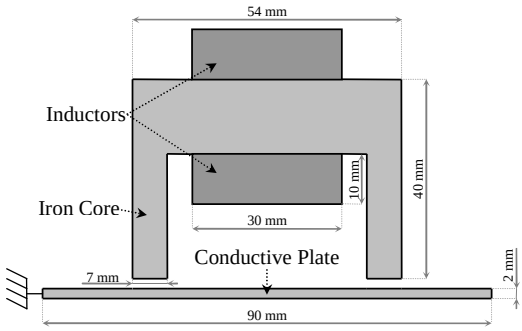

Fig. 6. Electro-magnet above a plate

As an illustration of the fact that the forces given by the virtual work method are strongly linked to the mesh on which they are computed, those latter are shown in Fig. 7 for both M_{mag} (left) and M_{mech1} (right). The corresponding force density distributions are shown in Fig. 8.

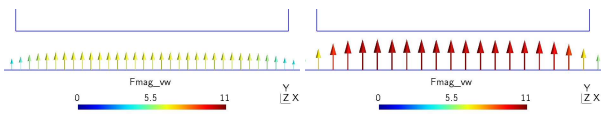

Fig. 7. Force given by virtual work method under a tooth using M_{mag} *(left)* and M_{mech1} *(right)*

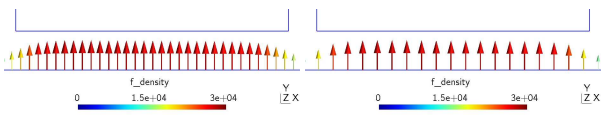

Fig. 8. Force density evaluated from the force given by virtual work method under a tooth using M_{mag} *(left)* and M_{mech1} *(right)*

Results of the force density projection operation are shown in Fig. 9. A good accuracy between the force density computed on M_{mag} and the projected ones is obtained. Then, the projected force densities are used for the magneto-elastic computation. The displacement of the free side of the plate for both computations – using M_{mech1} and M_{mech2} – are presented in table II. The mechanical computation using dedicated meshes – and so projected source terms – gives very accurate results in comparison with the reference solution.

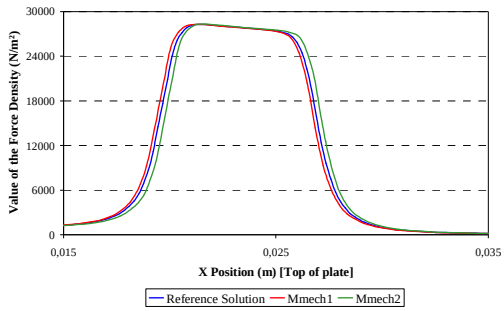

Fig. 9. Force density versus x-position along the top of the plate, under a tooth

TABLE II

MESH NAME	DISPLACEMENT VALUE (mm)	RELATIVE ERROR (%)
M_{mech1}	0.30037	0.89
M_{mech2}	0.29500	2.67

VI. CONCLUSION

A Galerkin projection method has been used for the projection of source quantities in both magneto-thermal and magneto-mechanical problems using dedicated meshes for each physical field computation. It has been shown that the projected quantities are in good agreement with the original ones and can be used as source terms for multi-physical problems using dedicated meshes.

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