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On the decidability of a fragment of preferential LTL

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Abstract

Linear Temporal Logic (LTL) has found extensive applications in Computer Science and Artificial Intelligence, notably as a formal framework for representing and verifying computer systems that vary over time. Non-monotonic reasoning, on the other hand, allows us to formalize and reason with exceptions and the dynamics of information. The goal of this paper is therefore to enrich temporal formalisms with non-monotonic reasoning features. We do so by investigating a preferential semantics for defeasible LTL along the lines of that extensively studied by Kraus et al. in the propositional case and recently extended to modal and description logics. The main contribution of the paper is a decidability result for a meaningful fragment of preferential LTL that can serve as the basis for further exploration of defeasibility in temporal formalisms.

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1 Introduction

Specification and verification of dynamic computer systems is an important task, given the increasing number of new computer technologies being developed. Recent examples include blockchain technology and various existing tools for home automation of the different production chains provided by Industry 4.0. Therefore, it is fundamental to ensure that systems based on them have the desired behavior but, above all, satisfy safety standards. This becomes even more critical with the increasing deployment of artificial intelligence techniques as well as the need to explain their behaviors.

Several approaches for qualitative analysis of computer systems have been developed. Among the most fruitful are the different families of temporal logic. The success of these is due mainly to their simplified syntax compared to that of first-order logic, their intuitive syntax, semantics and their good computational properties. One of the members of this family is Linear Temporal Logic [15, 19], known as *LTL*, is widely used in formal verification and specification of computer programs.

Despite the success and wide use of linear temporal logic, it remains limited for modeling and reasoning about the real aspects of computer systems or those that depend on them. In fact, computer systems are not either 100% secure or 100% defective, and the properties we wish to check may have innocuous and tolerable exceptions, or conversely, exceptions that must be carefully addressed in order to guarantee the overall reliability of the system. Similarly, the expected behavior of a system may be correct not for all possible execution, but rather for its most “normal” or expected executions.

It turns out that *LTL*, because it is a logical formalism of the so-called classical type, whose underlying reasoning is that of mathematics and not that of common sense, does not allow at all to formalize the different nuances of the exceptions and even less to treat them. First of all, at the level of the object language (that of the logical symbols), it has operators behaving monotonically, and at the level of reasoning, possesses a notion of logical consequence which is monotonic too, and consequently, it is not adapted to the evolution of defeasible facts.

Non-monotonic reasoning (NMR), on the other hand, allows to formalize and reason with exceptions, it has been widely studied by the AI community for over 40 years now. Such is the case of Kraus et al. [12], known as the KLM approach.

However, the major contributions in this area are limited to the propositional framework. It is only recently that some approaches to non-monotonic reasoning, such as belief revision, default



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46 rules and preferential approaches, have been studied for more expressive logics than propositional
 47 logic, including modal [3, 5] and description logics [4]. The objective of our study is to establish a
 48 bridge between temporal formalisms for the specification and verification of computer systems and
 49 approaches to non-monotonic reasoning, in particular the preferential one, which satisfactorily solves
 50 the limitations raised above.

51 In this paper, we define a logical framework for reasoning about defeasible properties of program
 52 executions, we investigate the integration of preferential semantics in the case of *LTL*, hereby
 53 introducing preferential linear temporal logic *LTL*⁻. The remainder of the present paper is structured
 54 as follows: In Section 3 we set up the notation and appropriate semantics of our language. In
 55 Sections 4, 5 and 6, we investigate the satisfiability problem of this formalism. The appendix
 56 contains proofs of results in this paper. The remaining proofs can be viewed anonymously in
 57 https://github.com/calleann/Preferential_LTL.

58 2 Preliminaries: LTL and the KLM approach to NMR

59 Let \mathcal{P} be a finite set of *propositional atoms*. The set of operators in the *Linear Temporal Logic* can be
 60 split into two parts: the set of *Boolean connectives* (\neg, \wedge), and that of *temporal operators* ($\Box, \Diamond, \bigcirc, \mathcal{U}$),
 61 where \Box reads as *always*, \Diamond as *eventually*, \bigcirc as *next* and \mathcal{U} as *until*. The set of well-formed sentences
 62 expressed in *LTL* is denoted by \mathcal{L} . Sentences of \mathcal{L} are built up according to the following grammar:
 63 $\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha' \mid \alpha \vee \alpha' \mid \Box\alpha \mid \Diamond\alpha \mid \bigcirc\alpha \mid \alpha\mathcal{U}\alpha'$.

64 Let the set of natural numbers \mathbb{N} denote time points. A *temporal interpretation* I is a mapping
 65 function $V : \mathbb{N} \rightarrow 2^{\mathcal{P}}$ which associates each time point $t \in \mathbb{N}$ with a set of propositional atoms
 66 $V(t)$ corresponding to the set of propositions that are true in t . (Propositions not belonging to $V(t)$
 67 are assumed to be false at the given time point.) The truth conditions of LTL sentences are defined as
 68 follows, where I is a temporal interpretation and t a time point in I :

- 69 ■ $I, t \models p$ if $p \in V(t)$; $I, t \models \neg\alpha$ if $I, t \not\models \alpha$;
- 70 ■ $I, t \models \alpha \wedge \alpha'$ if $I, t \models \alpha$ and $I, t \models \alpha'$; $I, t \models \alpha \vee \alpha'$ if $I, t \models \alpha$ or $I, t \models \alpha'$;
- 71 ■ $I, t \models \Box\alpha$ if $I, t' \models \alpha$ for all $t' \in \mathbb{N}$ s.t. $t' \geq t$; $I, t \models \Diamond\alpha$ if $I, t' \models \alpha$ for some $t' \in \mathbb{N}$ s.t. $t' \geq t$;
- 72 ■ $I, t \models \bigcirc\alpha$ if $I, t+1 \models \alpha$;
- 73 ■ $I, t \models \alpha\mathcal{U}\alpha'$ if $I, t' \models \alpha'$ for some $t' \geq t$ and for all $t \leq t'' < t'$ we have $I, t'' \models \alpha$.

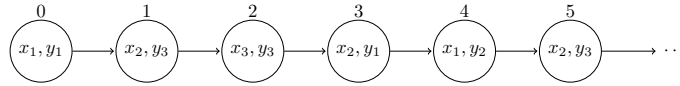
74 We say $\alpha \in \mathcal{L}$ is *satisfiable* if there are I and $t \in \mathbb{N}$ such that $I, t \models \alpha$.

75 We now give a brief outline to Kraus et al.'s [12] approach to non-monotonic reasoning. A
 76 propositional *defeasible consequence relation* \sim [12] is defined as a binary relation on sentences of
 77 an underlying propositional logic. The semantics of preferential consequence relation is in terms of
 78 *preferential models*: A preferential model on a set of atomic propositions \mathcal{P} is a tuple $\mathcal{P} \stackrel{\text{def}}{=} (S, l, \prec)$
 79 where S is a set of elements called states, $l : S \rightarrow 2^{\mathcal{P}}$ is a mapping which assigns to each state s a
 80 single world $m \in 2^{\mathcal{P}}$ and \prec is a *strict partial order* on S satisfying smoothness condition. Intuitively,
 81 the states that are lower down in the ordering are more plausible, normal or in a general case preferred,
 82 than those that are higher up. A statement of the form $\alpha \sim \beta$ holds in a preferential model iff the
 83 minimal α -states are also β -states.

84 3 Preferential LTL

85 In this paper, we introduce a new formalism for reasoning about time that is able to distinguish
 86 between normal and exceptional points of time. We do so by investigating a defeasible extension of
 87 *LTL* with a preferential semantics. The following example introduces a case scenario we shall be
 88 using in the remainder of this section, with the purpose of giving a motivation for this formalism and
 89 better illustrating the definitions in what follows.

90 ► **Example 1.** We have a computer program in which the values of its variables change with time.
 91 In particular, the agent wants to check two parameters, say x and y . These two variables take one
 92 and only one value between 1 and 3 on each iteration of the program. We represent the set of atomic
 93 propositions by $\mathcal{P} = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ where x_i (resp. y_i) for all $i \in \{1, 2, 3\}$ is true iff the
 94 variable x (resp. y) has the value i in a current iteration. Figure 1 depicts a temporal interpretation
 95 corresponding to a possible behaviour of such a program:



■ **Figure 1** LTL interpretation V (for $t > 5$, $V(t) = V(5) = \{x_2, y_3\}$)

96 Under normal circumstances, the program assigns the value 3 to y whenever $x = 2$. We can
 97 express this fact using classical LTL as follows: $\Box(x_2 \rightarrow y_3)$, with $x_2 \rightarrow y_3$ is defined by $\neg x_2 \vee y_3$.
 98 Nevertheless, the agent notices that there is one exceptional iteration (Iteration 3) where the program
 99 assigns the value 1 to y when $x = 2$.

100 Some might consider that the current program is defective at some points of time. In LTL, the
 101 statement $\Box(x_2 \rightarrow y_3) \wedge \Diamond(x_2 \wedge y_1)$ will always be false, since y cannot have two different values
 102 in an iteration where $x = 2$. Nonetheless we want to propose a logical framework that is exception
 103 tolerant for reasoning about a system's behaviour. In order to express this general tendency ($x_2 \rightarrow y_3$)
 104 while taking into account that there might be some exceptional iterations which do not crash the
 105 program. We base our semantic constructions on the preferential approach [16, 12].

106 3.1 Introducing defeasible temporal operators

107 Britz & Varzinczak [5] introduced new modal operators called defeasible modalities. In their setting,
 108 defeasible operators, unlike their classical counterparts, are able to single out normal worlds from
 109 those that are less normal or exceptional in the reasoner's mind. Here we extend the vocabulary of
 110 classical LTL with the *defeasible temporal operators* \Box and \Diamond . Sentences of the resulting logic LTL^\sim
 111 are built up according to the following grammar:

$$112 \quad \alpha ::= p \mid \neg \alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box \alpha \mid \Diamond \alpha \mid \bigcirc \alpha \mid a\mathcal{U}\alpha \mid \Box \alpha \mid \Diamond \alpha$$

113 The intuition behind these new operators is the following: \Box reads as *defeasible always* and \Diamond reads
 114 as *defeasible eventually*.

115 ► **Example 2.** Going back to our example 1, we can describe the normal behaviour of the program
 116 using the statement $\Box(x_2 \rightarrow y_3) \wedge \Diamond(x_2 \wedge y_1)$. In all normal future time points, the program assigns
 117 the value 3 to y when $x = 2$. Although unlikely, there are some exceptional time points in the future
 118 where $x = 2$ and $y = 1$. But those are 'ignored' by the defeasible always operator.

119 The set of all well-formed LTL^\sim sentences is denoted by \mathcal{L}^\sim . It is worth to mention that any
 120 well-formed sentence $\alpha \in \mathcal{L}$ is a sentence of \mathcal{L}^\sim . We denote a subset of our language that contains
 121 only Boolean connectives, the two defeasible operators \Box , \Diamond and their classical counterparts by
 122 \mathcal{L}^* . Next we shall discuss how to interpret statements that have this defeasible aspect and how to
 123 determine the truth values of each well-formed sentence in \mathcal{L}^\sim .

124 3.2 Preferential semantics

125 First of all, in order to interpret the sentences of \mathcal{L}^\sim we consider, as stated on the preliminaries, $(\mathbb{N}, <)$
 126 to be a temporal structure. Hence, a temporal interpretation that associates each time point t with a
 127 truth assignment of all propositional atoms.

128 The preferential component of the interpretation of our language is directly inspired by the
 129 preferential semantics proposed by Shoham [17] and used in the KLM approach [12]. The preference
 130 relation \prec is a strict partial order on our points of time. Following Kraus et al. [12], $t \prec t'$ means
 131 that t is more preferred than t' . The reasoner has now the tools to express the preference between
 132 points of time by comparing them w.r.t. each other, with time points lower down the order being more
 133 preferred than those higher up.

134 ► **Definition 3 (Minimality w.r.t. \prec).** Let \prec be a strict partial order on a set \mathbb{N} and $N \subseteq \mathbb{N}$. The
 135 set of the minimal elements of N w.r.t. \prec , denoted by $\min_{\prec}(N)$, is defined by $\min_{\prec}(N) \stackrel{\text{def}}{=} \{t \in N \mid$
 136 there is no $t' \in N$ such that $t' \prec t\}$.

137 ► **Definition 4 (Well-founded set).** Let \prec be a strict partial order on a set \mathbb{N} . We say \mathbb{N} is
 138 well-founded w.r.t. \prec iff $\min_{\prec}(N) \neq \emptyset$ for every $\emptyset \neq N \subseteq \mathbb{N}$.

139 ► **Definition 5 (Preferential temporal interpretation).** An LTL^{\sim} interpretation on a set of pro-
 140 positional atoms \mathcal{P} , also called preferential temporal interpretation on \mathcal{P} , is a pair $I \stackrel{\text{def}}{=} (V, \prec)$ where
 141 V is a temporal interpretation on \mathcal{P} , and $\prec \subseteq \mathbb{N} \times \mathbb{N}$ is a strict partial order on \mathbb{N} such that \mathbb{N} is
 142 well-founded w.r.t. \prec . We denote the set of preferential temporal interpretations by \mathcal{I} .

143 In what follows, given a preference relation \prec and a time point $t \in \mathbb{N}$, the set of *preferred time*
 144 *points relative to t* is the set $\min_{\prec}([t, +\infty[)$ which is denoted in short by $\min_{\prec}(t)$. It is also worth
 145 to point out that given a preferential interpretation $I = (V, \prec)$ and \mathbb{N} , the set $\min_{\prec}(t)$ is always a
 146 non-empty subset of $[t, +\infty[$ at any time point $t \in \mathbb{N}$.

147 Preferential temporal interpretations provide us with an intuitive way of interpreting sentences
 148 of \mathcal{L}^{\sim} . Let $\alpha \in \mathcal{L}^{\sim}$, let $I = (V, \prec)$ be a preferential interpretation, and let t be a time point in I in \mathbb{N} .
 149 Satisfaction of α at t in I , denoted $I, t \models \alpha$, is defined as follows:

- 150 ■ $I, t \models \Box\alpha$ if $I, t' \models \alpha$ for all $t' \in \min_{\prec}(t)$;
- 151 ■ $I, t \models \Diamond\alpha$ if $I, t' \models \alpha$ for some $t' \in \min_{\prec}(t)$.

152 The truth values of Boolean connectives and classical modalities are defined as in LTL . The
 153 intuition behind a sentence like $\Box\alpha$ is that α holds in *all* preferred time points that come after t . $\Diamond\alpha$
 154 intuitively means that α holds on at least one preferred time point relative in the future of t .

155 We say $\alpha \in \mathcal{L}^{\sim}$ is *preferentially satisfiable* if there is a preferential temporal interpretation I and
 156 a time point t in \mathbb{N} such that $I, t \models \alpha$. We can show that $\alpha \in \mathcal{L}^{\sim}$ is *preferentially satisfiable* iff there
 157 is a preferential temporal interpretation I s.t. $I, 0 \models \alpha$. A sentence $\alpha \in \mathcal{L}^{\sim}$ is *valid* (denoted by $\models \alpha$)
 158 iff for all temporal interpretation I and time points t in \mathbb{N} , we have $I, t \models \alpha$.

159 ► **Example 6.** Going back to Example 1, we can see that the time points 5 and 1 are more “normal”
 160 than iteration 3. By adding preferential preference $\prec := \{(5, 3), (1, 3)\}$, we denote the preferential
 161 temporal interpretation by $I = (V, \prec)$. We have that $I, 0 \not\models \Box(x_2 \rightarrow y_3) \wedge \Diamond(x_2 \wedge y_1)$ and
 162 $I, 0 \models \Box(x_2 \rightarrow y_3) \wedge \Diamond(x_2 \wedge y_1)$.

163 We can see that the addition of \prec relation preserves the truth values of all classical temporal
 164 sentences. Moreover, for every $\alpha \in \mathcal{L}$, we have that α is satisfiable in LTL if and only if α is
 165 preferentially satisfiable in LTL^{\sim} .

166 We discuss some properties of these defeasible modalities next. In what follows, let α, β be
 167 well-formed sentences in \mathcal{L}^{\sim} . We have duality between our defeasible operators: $\models \Box\alpha \leftrightarrow \neg \Diamond\neg\alpha$.
 168 We also have $\models \Box\alpha \rightarrow \Box\alpha$ and $\models \Diamond\alpha \rightarrow \Diamond\alpha$. Intuitively, This property states that if a statement
 169 holds in all of future time points of any given point of time t , it holds on all our *future preferred* time
 170 points. As intended, this property establishes the defeasible always as “weaker” than the classical
 171 always. It can commonly be accepted since the set of all preferred future states are in the future. This

172 is why we named \Box *feasible always*. On the other hand, we see that \Diamond is “stronger” than classical
 173 eventually, the statement within \Diamond holds at a preferable future.

174 The axiom of distributivity (K) can be stated in terms of our defeasible operators. We can also
 175 verify the validity of these two statements $\models \Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta)$ and $\models (\Box\alpha \vee \Box\beta) \rightarrow$
 176 $\Box(\alpha \vee \beta)$, the converse of the second statement is not always true.

177 The reflexivity axiom (T) for the classical operators does not hold in the case of defeasible
 178 modalities. We can easily find an interpretation $I = (V, \prec)$ where $I, t \not\models \Box\alpha \rightarrow \alpha$. Indeed, since we
 179 can have $t \notin \min_{\prec}(t)$ for a temporal point t , we can have $I, t \models \Box\alpha$ and $I, t \models \neg\alpha$.

180 One thing worth pointing out is the set of future preferred time points changes dynamically as we
 181 move forward in time. Given three time points $t_1 \leq t_2 \leq t_3$, $t_3 \notin \min_{\prec}(t_1)$ whilst $t_3 \in \min_{\prec}(t_2)$
 182 could be true in some cases. Hence, if $I, t \models \Box\Box\alpha$ does not imply that for all $t' \in \min_{\prec}(t)$,
 183 $I, t' \models \Box\alpha$. Therefore, the transitivity axiom (4) does not hold also in our defeasible modalities. On
 184 the other hand, given those three time points, $t_3 \notin \min_{\prec}(t_1)$ implies that $t_3 \notin \min_{\prec}(t_2)$.

185 And since we do not have a version of the axioms (T) and (4) for our defeasible operators, we do
 186 not have the collapsing property on the case \Box, \Diamond . Redundant sentences in the case modal sentences
 187 such as $\Box\Box \dots \Box\alpha$ can be reduced to $\Box\alpha$. It is not the case for our preferential operators \Box and \Diamond .

188 3.3 State-dependent preferential interpretations

189 We define a class of well-behaved LTL^{\sim} interpretations that are useful in the remainder of the paper.

190 ► **Definition 7 (State-dependent preferential interpretations).** *Let $I = (V, \prec) \in \mathfrak{I}$. I is state-*
 191 *dependent preferential interpretation iff for every $i, j, i', j' \in \mathbb{N}$, if $V(i') = V(i)$ and $V(j') = V(j)$,*
 192 *then $(i, j) \in \prec$ iff $(i', j') \in \prec$.*

193 In what follows, \mathfrak{I}^{sd} denotes the set of all state-dependent interpretations. The intuition behind
 194 setting up this restriction is to have a more compact form of expressing preference over time points. In
 195 a way, time points with similar valuations are considered to be identical with regards to \prec , they express
 196 the same preferences towards other time points. Moreover, we have some interesting properties that
 197 do not in the general case. In particular, we have the following property :

198 ► **Proposition 8.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i, i', j, j' \in \mathbb{N}$ s.t. $i \leq i'$, $i' \leq j'$ and $j \in \min_{\prec}(i)$.*
 199 *If $V(j) = V(j')$, then $j' \in \min_{\prec}(i')$.*

200 This property is specific to the class of state-dependent interpretations. However, the following
 201 proposition is true for every $I \in \mathfrak{I}$.

202 ► **Proposition 9.** *Let $I = (V, \prec) \in \mathfrak{I}$ and let $i, j \in \mathbb{N}$ s.t. $j \in \min_{\prec}(i)$. For all $i \leq i' \leq j$, we*
 203 *have $j \in \min_{\prec}(i')$.*

204 4 A useful representation of preferential structures

205 One of the objectives of this paper is to establish some computational properties about the satisfiability
 206 problem. In order to do this, we introduce into the sequel different structures inspired by the approach
 207 followed by Sistla and Clarke in [18]. They observe that in every LTL interpretation, there is a time
 208 point t after which every t -successor’s valuation occurs infinitely many times. This is an obvious
 209 consequence of having an infinite set of time points and a finite number of possible valuations. That
 210 is the case also for LTL^{\sim} interpretations.

211 ► **Lemma 10.** *Let $I = (V, \prec) \in \mathfrak{I}$. There exists a $t \in \mathbb{N}$ s.t. for all $l \in [t, +\infty[$, there is a $k > l$*
 212 *where $V(l) = V(k)$.*

213 For an interpretation $I \in \mathfrak{I}$, we denote the first time point where the condition set in Lemma 10 is
214 satisfied by t_I . We can split each temporal structure into two intervals: an initial and a final part.

215 ► **Definition 11.** Let $I = (V, \prec) \in \mathfrak{I}$. We define: $init(I) \stackrel{\text{def}}{=} [0, t_I[$; $final(I) \stackrel{\text{def}}{=} [t_I, +\infty[$;
216 $range(I) \stackrel{\text{def}}{=} \{V(i) \mid i \in final(I)\}$; $val(I) \stackrel{\text{def}}{=} \{V(i) \mid i \in \mathbb{N}\}$; $size(I) \stackrel{\text{def}}{=} length(init(I)) +$
217 $card(range(I))$, where $length(\cdot)$ denotes the length of a sequence and $card(\cdot)$ set cardinality.

218 In the size of I we count the number of time points in the initial part and the number of valuations
219 contained in the final part. In what follows, we discuss some properties concerning these notions and
220 state dependent interpretations.

221 ► **Proposition 12.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i \leq j \leq i' \leq j'$ be time points in $final(I)$ s.t.
222 $V(j) = V(j')$. Then we have $j \in \min_{\prec}(i)$ iff $j' \in \min_{\prec}(i')$.

223 ► **Lemma 13.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $i \leq i'$ be time points of $final(I)$ where $V(i) = V(i')$.
224 Then for every $\alpha \in \mathcal{L}^*$, we have $I, i \models \alpha$ iff $I, i' \models \alpha$.

225 What we have in Lemma 13 is that given an interpretation $I \in \mathfrak{I}^{sd}$, points of time in $final(I)$ that
226 have the same valuations satisfy exactly the same sentences.

227 ► **Definition 14 (Faithful Interpretations).** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be
228 two interpretations over the same set of atoms \mathcal{P} . We say that I, I' are faithful interpretations if
229 $val(I) = val(I')$ and, for all $i, j, i', j' \in \mathbb{N}$ s.t. $V'(i') = V(i)$ and $V'(j') = V(j)$, we have $(i, j) \in \prec$
230 iff $(i', j') \in \prec'$.

231 Throughout this paper, we write $init(I) \doteq init(I')$ as shorthand for the condition that states:
232 $length(init(I)) = length(init(I'))$ and for each $i \in init(I)$ we have $V(i) = V'(i)$.

233 ► **Lemma 15.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over \mathcal{P}
234 such that $V'(0) = V(0)$ (in case $init(I)$ is empty), $init(I) \doteq init(I')$, and $range(I) = range(I')$.
235 Then for all $\alpha \in \mathcal{L}^*$, we have that $I, 0 \models \alpha$ iff $I', 0 \models \alpha$.

236 Lemma 15 implies that the ordering of time points in $final(\cdot)$ does not matter, and what matters is
237 the $range(\cdot)$ of valuations contained within it. It is worth to mention that Lemma 13 and 15 hold only
238 in the case interpretations in \mathfrak{I}^{sd} and they are not always true in the general case.

239 Sistla & Clarke [18] introduced the notion of acceptable sequences. The general purpose behind
240 it is the ability to build, from an initial interpretation, other interpretations. We adapt this notion for
241 preferential temporal structures. We then introduce the notion of pseudo-interpretations that will
242 come in handy in showing decidability of the satisfiability problem in \mathcal{L}^* in the upcoming section.

243 In the sequel, the term temporal sequence or sequence in short, will denote a sequence of ordered
244 integer numbers. A sequence allows to represent a set of time points. Sometimes, we will consider
245 integer intervals as sequences. Moreover, given two sequences N_1, N_2 , the union of N_1 and N_2 ,
246 denoted by $N_1 \cup N_2$, is the sequence containing only elements of N_1 and N_2 . An acceptable sequence
247 is a temporal sequence that is built relatively to a preferential temporal interpretation I as follows:

248 ► **Definition 16 (Acceptable sequence w.r.t. I).** Let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence of
249 temporal time points. N is an acceptable sequence w.r.t. I iff for all $i \in N \cap final(I)$ and for all
250 $j \in final(I)$ s.t. $V(i) = V(j)$, we have $j \in N$.

251 The particularity we are looking for is that any picked time point in $init(\cdot)$ (resp. $final(\cdot)$)
252 will remain in the initial (resp. final) part of the new interpretation. It is worth pointing out that
253 an acceptable sequence w.r.t. a preferential temporal interpretation can be either finite or infinite.

254 Moreover, \mathbb{N} is an acceptable sequence w.r.t. any interpretation $I \in \mathfrak{I}$. The purpose behind the notion
255 of acceptable sequence is to construct new interpretations starting from an LTL^{\sim} interpretation.

256 Given N an acceptable sequence w.r.t. I , if N has a time point t in $final(I)$, then all time points
257 t' that have the same valuation as t must be in N . Thus, we have an infinite sequence of time points.
258 As such, we can define an initial part and a final part, in a similar way as LTL^{\sim} interpretations. We
259 let $init(I, N)$ be the largest subsequence of N that is a subsequence of $init(I)$. Note that if N does
260 not contain any time point of $final(I)$, then N is finite.

261 We now define the notions $init(\cdot)$, $final(\cdot)$, $range(\cdot)$, and $size(\cdot)$ for acceptable sequences.

262 ► **Definition 17.** Let $I = (V, \prec) \in \mathfrak{I}$, and let N be an acceptable sequence w.r.t. I . We define:
263 $init(I, N) \stackrel{\text{def}}{=} N \cap init(I)$; $final(I, N) \stackrel{\text{def}}{=} N \setminus init(I, N)$; $range(I, N) \stackrel{\text{def}}{=} \{V(t) \mid t \in final(I, N)\}$;
264 $val(I, N) \stackrel{\text{def}}{=} \{V(t) \mid t \in N\}$; $size(I, N) \stackrel{\text{def}}{=} length(init(I, N)) + card(range(I, N))$.

265 It is worth mentioning that, thanks to Definition 16, given an acceptable sequence w.r.t. I , we
266 have $size(I, N) \leq size(I)$.

267 ► **Definition 18 (Pseudo-interpretation over N).** Let $I = (V, \prec) \in \mathfrak{I}$ and N be an acceptable
268 sequence w.r.t. I . The pseudo-interpretation over N is the tuple $I^N \stackrel{\text{def}}{=} (N, V^N, \prec^N)$ where:

- 269 ■ $V^N : N \rightarrow 2^{\mathcal{P}}$ is a valuation function over N , where for all $i \in N$, we have $V^N(i) = V(i)$,
- 270 ■ $\prec^N \subseteq N \times N$, where for all $(i, j) \in N^2$, we have $(i, j) \in \prec^N$ iff $(i, j) \in \prec$

271 The truth values of \mathcal{L}^* sentences in pseudo-interpretations are defined in a similar fashion as
272 for preferential temporal interpretations. With $\models_{\mathcal{P}}$ we denote the truth values of sentences in a
273 pseudo-interpretation. We highlight truth values for classical and defeasible modalities.

- 274 ■ $I^N, t \models_{\mathcal{P}} \Box \alpha$ if $I^N, t' \models_{\mathcal{P}} \alpha$ for all $t' \in N$ s.t. $t' \geq t$;
- 275 ■ $I^N, t \models_{\mathcal{P}} \Diamond \alpha$ if $I^N, t' \models_{\mathcal{P}} \alpha$ for some $t' \in N$ s.t. $t' \geq t$;
- 276 ■ $I^N, t \models_{\mathcal{P}} \Box \alpha$ if for all $t' \in N$ s.t. $t' \in \min_{\prec^N}(t)$, we have $I^N, t' \models_{\mathcal{P}} \alpha$;
- 277 ■ $I^N, t \models_{\mathcal{P}} \Diamond \alpha$ if $I^N, t' \models_{\mathcal{P}} \alpha$ for some $t' \in N$ s.t. $t' \in \min_{\prec^N}(t)$.

278 ► **Proposition 19.** Let $I = (V, \prec) \in \mathfrak{I}$, N_1, N_2 be two acceptable sequences w.r.t. I . Then $N_1 \cup N_2$
279 is an acceptable sequence w.r.t. I s.t. $size(I, N_1 \cup N_2) \leq size(I, N_1) + size(I, N_2)$.

280 ► **Proposition 20.** Let $I = (V, \prec) \in \mathfrak{I}$ and N be an acceptable sequence w.r.t. I . If for all distinct
281 $t, t' \in N$, we have $V(t') = V(t)$ only when both $t, t' \in final(I, N)$, then $size(I, N) \leq 2^{|\mathcal{P}|}$.

282 5 Bounded-model property

283 The main contribution of this paper is to establish certain computational properties regarding the
284 satisfiability problem in \mathcal{L}^* . The algorithmic problem is as follows: Given an input sentence $\alpha \in \mathcal{L}^*$,
285 decide whether α is preferentially satisfiable. In this section, we show that this problem is decidable.

286 The proof is based on the one given by Sistla and Clarke to show the complexity of propositional
287 linear temporal logic [18]. Let \mathcal{L}^* be the fragment of \mathcal{L}^{\sim} that contains only Boolean connectives and
288 temporal operators ($\Box, \Box, \Diamond, \Diamond$). Let $\alpha \in \mathcal{L}^*$, with $|\alpha|$ we denote the number of symbols within α .
289 The main result of the present paper is summarized in the following theorem, of which the proof will
290 be given in the remainder of the section.

291 ► **Theorem 21 (Bounded-model property).** If $\alpha \in \mathcal{L}^*$ is \mathfrak{I}^{sd} -satisfiable, then we can find an
292 interpretation $I \in \mathfrak{I}^{sd}$ such that $I, 0 \models \alpha$ and $size(I) \leq |\alpha| \times 2^{|\mathcal{P}|}$.

293 Hence, given a satisfiable sentence $\alpha \in \mathcal{L}^*$, there is an interpretation satisfying α of which the size
294 is bounded. Since α is \mathfrak{I}^{sd} -satisfiable, we know $I, 0 \models \alpha$. From I we can construct an interpretation
295 I' also satisfying α , i.e., $I', 0 \models \alpha$, which is bounded on its size by $|\alpha| \times 2^{|\mathcal{P}|}$.

296 The goal of this section is to show how to build said bounded interpretation. Let $\alpha \in \mathcal{L}^*$ and
 297 let $I \in \mathfrak{I}^{sd}$ be s.t. $I, 0 \models \alpha$. The first step is to characterize an acceptable sequence N w.r.t. I
 298 such that N is bounded first of all, and “keeps” the satisfiability of the sub-sentences α_1 contained
 299 in α i.e., if $I, t \models \alpha_1$, then $I^N, t \models_{\mathcal{D}} \alpha_1$ (see Definition 18). We do so by building inductively a
 300 bounded pseudo-interpretation step by step by selecting what to take from the initial interpretation I
 301 for each sub-sentence α_1 contained in α to be satisfied. In what follows, we introduce the notion of
 302 *Anchors*(\cdot) as a strategy for picking out the desired time points from I . Definitions 23–25 tell us how
 303 to pick said time points.

304 ► **Definition 22 (Acceptable sequence transformation).** Let $I = (V, \prec) \in \mathfrak{I}$ and let N be a
 305 sequence of time points. Let N' be the sequence of all time points t' in $final(I)$ for which there is
 306 $t \in N \cap final(I)$ with $V(t') = V(t)$. With $AS(I, N) \stackrel{\text{def}}{=} N \cup N'$ we denote the acceptable sequence
 307 transformation of N w.r.t. I .

308 The sequence $AS(I, N)$ is the acceptable sequence transformation of N w.r.t. I . In the previous
 309 definition, N' is the sequence of all time points t' having the same valuation as some time point $t \in N$
 310 that is in $final(I)$. It is also worth to point out that N' can be empty in the case of there being no time
 311 point $t \in N$ that is in $final(I)$. N is then a finite acceptable sequence w.r.t. I where $AS(I, N) = N$.
 312 This notation is mainly used to ensure that we are using the acceptable version of any sequence.

313 ► **Definition 23 (Chosen occurrence w.r.t. α).** Let $I = (V, \prec) \in \mathfrak{I}$, $\alpha \in \mathcal{L}^{\sim}$ and N be an
 314 acceptable sequence w.r.t. I s.t. there exists a time point t in N with $I, t \models \alpha$. The chosen occurrence
 315 satisfying α in N , denoted by $t_{\alpha}^{I, N}$, is defined as follows:

$$316 \quad t_{\alpha}^{I, N} \stackrel{\text{def}}{=} \begin{cases} \min_{<} \{t \in final(I, N) \mid I, t \models \alpha\}, & \text{if } \{t \in final(I, N) \mid I, t \models \alpha\} \neq \emptyset \\ \max_{<} \{t \in init(I, N) \mid I, t \models \alpha\}, & \text{otherwise} \end{cases}$$

317 Notice that $<$ above denotes the natural ordering of the underlying temporal structure

318 The strategy to pick out a time point satisfying a given sentence α in N is as follows. If said
 319 sentence is in the final part, we pick the first time point that satisfies it, since we have the guarantee to
 320 find infinitely many time points having the same valuations as $t_{\alpha}^{I, N}$ that also satisfy α (see Lemma
 321 13). If not, we pick the last occurrence in the initial part that satisfies α . Thanks to Definition 23, we
 322 can limit the number of time points taken that satisfy the same sentence.

323 ► **Definition 24 (Selected time points).** Let $I = (V, \prec) \in \mathfrak{I}$, N be an acceptable sequence w.r.t.
 324 I and $\alpha \in \mathcal{L}^{\sim}$ s.t. there is t in N s.t. $I, t \models \alpha$. With $ST(I, N, \alpha) \stackrel{\text{def}}{=} AS(I, (t_{\alpha}^{I, N}))$ we denote the
 325 selected time points of N and α w.r.t. I . (Note that $(t_{\alpha}^{I, N})$ is a sequence of only one element.)

326 Given a sentence $\alpha \in \mathcal{L}^{\sim}$ and an acceptable sequence N w.r.t. I s.t. there is at least one time
 327 point t where $I, t \models \alpha$, the sequence $ST(I, N, \alpha)$ is the acceptable sequence transformation of the
 328 sequence $(t_{\alpha}^{I, N})$. If $t_{\alpha}^{I, N} \in init(I)$, the sequence $ST(I, N, \alpha)$ is the sequence $(t_{\alpha}^{I, N})$. Otherwise, the
 329 sequence $ST(I, N, \alpha)$ is the sequence of all time points t in $final(I)$ that have the same valuation as
 330 $t_{\alpha}^{I, N}$. In both cases, we can see that $size(I, ST(I, N, \alpha)) = 1$.

331 Given an interpretation $I = (V, \prec)$ and N an acceptable sequence w.r.t. I , the *representative*
 332 *sentence* of a valuation v is formally defined as $\alpha_v \stackrel{\text{def}}{=} \bigwedge \{p \mid p \in v\} \wedge \bigwedge \{\neg p \mid p \notin v\}$.

333 ► **Definition 25 (Distinctive reduction).** Let $I = (V, \prec) \in \mathfrak{I}$ and let N be an acceptable sequence
 334 w.r.t. I . With $DR(I, N) \stackrel{\text{def}}{=} \bigcup_{v \in val(I, N)} ST(I, N, \alpha_v)$ we denote the distinctive reduction of N .

335 Given an acceptable sequence N w.r.t. I , $DR(I, N)$ is the sequence containing the chosen
 336 occurrence $t_{\alpha_v}^{I, N}$ that satisfies the representative α_v in N for each $v \in val(I, N)$. In other words,
 337 we pick the selected time points for each possible valuation in $val(I, N)$. There are two interesting

338 results with regard to $DR(I, N)$. The first one is that $DR(I, N)$ is an acceptable sequence w.r.t. I .
 339 This can easily be proven since $ST(I, N, \alpha_v)$ is also an acceptable sequence w.r.t. I , and the union
 340 of all $ST(I, N, \alpha_v)$ is an acceptable sequence w.r.t. I (see Proposition 19). The second result is that
 341 $size(I, DR(I, N)) \leq 2^{|\mathcal{P}|}$. Indeed, thanks to Proposition 19, we can see that $size(I, DR(I, N)) \leq$
 342 $\sum_{v \in val(I, N)} size(ST(I, N, \alpha_v))$. Moreover, we have $size(I, ST(I, N, \alpha_v)) = 1$ for each $v \in$
 343 $val(I, N)$. On the other hand, there are at most $2^{|\mathcal{P}|}$ possible valuations in $val(I, N)$. Thus, we can
 344 assert that $\sum_{v \in val(I, N)} size(I, ST(I, N, \alpha_v)) \leq 2^{|\mathcal{P}|}$, and then we have $size(I, DR(I, N)) \leq 2^{|\mathcal{P}|}$.

345 ► **Definition 26 (Anchors).** Let a sentence $\alpha \in \mathcal{L}^*$ starting with a temporal operator, let $I = (V, \prec$
 346 $) \in \mathcal{J}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$.
 347 The sequence $Anchors(I, T, \alpha)$ is defined as: Let $\alpha_1 \in \mathcal{L}^*$.

$$\begin{aligned} Anchors(I, T, \diamond\alpha_1) &\stackrel{\text{def}}{=} ST(I, \mathbb{N}, \alpha_1); \\ Anchors(I, T, \square\alpha_1) &\stackrel{\text{def}}{=} \emptyset; \\ Anchors(I, T, \heartsuit\alpha_1) &\stackrel{\text{def}}{=} \bigcup_{t \in T} ST(I, AS(I, \min_{\prec}(t)), \alpha_1); \\ Anchors(I, T, \boxtimes\alpha_1) &\stackrel{\text{def}}{=} DR(I, \bigcup_{t \in T} AS(I, \min_{\prec}(t))). \end{aligned}$$

349 Given an acceptable sequence T w.r.t. $I \in \mathcal{J}^{sd}$ where all of its time points satisfy α , where α is a
 350 sentence starting with a temporal operator, $Anchors(I, T, \alpha)$ is an acceptable sequence w.r.t. I . This
 351 is due thanks to the notion of selected time points and distinctive reduction (see Definition 24 and 25).
 352 $Anchors(I, T, \alpha)$ contains the selected time points satisfying the sub-sentence α_1 of α (except for
 353 $\square\alpha_1$). Our goal is to have the selected time points that satisfy α_1 for each $t \in T$.

354 It is worth to point out that the choice of $Anchors(I, T, \square\alpha_1) = \emptyset$ is due to the fact α_1 is satisfied
 355 starting from the first time $t_0 \in T$ i.e., for all $t \geq t_0$, we have $I, t \models \alpha$. So no matter what time point
 356 t we pick after t_0 , we have $I, t \models \alpha_1$. On the other hand, by the nature of the semantics of $\boxtimes\alpha_1$,
 357 all $t \in \bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$ satisfy α_1 . The acceptable sequence $Anchors(I, T, \boxtimes\alpha_1)$ contains
 358 only the selected time points for each distinct valuation in $\bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$.

359 The following are some properties of $Anchors(\cdot)$ that are worth mentioning:

360 ► **Lemma 27.** Let $\alpha_1 \in \mathcal{L}^*$ be a sentence starting with a temporal operator, $I = (V, \prec) \in$
 361 \mathcal{J}^{sd} and let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I, t \models$
 362 $\heartsuit\alpha_1$. Then for all $t, t' \in Anchors(I, T, \heartsuit\alpha_1)$ s.t. $V(t) = V(t')$ and $t \neq t'$, we have $t, t' \in$
 363 $final(I, Anchors(I, T, \heartsuit\alpha_1))$.

364 ► **Proposition 28.** Let $\alpha \in \mathcal{L}^*$ be a sentence starting with a temporal operator, $I = (V, \prec) \in \mathcal{J}^{sd}$.
 365 Let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I, t \models \alpha$. Then, we
 366 have $size(I, Anchors(I, T, \alpha)) \leq 2^{|\mathcal{P}|}$.

367 ► **Proposition 29.** Let $\alpha_1 \in \mathcal{L}^*$, $I = (V, \prec) \in \mathcal{J}^{sd}$, let T be a non-empty acceptable sequence
 368 w.r.t. I s.t. for all $t \in T$ we have $I, t \models \boxtimes\alpha_1$, with $\alpha_1 \in \mathcal{L}^*$. For all acceptable sequences N w.r.t. I
 369 s.t. $Anchors(I, T, \boxtimes\alpha_1) \subseteq N$ and for all $t_i \in N \cap T$, we have the following: Let $I^N = (V^N, \prec^N)$
 370 be the pseudo-interpretation over N and $t' \in N$, if $t' \notin \min_{\prec}(t_i)$, then $t' \notin \min_{\prec^N}(t_i)$.

371 Proposition 29 helps us mitigate the dynamic nature of $\min_{\prec}(t_i)$. The selected time points help
 372 us circumvent adding time points that were not originally “preferred” w.r.t. t_i in I , and becoming
 373 preferred in the reduced structure I^N that we want to build. The strategy of building $Anchors(\cdot)$ is
 374 explained by the fact that we want to preserve the truth values of defeasible sub-sentences of α in the
 375 bounded interpretation.

376 With $Anchors(\cdot)$ defined, we introduce the notion of $Keep(\cdot)$. $Keep(\cdot)$ will help us compute
 377 recursively starting from the initial satisfiable sentence α down to its literals, the selected time points
 378 to pick in order to build our pseudo-interpretation.

379 ► **Definition 30 (Keep).** Let $\alpha \in \mathcal{L}^*$ be in NNF, $I = (V, \prec) \in \mathfrak{J}^{sd}$, and let T be an acceptable
 380 sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. The sequence $Keep(I, T, \alpha)$ is defined as \emptyset , if
 381 $T = \emptyset$; otherwise it is recursively defined as follows:

- 382 ■ $Keep(I, T, \ell) \stackrel{\text{def}}{=} \emptyset$, where ℓ is a literal;
- 383 ■ $Keep(I, T, \alpha_1 \wedge \alpha_2) \stackrel{\text{def}}{=} Keep(I, T, \alpha_1) \cup Keep(I, T, \alpha_2)$;
- 384 ■ $Keep(I, T, \alpha_1 \vee \alpha_2) \stackrel{\text{def}}{=} Keep(I, T_1, \alpha_1) \cup Keep(I, T_2, \alpha_2)$, where $T_1 \subseteq T$ (resp. $T_2 \subseteq T$) is the
 385 sequence of all $t_1 \in T$ (resp. $t_2 \in T$) s.t. $I, t_1 \models \alpha_1$ (resp. $I, t_2 \models \alpha_2$);
- 386 ■ $Keep(I, T, \diamond \alpha_1) \stackrel{\text{def}}{=} Anchors(I, T, \diamond \alpha_1) \cup Keep(I, Anchors(I, T, \diamond \alpha_1), \alpha_1)$;
- 387 ■ $Keep(I, T, \square \alpha_1) \stackrel{\text{def}}{=} Keep(I, T, \alpha_1)$;
- 388 ■ $Keep(I, T, \heartsuit \alpha_1) \stackrel{\text{def}}{=} Anchors(I, T, \heartsuit \alpha_1) \cup Keep(I, Anchors(I, T, \heartsuit \alpha_1), \alpha_1)$;
- 389 ■ $Keep(I, T, \boxtimes \alpha_1) \stackrel{\text{def}}{=} Anchors(I, T, \boxtimes \alpha_1) \cup Keep(I, T', \alpha_1)$, where $T' = \bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$.

390 With $\mu(\alpha)$ we denote the number of classical and non-monotonic modalities in α .

391 ► **Proposition 31.** Let $\alpha \in \mathcal{L}^*$ be in NNF, $I = (V, \prec) \in \mathfrak{J}^{sd}$, and let T be a non-empty acceptable
 392 sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. Then, we have $size(I, Keep(I, T, \alpha)) \leq$
 393 $\mu(\alpha) \times 2^{|\mathcal{P}|}$.

394 Given an acceptable sequence N w.r.t. I , we need to make sure when a time point $t \in N$ in
 395 our acceptable sequence s.t. $I, t \models \alpha$, then $I^N, t \models_{\neq} \alpha$. The function $Keep(I, T, \alpha)$ returns the
 396 acceptable sequence of time s.t. if $Keep(I, T, \alpha) \subseteq N$ and $t \in T$, then said condition is met. We
 397 prove this in Lemma 32.

398 ► **Lemma 32.** Let $\alpha \in \mathcal{L}^*$ be in NNF, $I = (V, \prec) \in \mathfrak{J}^{sd}$, and let T be a non-empty acceptable
 399 sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. For all acceptable sequences N w.r.t. I , if
 400 $Keep(I, T, \alpha) \subseteq N$, then for every $t \in N \cap T$, we have $I^N, t \models_{\neq} \alpha$.

401 Since we build our pseudo-interpretation I^N by adding selected time points for each sub-sentence
 402 α_1 of α , we need to make sure that said sub-sentence remains satisfied in I^N . Lemma 32 ensures that.

403 ► **Definition 33 (Pseudo-interpretation transformation).** Let $I = (V, \prec) \in \mathfrak{J}^{sd}$ and let N be an
 404 infinite acceptable sequence w.r.t. I . The pseudo-interpretation $I^N = (V^N, \prec^N)$ can be transformed
 405 into a preferential interpretation $I' = (V', \prec')$ as follows:

- 406 ■ for all $i \geq 0$, we have $V'(i) = V^N(t_i)$;
- 407 ■ for all $i, j \geq 0$, $t_i, t_j \in N$, we have $(t_i, t_j) \in \prec^N$ iff $(i, j) \in \prec'$.

408 We can now prove our bounded-model theorem.

409 **Proof of Theorem 21.** We assume that $\alpha \in \mathcal{L}^*$ is \mathfrak{J}^{sd} -satisfiable. The first thing we notice is that
 410 $|\alpha| \geq \mu(\alpha) + 1$. Let α' be the NNF of the sentence α . As a consequence of the duality rules of \mathcal{L}^* ,
 411 we can deduce that $\mu(\alpha') = \mu(\alpha)$. Let $I = (V, \prec) \in \mathfrak{J}^{sd}$ s.t. $I, 0 \models \alpha'$. Let $T_0 = AS(I, (0))$ be an
 412 acceptable sequence w.r.t. I . We can see that $size(I, T_0) = 1$. Since for all $t \in T_0$ we have $I, t \models \alpha'$
 413 (see Lemma 13), we can compute recursively $U = Keep(I, T_0, \alpha')$. Thanks to Proposition 31, we
 414 conclude that U is an acceptable sequence w.r.t. I s.t. $size(I, U) \leq \mu(\alpha') \times 2^{|\mathcal{P}|}$. Let $N = T_0 \cup U$
 415 be the union of T_0 and U and let $I^N = (N, V^N, \prec^N)$ be its pseudo-interpretation over N . Thanks to
 416 Proposition 19, we have $size(I, N) \leq 1 + \mu(\alpha') \times 2^{|\mathcal{P}|}$. Thanks to Lemma 32, since $0 \in N \cap T_0$
 417 and $Keep(I, T_0, \alpha') \subseteq N$, we have $I^N, 0 \models_{\neq} \alpha'$. In case N is finite, we replicate the last time point
 418 t_n infinitely many times. Notice that $size(I, N)$ does not change if we replicate the last element.
 419 We can transform the pseudo interpretation I^N to $I' \in \mathfrak{J}^{sd}$ by changing the labels of N into a
 420 sequence of natural numbers minding the order of time points in N (see Definition 33). We can
 421 see that $size(I') = size(I, N)$ and $I', 0 \models \alpha$. Consequently, we have $size(I') \leq 1 + \mu(\alpha') \times 2^{|\mathcal{P}|}$.
 422 Hence, from a given interpretation I s.t. $I, 0 \models \alpha$ we can build an interpretation I' s.t. $I', 0 \models \alpha$ and
 423 $size(I') \leq 1 + \mu(\alpha') \times 2^{|\mathcal{P}|}$. Without loss of generality, we conclude that $size(I') \leq |\alpha| \times 2^{|\mathcal{P}|}$. ◀

6 The satisfiability problem in \mathcal{L}^*

We now provide an algorithm allowing to decide whether a sentence $\alpha \in \mathcal{L}^*$ is \mathcal{J}^{sd} -satisfiable or not. For this purpose, first we focus on particular interpretations of the class \mathcal{J}^{sd} , namely the ultimately periodic interpretations (UPI in short), and a finite representation of these interpretations, called ultimately periodic pseudo-interpretation (UPPI in short). As we will see in the second part of this section, to decide the \mathcal{J}^{sd} -satisfiability of a sentence $\alpha \in \mathcal{L}^*$, the proposed algorithm guesses a bounded UPPI in a first step. Then, it checks the satisfiability of α by the UPI of the guessed UPPI.

► **Definition 34 (UPI).** Let $I = (V, \prec) \in \mathcal{J}^{sd}$ and let $\pi = \text{card}(\text{range}(I))$. We say I is an ultimately periodic interpretation if:

- for every $t, t' \in [t_I, t_I + \pi[$ s.t. $t \neq t'$ (see Definition 10), we have $V(t) \neq V(t')$,
- for every $t \in [t_I, +\infty[$, we have $V(t) = V(t_I + (t - t_I) \bmod \pi)$.

A UPI I is a state dependent interpretation s.t. each time point's valuation in $\text{final}(I)$ is replicated periodically. Given a UPI, $\pi = \text{card}(\text{range}(I))$ denotes the length of the period and the interval $[t_I, t_I + \pi[$ is the first period which is replicated periodically throughout the final part. It is worth pointing out that for every $t \in \text{final}(I)$, we have $V(t) \in \{V(t') \mid t' \in [t_I, t_I + \pi[\}$, which is one of the consequences of the definition above. Thanks to Lemma 15, we can prove the following proposition.

► **Proposition 35.** Let \mathcal{P} be a set of atomic propositions, $I = (V, \prec) \in \mathcal{J}^{sd}$, $i = \text{length}(\text{init}(I))$ and $\pi = \text{card}(\text{range}(I))$. There exists an ultimately periodic interpretation $I' = (V', \prec') \in \mathcal{J}^{sd}$ s.t. I, I' are faithful interpretations over \mathcal{P} (Definition 14), $\text{init}(I') \doteq \text{init}(I)$, $\text{range}(I') = \text{range}(I)$ and $V'(0) = V(0)$. Moreover, for all $\alpha \in \mathcal{L}^*$, we have $I, 0 \models \alpha$ iff $I', 0 \models \alpha$.

It is worth to point out that the size of an interpretation and that of its UPI counterparts are equal. It can easily be seen that these interpretations have the same initial part and the same range of valuations in the final part. I' from the aforementioned proposition is obtained from I by keeping the same initial part, and placing each distinct valuation of $\text{range}(I)$ in the interval $[t_I, t_I + \pi[$ and replicating this interval infinitely many times. Moreover, the preference relation \prec' arranges valuations in the same way as \prec . We can see that I and I' are faithful and that $\text{init}(I') \doteq \text{init}(I)$, $\text{range}(I') = \text{range}(I)$ and $V'(0) = V(0)$. Therefore, I and I' satisfy the same sentences.

► **Definition 36 (UPPI).** A model structure is a tuple $M = (i, \pi, V_M, \prec_M)$ where: i, π are two integers such that $i \geq 0$ and $\pi > 0$ (where i is intended to be the starting point of the period, π is the length of the period); $V_M : [0, i + \pi[\rightarrow 2^{\mathcal{P}}$, and $\prec_M \subseteq 2^{\mathcal{P}} \times 2^{\mathcal{P}}$ is a strict partial order. Moreover, (I) for all $t \in [i, i + \pi[$, we have $V_M(t) \neq V_M(i - 1)$; and (II) for all distinct $t, t' \in [i, i + \pi[$, we have $V_M(t) \neq V_M(t')$.

The reason behind setting properties (I) and (II) is that we can build a UPPI from a UPI, and back.

Given a UPPI $M = (i, \pi, V_M, \prec_M)$, we define the size of M by $\text{size}(M) \stackrel{\text{def}}{=} i + \pi$. From a UPPI we define a UPI in the following way:

► **Definition 37.** Given a UPPI $M = (i, \pi, V_M, \prec_M)$, let $\text{l}(M) \stackrel{\text{def}}{=} (V, \prec)$, where for every $t \geq 0$, $V(t) \stackrel{\text{def}}{=} V_M(t)$, if $t < i$, and $V(t) \stackrel{\text{def}}{=} V_M(i + (t - i) \bmod \pi)$, otherwise, and $\prec \stackrel{\text{def}}{=} \{(t, t') \mid (V(t), V(t')) \in \prec_M\}$.

Given a UPPI $M = (i, \pi, V_M, \prec_M)$, the interval $[0, i[$ of a UPPI corresponds to the initial temporal part of the underlying interpretation $\text{l}(M)$ and $[i, i + \pi[$ represents a temporal period that is infinitely replicated in order to determine the final temporal part of the interpretation.

It is worth to point out that given a UPPI M , $\text{l}(M) = (V, \prec)$ is a UPI. Moreover, we have $\text{size}(\text{l}(M)) = \text{size}(M)$.

Now we extend the notion of preferred time points w.r.t a time point for a UPPI :

469 ► **Definition 38 (UPPI's preferred time points).** Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI and a
 470 time point $t \in [0, i + \pi[$. The set of preferred time points of t w.r.t. M , denoted by $\min_{\prec_M}(t)$,
 471 is defined as follows: $\min_{\prec_M}(t) \stackrel{\text{def}}{=} \{t' \in [\min_{<}\{t, i\}, i + \pi[\mid \text{there is no } t'' \in [\min_{<}\{t, i\}, i +$
 472 $\pi[\text{ with } (V_M(t''), V_M(t')) \in \prec_M\}$.

473 ► **Proposition 39.** Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI, $\mathfrak{l}(M) = (V, \prec)$ and $t, t', t_M, t'_M \in \mathbb{N}$ s.t.:

$$474 \quad t_M = \begin{cases} t, & \text{if } t < i; \\ i + (t - i) \bmod \pi, & \text{otherwise.} \end{cases} \quad t'_M = \begin{cases} t', & \text{if } t' < i; \\ i + (t' - i) \bmod \pi, & \text{otherwise.} \end{cases}$$

475 We have the following: $t' \in \min_{\prec}(t)$ iff $t'_M \in \min_{\prec_M}(t_M)$.

476 Now that UPPI is defined, we can move to the task of checking the satisfiability of a sentence
 477 α . We define for a UPPI $M = (i, \pi, V_M, \prec_M)$ and a sentence $\alpha \in \mathcal{L}^*$ a labelling function $\text{lab}_\alpha^M(\cdot)$
 478 which associates a set of sub-sentences of α to each $t \in [0, i + \pi[$.

479 ► **Definition 40 (Labelling function).** Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI, $\alpha \in \mathcal{L}^*$. The set of
 480 sub-sentences of α for $t \in [0, i + \pi[$, denoted by $\text{lab}_\alpha^M(t)$, is defined as follows:

- 481 ■ $p \in \text{lab}_\alpha^M(t)$ iff $p \in V_M(t)$; $\neg\alpha_1 \in \text{lab}_\alpha^M(t)$ iff $\alpha_1 \notin \text{lab}_\alpha^M(t)$;
- 482 ■ $\alpha_1 \wedge \alpha_2 \in \text{lab}_\alpha^M(t)$ iff $\alpha_1, \alpha_2 \in \text{lab}_\alpha^M(t)$; $\alpha_1 \vee \alpha_2 \in \text{lab}_\alpha^M(t)$ iff $\alpha_1 \in \text{lab}_\alpha^M(t)$ or $\alpha_2 \in \text{lab}_\alpha^M(t)$;
- 483 ■ $\diamond\alpha_1 \in \text{lab}_\alpha^M(t)$ iff $\alpha_1 \in \text{lab}_\alpha^M(t')$ for some $t' \in [\min_{<}\{t, i\}, i + \pi[$;
- 484 ■ $\square\alpha_1 \in \text{lab}_\alpha^M(t)$ iff $\alpha_1 \in \text{lab}_\alpha^M(t')$ for all $t' \in [\min_{<}\{t, i\}, i + \pi[$;
- 485 ■ $\diamond\alpha_1 \in \text{lab}_\alpha^M(t)$ iff $\alpha_1 \in \text{lab}_\alpha^M(t')$ for some $t' \in \min_{\prec_M}(t)$;
- 486 ■ $\square\alpha_1 \in \text{lab}_\alpha^M(t)$ iff $\alpha_1 \in \text{lab}_\alpha^M(t')$ for all $t' \in \min_{\prec_M}(t)$.

487 ► **Lemma 41.** Let a UPPI $M = (i, \pi, V_M, \prec_M)$, $\alpha \in \mathcal{L}^*$ and $t \in \mathbb{N}$, $\mathfrak{l}(M), 0 \models \alpha$ iff $\alpha \in \text{lab}_\alpha^M(0)$.

488 We accept a UPPI M as a model for $\alpha \in \mathcal{L}^*$ iff $\alpha \in \text{lab}_\alpha^M(0)$. Otherwise, M is rejected.

489 ► **Proposition 42.** Let $\alpha \in \mathcal{L}^*$. We have that α is \mathfrak{I}^{sd} -satisfiable iff there exists a UPPI M such
 490 that $\mathfrak{l}(M), 0 \models \alpha$ and $\text{size}(\mathfrak{l}(M)) \leq |\alpha| \times 2^{|\mathcal{P}|}$.

491 Hence, to decide the satisfiability of a sentence $\alpha \in \mathcal{L}^*$, we can first guess a UPPI M bounded by
 492 $|\alpha| \times 2^{|\mathcal{P}|}$. Next, using the labelling function of M , we check the satisfiability of α by the UPI $\mathfrak{l}(M)$.

493 ► **Theorem 43.** \mathfrak{I}^{sd} -satisfiability problem for \mathcal{L}^* sentences is decidable.

494 **7** Concluding remarks

495 The contributions of this paper are as follows: we introduced the formalism of LTL^\sim with its
 496 expressive syntax and intuitive semantics. We defined also the class of state-dependent interpretations
 497 \mathfrak{I}^{sd} and the fragment \mathcal{L}^* . We then showed that \mathfrak{I}^{sd} -satisfiability in \mathcal{L}^* is a decidable problem.

498 It is worth pointing out that it is hard to define a tableaux method for our logic similar to
 499 Wolper's [19]. The main reason is that we do not have defeasible versions of the axioms (T) and (4),
 500 and therefore nested defeasible modalities cannot be reduced as in the classical case. Furthermore, at
 501 present we have $\not\models \square\alpha \leftrightarrow \alpha \wedge \bigcirc \square\alpha$ and $\not\models \diamond\alpha \leftrightarrow \alpha \vee \bigcirc \diamond\alpha$. That is why we decided to tackle the
 502 satisfiability problem of our logic before establishing a semantic tableaux for LTL^\sim .

503 Among the immediate next steps is the introduction of defeasible counterparts to \bigcirc and \mathcal{U} . We
 504 shall also investigate the addition of \sim -statements to our logic.

8 The fragment \mathcal{L}_1

505

506 In this section, we prove that we have another fragment whose satisfiability is a decidable problem.
 507 The vocabulary of the fragment \mathcal{L}_1 consists of a finite set of literals \mathcal{P} ($l \in \mathcal{P}$ can be an atomic
 508 proposition or its negation) and we limit the set of operators to the following: $(\wedge, \vee, \diamond, \heartsuit, \circ)$.
 509 Sentences in \mathcal{L}_1 can be recursively defined as such:

$$510 \quad \alpha ::= \top \mid \perp \mid l \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \diamond \alpha \mid \circ \alpha \mid \heartsuit \alpha$$

511 Notice that the negation can only be on the literal level.

512 ► **Definition 44 (Sub-sequence).** Let N, N' be two ordered sequences of natural numbers, N' is
 513 a subsequence of N iff $N' \subseteq N$.

514 ► **Definition 45 (Pseudo-interpretation over N).** Let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence of
 515 \mathbb{N} . The pseudo-interpretation over N is the pair $I^N \stackrel{\text{def}}{=} (V^N, \prec^N)$ where:

- 516 ■ $V^N : N \rightarrow 2^{\mathcal{P}}$ is a valuation function over N , where for all $i \in N$, we have $V^N(i) = V(i)$,
- 517 ■ $\prec^N \subseteq N \times N$, where for all $(i, j) \in N^2$, we have $(i, j) \in \prec^N$ iff $(i, j) \in \prec$.

518 The size of a pseudo interpretation is the number of time points in N . Namely, the size of I^N is
 519 $\text{size}(I, N) \stackrel{\text{def}}{=} \text{length}(N)$.

520 The truth values of \mathcal{L}^* sentences in pseudo-interpretations are defined in a similar fashion as
 521 for preferential temporal interpretations. With \models we denote the truth values of sentences in a
 522 pseudo-interpretation. Let $t \in N$:

- 523 ■ $I^N, t \models l$ if $I, t \models l$;
- 524 ■ $I^N, t \models \diamond \alpha$ if $I^N, t' \models \alpha$ for some $t' \in N$ s.t. $t' \geq t$;
- 525 ■ $I^N, t \models \circ \alpha$ if we have $t + 1 \in N$ and $I^N, t + 1 \models \alpha$;
- 526 ■ $I^N, t \models \heartsuit \alpha$ if $I^N, t' \models \alpha$ for some $t' \in N$ s.t. $t' \in \min_{\prec^N}(t)$.

527 **Proof.** ■ $\alpha_{bool} := p$. Since $I^N, i \models p$, we know that $p \in V^N(i)$ and therefore $p \in V(i)$. On the
 528 other hand, since we have $i \in N'$ and $p \in V(i)$, then we have $p \in V^{N'}(i)$. Therefore, we have
 529 $I^{N'}, i \models p$.

530 ■ $\alpha_{bool} := \neg p$. Since $I^N, i \models \neg p$, we know that $p \notin V^N(i)$ and therefore $p \notin V(i)$. On the
 531 other hand, since we have $i \in N'$ and $p \notin V(i)$, then we have $p \in V^{N'}(i)$. Therefore, we have
 532 $I^{N'}, i \models \neg p$.

533 ■ $\alpha_{bool} := \alpha_1 \wedge \alpha_2$. We have $I^N, i \models \alpha_1 \wedge \alpha_2$, that means $I^N, i \models \alpha_1$ and $I^N, i \models \alpha_2$. Since N'
 534 is a subsequence of N containing i , and using the induction hypothesis on α_1 and α_2 , we have
 535 $I^{N'}, i \models \alpha_1$ and $I^{N'}, i \models \alpha_2$. Therefore, we have $I^{N'}, i \models \alpha_1 \wedge \alpha_2$.

536 ■ $\alpha_{bool} := \alpha_1 \vee \alpha_2$. We have $I^N, i \models \alpha_1 \vee \alpha_2$, that means either $I^N, i \models \alpha_1$ or $I^N, i \models \alpha_2$. We
 537 suppose that $I^N, i \models \alpha_1$. Since N' is a subsequence of N containing i , and using the induction
 538 hypothesis on α_1 , we have $I^{N'}, i \models \alpha_1$. Therefore, we have $I^{N'}, i \models \alpha_1 \vee \alpha_2$. Same reasoning
 539 applies when $I^N, i \models \alpha_2$.

540

541 ► **Lemma 46.** Let $\alpha \in \mathcal{L}_1$, $I = (V, \prec) \in \mathfrak{I}$ and $N \subseteq \mathbb{N}$ s.t. $I^N, t \models \alpha$; there exists a **finite**
 542 sequence M containing t such that:

- 543 I $M \subseteq N$;
- 544 II $\text{size}(I, M) \leq |\alpha|$;
- 545 III for all sequences Q where $M \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha$.

546 **Proof.** Let $\alpha \in \mathcal{L}_1$, $I = (V, \prec) \in \mathcal{J}$ and $N \subseteq \mathbb{N}$ s.t. $I^N, t \models \alpha$; we use structural induction on the
547 length of α .

548 ■ Base case: $\alpha := l$ where L is a literal. Let $M := (t)$ be sequence containing only t , then M is a
549 finite sequence such that:

- 550 I Since $t \in N$, then $M \subseteq N$;
- 551 II $size(I, M) = 1 \leq |l|$;
- 552 III Since $I^N, t \models l$, then we have $I, t \models l$. Let Q be a sequence s.t. $M \subseteq Q \subseteq N$, we have $t \in Q$.
553 Therefore, we have $I^Q, t \models l$.

554 ■ $\alpha := \alpha_1 \wedge \alpha_2$. Since $I^N, t \models \alpha_1 \wedge \alpha_2$, we then have $I^N, t \models \alpha_1$ and $I^N, t \models \alpha_2$. Using the
555 induction hypothesis on α_1 , there exists a finite sequence M_1 containing t such that:

- 556 I $M_1 \subseteq N$;
- 557 II $size(I, M_1) \leq |\alpha_1|$;
- 558 III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha_1$.

559 Similarly, using the induction hypothesis on α_2 , there exists a finite sequence M_2 such that:

- 560 I $M_2 \subseteq N$;
- 561 II $size(I, M_2) \leq |\alpha_2|$;
- 562 III for all sequences Q where $M_2 \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha_2$.

563 Let $M := M_1 \cup M_2$. Since M_1 and M_2 contain t , then M is a finite sequence that contains t . We
564 also have:

- 565 I Since $M_1 \subseteq N$ and $M_2 \subseteq N$, then we have $M_1 \cup M_2 \subseteq N$;
- 566 II $size(I, M) = size(M_1 \cup M_2) \leq size(I, M_1) + size(I, M_2) \leq |\alpha_1| + |\alpha_2| \leq |\alpha_1 \wedge \alpha_2|$;
- 567 III Let $M \subseteq Q \subseteq N$. Since $M_1 \subseteq Q \subseteq N$, then we have $I^Q, t \models \alpha_1$. Similarly, Since
568 $M_2 \subseteq Q \subseteq N$, then we have $I^Q, t \models \alpha_2$. Therefore, we have $I^Q, t \models \alpha_1 \wedge \alpha_2$.

569 ■ $\alpha := \alpha_1 \vee \alpha_2$. We have either $I^N, t \models \alpha_1$ or $I^N, t \models \alpha_2$. Using the induction hypothesis on α_1 ,
570 there exists a finite sequence M_1 containing t such that:

- 571 I $M_1 \subseteq N$;
- 572 II $size(I, M_1) \leq |\alpha_1|$;
- 573 III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha_1$.

574 Let $M := M_1$. Since M_1 contains t , then M is a finite sequence that contains t . We also have:

- 575 I Since $M = M_1 \subseteq N$;
- 576 II $size(I, M) = size(M_1) \leq |\alpha_1| \leq |\alpha_1 \vee \alpha_2|$;
- 577 III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha_1$. Therefore, $I, t \models \alpha_1 \vee \alpha_2$.

578 The reasoning is the same when $I^N, t \models \alpha_2$.

579 ■ $\alpha := \bigcirc \alpha_1$. Since $I^N, t \models \bigcirc \alpha_1$, then $t + 1 \in N$ and $I^N, t + 1 \models \alpha_1$. Using the induction
580 hypothesis on α_1 , there exists a finite sequence sequence containing $t + 1$ such that:

- 581 I $M_1 \subseteq N$;
 582 II $size(I, M_1) \leq |\alpha_1|$;
 583 III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t + 1 \models \alpha_1$.

584 Let $M := (t) \cup M_1$; then M is a finite sequence containing t such that:

- 585 I Since $M_1 \subseteq N$ and $t \in N$, then we have $M \subseteq N$;
 586 II $size(I, M) = 1 + size(I, M_1) \leq |\alpha_1|$;
 587 III Let Q be a sequence such that $M \subseteq Q \subseteq N$, we have $t, t + 1 \in M$. Since $M_1 \subseteq Q \subseteq N$,
 588 then $I^Q, t + 1 \models \alpha_1$. Therefore, we have $I^Q, t \models \alpha_1$.

- 589 ■ $\alpha := \diamond\alpha_1$. Since $I^N, t \models \diamond\alpha_1$, then $t' \in N$ and $I^N, t' \models \alpha_1$. Using the induction hypothesis on
 590 α_1 , there exists a finite sequence containing t' such that:

- 591 I $M_1 \subseteq N$;
 592 II $size(I, M_1) \leq |\alpha_1|$;
 593 III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t' \models \alpha_1$.

594 Let $M := (t) \cup M_1$; then M is a finite sequence containing t such that:

- 595 I Since $M_1 \subseteq N$ and $t \in N$, then we have $M \subseteq N$;
 596 II $size(I, M) = 1 + size(I, M_1) \leq |\diamond\alpha_1|$;
 597 III Let Q be a sequence such that $M \subseteq Q \subseteq N$, we have $t, t' \in M$. Since $M_1 \subseteq Q \subseteq N$, then
 598 $I^Q, t' \models \alpha_1$. Therefore, we have $I^Q, t \models \diamond\alpha_1$.

- 599 ■ $\alpha := \diamond\alpha_1$. Since $I^N, t \models \diamond\alpha_1$, there exists $t' \in N$ s.t. $t' \in \min_{\prec^N}(t)$. Using the induction
 600 hypothesis on α_1 , there exists a finite sequence M_1 containing t' such that:

- 601 I $M_1 \subseteq N$;
 602 II $size(I, M_1) \leq |\alpha_1|$;
 603 III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t' \models \alpha_1$.

604 Let $M := (t) \cup M_1$; then M is a finite sequence containing t such that:

- 605 I Since $M_1 \subseteq N$ and $t \in N$, then we have $M \subseteq N$;
 606 II $size(I, M) = 1 + size(I, M_1) \leq |\diamond\alpha_1|$;
 607 III Let Q be a sequence such that $M \subseteq Q \subseteq N$, we have $t, t' \in M$. Since $M_1 \subseteq Q \subseteq N$, then
 608 (i) $I^Q, t' \models \alpha_1$.
 609 We suppose that $t' \in \min_{\prec^Q}(t)$, there exists $t'' \in Q$ s.t. $(t'', t') \in \prec^Q$. Following this sup-
 610 position, we have $(t'', t') \in \prec$. Since $t', t'' \in N$, we have $(t'', t') \in \prec^N$, thus $t' \notin \min_{\prec^N}(t)$.
 611 This supposition conflicts with our assumption that $t' \in \min_{\prec^N}(t)$. Therefore we have (ii)
 612 $t' \in \min_{\prec^Q}(t)$. From (i) and (ii), we conclude that $I^Q, t \models \diamond\alpha_1$.

613
 614



615 ► **Definition 47** (Pseudo-interpretation transformation). Let $I = (V, \prec) \in \mathfrak{J}$, let $N =$
 616 $(t_0, t_1, t_2, \dots, t_{n-1})$ be a finite sequence. The pseudo-interpretation $I^N = (V^N, \prec^N)$ can be trans-
 617 formed into a preferential interpretation $I' \stackrel{\text{def}}{=} (V', \prec')$ as follows:

$$618 \quad V' : \begin{cases} V'(i) := V^N(t_i), & \text{if } 0 \leq i < n; \\ V'(i) := \top, & \text{otherwise.} \end{cases}$$

619 And for all $0 \leq i, j < n$ s.t. $(t_i, t_j) \in \prec^N$, we have $(i, j) \in \prec'$.

620 ► **Theorem 48** (Bounded Model property). Let $\alpha \in \mathcal{L}_1$ be \mathfrak{J} -satisfiable, there exists $I = (V, \prec$
 621 $) \in \mathfrak{J}$ s.t. $\text{size}(I) \leq |\alpha| + 1$.

622 Let $I^N := (V^N, \prec^N)$ be a pseudo-interpretation and let $I = (V', \prec')$ be its transformed inter-
 623 pretation. We can see that $\text{size}(I') = \text{size}(I, M) + 1$. The size of the initial part of I' is the sequence
 624 N and the final part has one distinct valuation which is \top .

625 **Proof.** Let α be a \mathfrak{J} -satisfiable sentence and let $I = (V, \prec) \in \mathfrak{J}$ where $I, 0 \models \alpha$ be an interpretation
 626 that satisfies α . Thanks to Lemma 46, since \mathbb{N} is a sequence and $0 \in \mathbb{N}$ s.t. $I, 0 \models \alpha$, then there is a
 627 sequence $M \subseteq N$ containing 0 where $\text{size}(I, M) \leq |\alpha|$ and $I^M, 0 \models \alpha$. We can transform it then
 628 to $I' = (V', \prec')$ by changing the labels of M into a sequence of natural numbers and looping the
 629 valuation \top after the transformed sequence M . We can see that $I', 0 \models \alpha$ and $\text{size}(I) \leq |\alpha| + 1$.
 630 ◀

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678 A Proofs of results in Section 3 and Section 4

679 ► **Proposition 8.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i, i', j, j' \in \mathbb{N}$ s.t. $i \leq i'$, $i' \leq j'$ and $j \in \min_{\prec}(i)$.
680 If $V(j) = V(j')$, then $j' \in \min_{\prec}(i')$.

681 **Proof.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let i, j, i', j' be four time points s.t. $i \leq i'$, $i' \leq j'$ and $j \in$
682 $\min_{\prec}(i)$. We assume that $V(j) = V(j')$ and we suppose that $j' \notin \min_{\prec}(i')$. Following our
683 supposition, $j' \notin \min_{\prec}(i')$ means that there exists $k \in [i', +\infty[$ where $(k, j') \in \prec$. From Definition
684 7, if $(k, j') \in \prec$ and $V(j) = V(j')$, then $(k, j) \in \prec$. Since $(k, j) \in \prec$, we have $j \notin \min_{\prec}(i)$.
685 This conflicts with our assumption of $j \in \min_{\prec}(i)$. We conclude that if $V(j) = V(j')$ then
686 $j' \in \min_{\prec}(i')$. ◀

687 ► **Proposition 9.** Let $I = (V, \prec) \in \mathfrak{I}$ and let $i, j \in \mathbb{N}$ s.t. $j \in \min_{\prec}(i)$. For all $i \leq i' \leq j$, we
688 have $j \in \min_{\prec}(i')$.

689 **Proof.** Let $I = (V, \prec) \in \mathfrak{I}$ and let $i, i', j \in \mathbb{N}$ s.t. $j \in \min_{\prec}(i)$ and $i \leq i' \leq j$. Since $j \in \min_{\prec}(i)$,
690 there is no $j' \in [i, +\infty[$ s.t. $(j', j) \in \prec$. Moreover, we have $i \leq i'$, we conclude that there is no
691 $j' \in [i', +\infty[$ s.t. $(j', j) \in \prec$. Therefore, we have $j \in \min_{\prec}(i')$. ◀

692 ► **Proposition 12.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i \leq j \leq i' \leq j'$ be time points in $\text{final}(I)$ s.t.
693 $V(j) = V(j')$. Then we have $j \in \min_{\prec}(i)$ iff $j' \in \min_{\prec}(i')$.

694 **Proof.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$. We have four time points $i \leq j \leq i' \leq j' \in \text{final}(I)$, this proof is
695 divided in two parts:

- 696 ■ For the only-if part, we suppose that $j \in \min_{\prec}(i)$ and we prove that $j' \in \min_{\prec}(i')$. We have
697 $i \leq i'$, $i' \leq j'$, $V(j) = V(j')$ and $j \in \min_{\prec}(i)$. Thanks to Proposition 8, $j' \in \min_{\prec}(i')$.
- 698 ■ For the if part, we suppose that $j' \in \min_{\prec}(i')$ and we prove that $j \in \min_{\prec}(i)$. We use a proof
699 by contradiction. We assume that $j' \in \min_{\prec}(i')$ and we suppose that $j \notin \min_{\prec}(i)$. This implies
700 that there exists $k \in [i, +\infty[$ such that $(k, j) \in \prec$.

- 701 ■ Case 1: $k \in [i', +\infty[$. From Definition 7, since $V(j) = V(j')$ and $(k, j) \in \prec$, then $(k, j') \in \prec$
702 thus $j' \notin \min_{\prec}(i')$. This conflicts with our assumption that $j' \in \min_{\prec}(i')$.
- 703 ■ Case 2: $k \in [i, i'[$. From Lemma 10, since $k \in \text{final}(I)$, then there exists $k' \in [i', +\infty[$
704 such that $V(k') = V(k)$. From Definition 7, since we have $V(j') = V(j)$, $V(k') = V(k)$
705 and $(k, j) \in \prec$, then $(k', j') \in \prec$, thus $j' \notin \min_{\prec}(i')$. This conflicts with our assumption that
706 $j' \in \min_{\prec}(i')$.

707

708 ► **Lemma 13.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $i \leq i'$ be time points of $\text{final}(I)$ where $V(i) = V(i')$.
709 Then for every $\alpha \in \mathcal{L}^*$, we have $I, i \models \alpha$ iff $I, i' \models \alpha$.

710 **Proof.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $i \leq i'$ in $\text{final}(I)$ such that $V(i) = V(i')$. We prove that $I, i \models \alpha$
711 iff $I, i' \models \alpha$ using structural induction on α .

- 712 ■ Base: α is an atomic proposition p . For the only-if part, we know that $I, i \models p$ iff $p \in V(i)$.
713 Since $V(i) = V(i')$, we have $p \in V(i')$, thus $I, i' \models p$. Same reasoning applies for the if part.
- 714 ■ $\alpha = \neg\alpha_1$. For the only-if part, we assume that $I, i \models \neg\alpha_1$ and suppose that $I, i' \not\models \neg\alpha_1$.
715 $I, i' \not\models \neg\alpha_1$ implies $I, i' \models \alpha_1$. Since the Lemma holds on α_1 and $I, i' \models \alpha_1$, we conclude that
716 $I, i \models \alpha_1$, conflicting with our assumption. We follow the same reasoning for the if part.
- 717 ■ $\alpha = \alpha_1 \wedge \alpha_2$. For the only-if part, $I, i \models \alpha_1 \wedge \alpha_2$ means that $I, i \models \alpha_1$ and $I, i \models \alpha_2$. Since the
718 Lemma holds on both α_1 and α_2 , we have $I, i' \models \alpha_1$ and $I, i' \models \alpha_2$. Thus $I, i' \models \alpha_1 \wedge \alpha_2$. The
719 same reasoning applies for the if part.
- 720 ■ $\alpha = \Diamond\alpha_1$. For the only-if part, we assume that $I, i \models \Diamond\alpha_1$. Following our assumption, it means
721 that there exists $j \in [i, +\infty[$ s.t. $I, j \models \alpha_1$. Thanks to Lemma 10. Since $j \in \text{final}(I)$, there
722 exists $j' \in [i', +\infty[$ where $V(j') = V(j)$. Thanks to the induction hypothesis, if $V(j) = V(j')$
723 and $I, j \models \alpha_1$ then $I, j' \models \alpha_1$, we conclude that $I, i' \models \Diamond\alpha_1$.
- 724 For the if part, we assume that $I, i' \models \Diamond\alpha_1$. $I, i' \models \Diamond\alpha_1$ means that there exists $j' \in [i', +\infty[$ s.t.
725 $I, j' \models \alpha_1$. We know that $[i', +\infty[\subseteq [i, +\infty[$, we conclude that $I, i \models \Diamond\alpha_1$.
- 726 ■ $\alpha = \Diamond\alpha_1$. For the only-if part, we assume that $I, i \models \Diamond\alpha_1$. Following our assumption, $I, i \models \Diamond\alpha_1$
727 means that there exists $j \in [i, +\infty[$ s.t. $j \in \min_{\prec}(i)$ and $I, j \models \alpha_1$. Thanks to Lemma 10.
728 Since $j \in \text{final}(I)$, there exists $j' \in [i', +\infty[$ such that $V(j') = V(j)$. Thanks to the induction
729 hypothesis, if $V(j) = V(j')$ and $I, j \models \alpha_1$ then (I) $I, j' \models \alpha_1$. Thanks to Proposition 8,
730 $V(j) = V(j')$, $i \leq i'$, $i' \leq j'$ and $j \in \min_{\prec}(i)$ means that (II) $j' \in \min_{\prec}(i')$. From (I) and (II),
731 we conclude that $I, i' \models \Diamond\alpha_1$.
- 732 For the if part, we assume that $I, i' \models \Diamond\alpha_1$. $I, i' \models \Diamond\alpha_1$ means that there is a $j' \in [i', +\infty[$
733 such that $j' \in \min_{\prec}(i')$ and (I) $I, j' \models \alpha_1$. We need to prove that $j' \in \min_{\prec}(i)$. We suppose
734 that $j' \notin \min_{\prec}(i)$. It means that there exists $k \in [i, +\infty[$ such that $(k, j') \in \prec$. From Lemma
735 10, since $k \in \text{final}(I)$, that means there is $k' \in [i', +\infty[$ such that $V(k) = V(k')$. Following
736 the condition set in Definition 7, since $(k, j') \in \prec$ and $V(k') = V(k)$, then $(k', j') \in \prec$ and thus
737 $j' \notin \min_{\prec}(i')$, conflicting with our assumption of $j' \in \min_{\prec}(i')$, thus (II) $j' \in \min_{\prec}(i)$.
738 From (I) and (II), we conclude that $I, i \models \Diamond\alpha_1$.

739

740 The result of Lemma 15 can be found in Section D.

741 ► **Proposition 19.** Let $I = (V, \prec) \in \mathfrak{I}$, N_1, N_2 be two acceptable sequences w.r.t. I . Then $N_1 \cup N_2$
742 is an acceptable sequence w.r.t. I s.t. $\text{size}(I, N_1 \cup N_2) \leq \text{size}(I, N_1) + \text{size}(I, N_2)$.

743 **Proof.** Let $I = (V, \prec) \in \mathfrak{I}$, N_1, N_2 be two acceptable sequences w.r.t. I and let $I^{N_1} = (V^{N_1}, \prec^{N_1})$,
744 $I^{N_2} = (V^{N_2}, \prec^{N_2})$ be two pseudo interpretations over N_1 and N_2 respectively. We assume that
745 $N = N_1 \cup N_2$.

746 We suppose that N is not an acceptable sequence w.r.t. I . It means that there exist two time points
 747 $t, t' \in \text{final}(I)$ s.t. $V(t) = V(t')$ where $t \in N$ and $t' \notin N$. Since $t \in N$, t is either an element of N_1
 748 or N_2 . We consider that $t \in N_1$. By Definition 16, since $t \in N_1$ and N_1 is an acceptable sequence
 749 w.r.t. I , all time points of $\text{final}(I)$ that have the same valuation as t are in N_1 . Since $t' \in \text{final}(I)$
 750 and $V(t') = V(t)$, then $t' \in N_1$, and therefore $t' \in N$. This conflicts with the supposition of $t' \notin N$.
 751 Same reasoning applies if we take $t \in N_2$. We conclude that for all $t \in N$ s.t. $t \in \text{final}(I)$, all
 752 $t' \in \text{final}(I)$ s.t. $V(t') = V(t)$ are also in N . Thus, N is an acceptable sequence w.r.t. I .

753 In order to prove that $\text{size}(I, N) \leq \text{size}(I, N_1) + \text{size}(I, N_2)$, we need to prove that $\text{init}(I, N) \subseteq$
 754 $\text{init}(I, N_1) \cup \text{init}(I, N_2)$ and $\text{range}(I, N) \subseteq \text{range}(I, N_1) \cup \text{range}(I, N_2)$. Let $t \in N$ be a time
 755 point s.t. $t \in \text{init}(I, N)$. By the definition of $\text{init}(I, N)$, we know that $t \in \text{init}(I)$. Since N is a
 756 sequence containing only elements of N_1 or N_2 , the time point t is either in N_1 or N_2 . By definition
 757 of $\text{init}(I, N_1)$, if $t \in N_1$ and $t \in \text{init}(I)$, then $t \in \text{init}(I, N_1)$. The same goes in the case of $t \in N_2$.
 758 We conclude that if $t \in \text{init}(I, N)$, then $t \in \text{init}(I, N_1) \cup \text{init}(I, N_2)$.

759 Following the same line of thought, we can prove that $\text{final}(I, N) \subseteq \text{final}(I, N_1) \cup \text{final}(I, N_2)$
 760 and consequently we can prove that $\text{range}(I, N) \subseteq \text{range}(I, N_1) \cup \text{range}(I, N_2)$.

761 Since $\text{init}(I^N) \subseteq \text{init}(I^{N_1}) \cup \text{init}(I^{N_2})$, then $\text{length}(\text{init}(I^N)) \leq \text{length}(\text{init}(I^{N_1})) + \text{length}(\text{init}(I^{N_2}))$.
 762 Similarly, if $\text{range}(I^N) \subseteq \text{range}(I^{N_1}) \cup \text{range}(I^{N_2})$, then $\text{card}(\text{range}(I^N)) \leq \text{card}(\text{range}(I^{N_1})) +$
 763 $\text{card}(\text{range}(I^{N_2}))$. Therefore, we conclude that $\text{size}(I^N) \leq \text{size}(I^{N_1}) + \text{size}(I^{N_2})$. ◀

764 ▶ **Proposition 20.** Let $I = (V, \prec) \in \mathcal{J}$ and N be an acceptable sequence w.r.t. I . If for all distinct
 765 $t, t' \in N$, we have $V(t') = V(t)$ only when both $t, t' \in \text{final}(I, N)$, then $\text{size}(I, N) \leq 2^{|\mathcal{P}|}$.

766 **Proof.** Let $I = (V, \prec) \in \mathcal{J}$ and N be an acceptable sequence w.r.t. I . We assume that for all
 767 $t, t' \in N$ s.t. we have $V(t') = V(t)$ only when both $t, t' \in \text{final}(I, N)$. Two cases are possible:

- 768 ■ $\text{init}(I, N)$ is empty. Since $\text{card}(\text{range}(I, N)) \leq 2^{|\mathcal{P}|}$, we conclude that $\text{size}(I, N) \leq 2^{|\mathcal{P}|}$.
- 769 ■ $\text{init}(I, N)$ is not empty. Going back to our assumption, we can see that for all $t \in \text{init}(I, N)$ and
 770 $t' \in N$ s.t. $t' \neq t$ we have $V(t') \neq V(t)$. If $\text{init}(I, N)$ has n time points having distinct valuations,
 771 then $\text{range}(\text{final}(I, N))$ has at most $2^{|\mathcal{P}|} - n$ valuations. Therefore, we have $\text{size}(I, N) \leq 2^{|\mathcal{P}|}$.

772 ◀

773 B Proofs of results in Section 5

774 ▶ **Lemma 27.** Let $\alpha_1 \in \mathcal{L}^*$ be a sentence starting with a temporal operator, $I = (V, \prec) \in$
 775 \mathcal{J}^{sd} and let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I, t \models$
 776 $\diamond\alpha_1$. Then for all $t, t' \in \text{Anchors}(I, T, \diamond\alpha_1)$ s.t. $V(t) = V(t')$ and $t \neq t'$, we have $t, t' \in$
 777 $\text{final}(I, \text{Anchors}(I, T, \diamond\alpha_1))$.

778 **Proof.** Let $\alpha_1 \in \mathcal{L}^*$, let T be a non-empty acceptable sequence w.r.t. $I \in \mathcal{J}^{sd}$ where for all $t \in T$ we
 779 have $I, t \models \diamond\alpha_1$. Just as a reminder, we have $\text{Anchors}(I, T, \diamond\alpha_1) = \bigcup_{t_i \in T} ST(I, AS(I, \min_{\prec}(t_i)), \alpha_1)$.
 780 Thus, there exists $t_i \in T$ such that $t \in ST(I, AS(I, \min_{\prec}(t_i)), \alpha_1)$. Suppose that there ex-
 781 ist $t, t' \in \text{Anchors}(I, T, \diamond\alpha_1)$ with $t \neq t'$ such that t is in $\text{init}(I, \text{Anchors}(I, T, \diamond\alpha_1))$ and
 782 $V(t) = V(t')$. Notice that $t \in \text{init}(I)$, since $t \in \text{init}(I, \text{Anchors}(I, T, \diamond\alpha_1))$. Without loss of
 783 generality, we assume that $t < t'$. From Definition 24, we have $t \in AS(I, (\mathbf{t}_{\alpha_1}^{I, AS(I, \min_{\prec}(t_i))}))$.
 784 Thanks to Definition 22 and Definition 23, the fact that $t' \in \text{init}(I)$, we can see that : (1) there is no
 785 $t'' \in \text{final}(I, AS(I, \min_{\prec}(t_i)))$ s.t. $I, t'' \models \alpha_1$ and (2) $t = \mathbf{t}_{\alpha_1}^{I, AS(I, \min_{\prec}(t_i))} = \max_{\prec}\{t'' \in$
 786 $\text{init}(I, AS(I, \min_{\prec}(t_i))) \mid I, t'' \models \alpha_1\}$. On the other hand, thanks to Proposition 8, since
 787 $t' < t''$ and $t' \in \min_{\prec}(t_i)$, we have $t'' \in \min_{\prec}(t_i)$. Hence $t'' \in AS(I, \min_{\prec}(t_i))$. Since $t'' \in$
 788 $\text{Anchors}(I, T, \diamond\alpha_1)$, we also have $I, t'' \models \alpha_1$. From this and the property (1) we can assert that t'
 789 does not belong to $\text{final}(I, AS(I, \min_{\prec}(t_i)))$. It follows that $t' \in \text{init}(I, AS(I, \min_{\prec}(t_i)))$. From

790 the property (2) we can assert that $t \geq t'$, which leads to a contradiction since $t < t'$. Therefore, for all
791 $t, t' \in \text{Anchors}(I, T, \diamond\alpha_1)$ s.t. $V(t) = V(t')$, we must have $t, t' \in \text{final}(\text{Anchors}(I, T, \diamond\alpha_1))$. ◀

792 ► **Proposition 28.** *Let $\alpha \in \mathcal{L}^*$ be a sentence starting with a temporal operator, $I = (V, \prec) \in \mathfrak{J}^{sd}$.
793 Let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I, t \models \alpha$. Then, we
794 have $\text{size}(I, \text{Anchors}(I, T, \alpha)) \leq 2^{|\mathcal{P}|}$.*

795 **Proof.** Let $I = (V, \prec) \in \mathfrak{J}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$
796 we have $I, t \models \alpha$. We show that is the case for our temporal operators:

- 797 ■ Since $\text{size}(I, \text{Anchors}(I, T, \Box\alpha_1)) = \text{size}(I, \emptyset) = 0$, we conclude that $\text{size}(I, \text{Anchors}(I, T, \Box\alpha_1)) \leq 2^{|\mathcal{P}|}$.
- 798 ■ Since $\text{size}(I, \text{Anchors}(I, T, \diamond\alpha_1)) = \text{size}(I, \text{ST}(I, \mathbb{N}, \alpha_1)) = 1$, we conclude that $\text{size}(I, \text{Anchors}(I, T, \diamond\alpha_1)) \leq$
799 $2^{|\mathcal{P}|}$.
- 800 ■ T is an acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \diamond\alpha_1$. From Proposition 27, for
801 all $t'_i, t'_j \in \text{Anchors}(I, T, \diamond\alpha_1)$ s.t. $V(t'_i) = V(t'_j)$ we have $t'_i, t'_j \in \text{final}(I, \text{Anchors}(I, T, \diamond\alpha_1))$.
802 From Proposition 20, we can conclude that $\text{size}(\text{Anchors}(I, T, \diamond\alpha_1)) \leq 2^{|\mathcal{P}|}$.
- 803 ■ Going back to Definition 26, we have $\text{Anchors}(I, T, \Box\alpha_1) = \text{DR}(I, \bigcup_{t_i \in T} \text{AS}(I, \min_{\prec}(t_i)))$.
804 We denote the acceptable sequence $\bigcup_{t_i \in T} \text{AS}(I, \min_{\prec}(t_i))$ by N . From Definition 25 we
805 have $\text{Anchors}(I, T, \Box\alpha_1) = \text{DR}(I, N) = \bigcup_{v \in \text{val}(I, N)} \text{ST}(I, N, \alpha_v)$. Moreover, we know that
806 $\text{size}(\text{ST}(I, N, \alpha_v)) = 1$ for all $v \in \text{val}(I, N)$. Consequently, thanks to Proposition 19, we have
807 $\text{size}(\bigcup_{v \in \text{val}(I, N)} \text{ST}(I, N, \alpha_v)) \leq \text{card}(\text{val}(I, N))$. We can see that $\text{card}(\text{val}(I, N)) \leq 2^{|\mathcal{P}|}$,
808 we can conclude that $\text{size}(\text{Anchors}(I, T, \Box\alpha_1)) = \text{size}(\bigcup_{v \in \text{val}(I, N)} \text{ST}(I, N, \alpha_v)) \leq 2^{|\mathcal{P}|}$.
809 ◀

810 ► **Proposition 29.** *Let $\alpha_1 \in \mathcal{L}^*$, $I = (V, \prec) \in \mathfrak{J}^{sd}$, let T be a non-empty acceptable sequence
811 w.r.t. I s.t. for all $t \in T$ we have $I, t \models \Box\alpha_1$, with $\alpha_1 \in \mathcal{L}^*$. For all acceptable sequences N w.r.t. I
812 s.t. $\text{Anchors}(I, T, \Box\alpha_1) \subseteq N$ and for all $t_i \in N \cap T$, we have the following: Let $I^N = (V^N, \prec^N)$
813 be the pseudo-interpretation over N and $t' \in N$, if $t' \notin \min_{\prec}(t_i)$, then $t' \notin \min_{\prec^N}(t_i)$.*

814 **Proof.** Let $I = (V, \prec) \in \mathfrak{J}^{sd}$, let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we
815 have $I, t \models \Box\alpha_1$, with $\alpha_1 \in \mathcal{L}^*$. Let N be an acceptable sequence w.r.t. I s.t. $\text{Anchors}(I, T, \Box\alpha_1) \subseteq$
816 N . Let $t_i \in N \cap T$. Let $t' \in N$ be a time point s.t. $t' \notin \min_{\prec}(t_i)$, we discuss these two cases:

- 817 ■ $t' \notin [t_i, +\infty[$: Since $t' \notin [t_i, +\infty[$, then $t' \notin [t_i, +\infty[\cap N$. Therefore, we conclude that
818 $t' \notin \min_{\prec^N}(t_i)$.
- 819 ■ $t' \in [t_i, +\infty[$: Since \prec satisfies the well-foundedness condition, $t' \notin \min_{\prec}(t_i)$ implies that there
820 exists a time point $t'' \in \min_{\prec}(t_i)$ s.t. $(t'', t') \in \prec$. Let $\alpha_{t''}$ be the representative sentence of
821 $V(t'')$. For the sake of readability, we shall denote the sequence $\bigcup_{t \in T} \text{AS}(I, \min_{\prec}(t))$ with M .
822 Notice that there exists $V \in \text{val}(I, M)$ such that $V = V(t'')$ since $t_i \in T$ and $t'' \in \min_{\prec}(t_i)$.
823 Thanks to Definition 25, since $\text{DR}(I, M) = \bigcup_{v \in \text{val}(I, M)} \text{ST}(I, M, \alpha_v)$ and $V(t'') \in \text{val}(I, M)$,
824 we can find $t''' \in \text{ST}(I, M, \alpha_{t''})$ where $t''' \in \text{DR}(I, M) \subseteq N$, $V(t''') = V$ and $t''' \geq t''$. Since
825 $(t'', t') \in \prec$, $I \in \mathfrak{J}^{sd}$ and $V(t''') = V(t'')$, we have $(t''', t') \in \prec$. Moreover, we have $t''', t' \in N$,
826 and therefore $(t''', t') \in \prec^N$. Since $t''' \in [t_i, +\infty[\cap N$ and $(t''', t') \in \prec^N$, we conclude that
827 $t' \notin \min_{\prec^N}(t_i)$.
828 ◀

829 ► **Proposition 31.** *Let $\alpha \in \mathcal{L}^*$ be in NNF, $I = (V, \prec) \in \mathfrak{J}^{sd}$, and let T be a non-empty acceptable
830 sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. Then, we have $\text{size}(I, \text{Keep}(I, T, \alpha)) \leq$
831 $\mu(\alpha) \times 2^{|\mathcal{P}|}$.*

832 **Proof.** Let $I = (V, \prec) \in \mathcal{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$
 833 we have $I, t \models \alpha$ which $\alpha \in \mathcal{L}^*$.

834 We use structural induction on T and α in order to prove this property.

- 835 ■ Base $\alpha = p$ or $\alpha = \neg p$. $Keep(I, T, \alpha) = \emptyset$. Since $size(I, \emptyset) = 0 \leq \mu(\alpha) \times 2^{|\mathcal{P}|} = 0$, then the
 836 property holds on atomic propositions.
- 837 ■ $\alpha = \alpha_1 \wedge \alpha_2$. Since $I, t \models \alpha_1 \wedge \alpha_2$ for all $t \in T$, we can assert that $I, t \models \alpha_1$ and $I, t \models \alpha_2$. By
 838 applying the induction hypothesis on T , α_1 and α_2 , we have $size(I, Keep(I, T, \alpha_1)) \leq \mu(\alpha_1) \times$
 839 $2^{|\mathcal{P}|}$ and $size(I, Keep(I, T, \alpha_2)) \leq \mu(\alpha_2) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 19, $size(Keep(I, T, \alpha_1 \wedge$
 840 $\alpha_2)) \leq (\mu(\alpha_1) + \mu(\alpha_2)) \times 2^{|\mathcal{P}|}$. We conclude that $size(I, Keep(I, T, \alpha_1 \wedge \alpha_2)) \leq (\mu(\alpha_1 \wedge$
 841 $\alpha_2)) \times 2^{|\mathcal{P}|}$.
- 842 ■ $\alpha = \alpha_1 \vee \alpha_2$. Since $I, t \models \alpha_1 \vee \alpha_2$ for all $t \in T$ then $I, t \models \alpha_1$ or $I, t \models \alpha_2$. Consider the
 843 sequence T_1 (resp. T_2) containing all $t_1 \in T$ (resp. $t_2 \in T$) s.t. $I, t_1 \models \alpha_1$ (resp. $I, t_2 \models \alpha_2$).
 844 Using induction hypothesis on T_1, T_2, α_1 and α_2 , we have $size(I, Keep(I, T_1, \alpha_1)) \leq \mu(\alpha_1) \times$
 845 $2^{|\mathcal{P}|}$ and $size(I, Keep(I, T_2, \alpha_2)) \leq \mu(\alpha_2) \times 2^{|\mathcal{P}|}$. We conclude in the same way as the last case
 846 that $size(I, Keep(I, T, \alpha_1 \vee \alpha_2)) \leq (\mu(\alpha_1 \vee \alpha_2)) \times 2^{|\mathcal{P}|}$.
- 847 ■ $\alpha = \diamond \alpha_1$. First of all, we proved in Proposition 28 that (I) $size(I, Anchors(I, T, \diamond \alpha_1)) \leq 2^{|\mathcal{P}|}$.
 848 On the other hand, thanks to Definition 26 it is easy to see that $size(I, Anchors(I, T, \diamond \alpha_1))$ is a
 849 non-empty acceptable sequence w.r.t. I s.t. for all $t' \in Anchors(I, T, \diamond \alpha_1)$ we have $I, t' \models \alpha_1$.
 850 By the induction hypothesis on $Anchors(I, T, \diamond \alpha_1)$ and α_1 , we have (II) $size(I, Keep(I, Anchors(I, T, \diamond \alpha_1), \alpha_1)) \leq$
 851 $\mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 19, from (I) and (II) we conclude that $size(I, Keep(I, T, \diamond \alpha_1)) \leq$
 852 $(1 + \mu(\alpha_1)) \times 2^{|\mathcal{P}|} = \mu(\diamond \alpha_1) \times 2^{|\mathcal{P}|}$.
- 853 ■ $\alpha = \square \alpha_1$. As a result of semantics of the \square operator, we can see that for all $t \in T$ we have $I, t \models$
 854 α_1 . By the induction hypothesis on T and α_1 , we have $size(I, Keep(I, T, \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$.
 855 Since $Keep(I, T, \alpha_1) = Keep(I, T, \square \alpha_1)$ then $size(I, Keep(I, T, \square \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. We
 856 conclude that $size(I, Keep(I, T, \square \alpha_1)) \leq \mu(\square \alpha_1) \times 2^{|\mathcal{P}|}$.
- 857 ■ $\alpha = \heartsuit \alpha_1$. First of all, we proved in Proposition 28 that (I) $size(I, Anchors(I, T, \heartsuit \alpha_1)) \leq 2^{|\mathcal{P}|}$.
 858 On the other hand, thanks to Definition 26 it is easy to see that $Anchors(I, T, \heartsuit \alpha_1)$ is a non-empty
 859 acceptable sequence w.r.t. I s.t. for all $t' \in Anchors(I, T, \heartsuit \alpha_1)$ we have $I, t' \models \alpha_1$. By the induc-
 860 tion hypothesis on $Anchors(I, T, \heartsuit \alpha_1)$ and α_1 , we have (II) $size(I, Keep(I, Anchors(I, T, \heartsuit \alpha_1), \alpha_1)) \leq$
 861 $\mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 19, from (I) and (II), we conclude that $size(I, Keep(I, T, \heartsuit \alpha_1)) \leq$
 862 $(1 + \mu(\alpha_1)) \times 2^{|\mathcal{P}|} = \mu(\heartsuit \alpha_1) \times 2^{|\mathcal{P}|}$.
- 863 ■ $\alpha = \boxtimes \alpha_1$. First of all, we proved in Proposition 28 that (I) $size(I, Anchors(I, T, \boxtimes \alpha_1)) \leq 2^{|\mathcal{P}|}$.
 864 On the other hand, from definition30, we have $T' = \bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$. It is easy to see
 865 that for all $t' \in T'$ we have $I, t' \models \alpha_1$ and that T' is a non-empty acceptable sequence w.r.t. I .
 866 By the induction hypothesis on T' and α_1 , we have (II) $size(I, Keep(I, T', \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$.
 867 Thanks to Proposition 19, form (I) and (II) we conclude that $size(I, Keep(I, T, \boxtimes \alpha_1)) \leq (1 +$
 868 $\mu(\alpha_1)) \times 2^{|\mathcal{P}|} = \mu(\boxtimes \alpha_1) \times 2^{|\mathcal{P}|}$.

869 ◀

870 ► **Lemma 32.** Let $\alpha \in \mathcal{L}^*$ be in NNF, $I = (V, \prec) \in \mathcal{I}^{sd}$, and let T be a non-empty acceptable
 871 sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. For all acceptable sequences N w.r.t. I , if
 872 $Keep(I, T, \alpha) \subseteq N$, then for every $t \in N \cap T$, we have $I^N, t \models_{\emptyset} \alpha$.

873 **Proof.** Let $\alpha \in \mathcal{L}^*$ be in NNF, $I = (V, \prec) \in \mathcal{I}^{sd}$, and let T be a non-empty acceptable sequence
 874 w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. We consider N to be an acceptable sequence w.r.t. I s.t.
 875 $Keep(I, T, \alpha) \subseteq N$ and $t \in N \cap T$. Let $I^N = (N, V^N, \prec^N)$ be the pseudo-interpretation over N .

876 We use structural induction on T and α in order to prove this property.

- 877 ■ $\alpha = p$ or $\alpha = \neg p$. Since $I, t \models p$ (resp. $\neg p$), it means that $p \in V(t)$ (resp. $p \notin V(t)$). We know
 878 that $V^N(t) = V(t)$. We conclude that $I^N, t \models_{\emptyset} p$ (resp. $\neg p$).

- 879 ■ $\alpha = \alpha_1 \wedge \alpha_2$. Since $I, t \models \alpha_1 \wedge \alpha_2$ for all $t \in T$, we can assert that $I, t \models \alpha_1$ and $I, t \models$
880 α_2 . By applying the induction hypothesis on T , α_1 and α_2 , since $Keep(I, T, \alpha_1) \subseteq N$ and
881 $Keep(I, T, \alpha_2) \subseteq N$, therefore we have $I^N, t \models_{\neq} \alpha_1$ and $I^N, t \models_{\neq} \alpha_2$. Thus, we have
882 $I^N, t \models_{\neq} \alpha_1 \wedge \alpha_2$.
- 883 ■ $\alpha = \alpha_1 \vee \alpha_2$. Suppose that $I, t \models \alpha_1$ (the case $I, t \models \alpha_2$ can be treated in a similar way) and
884 consider the sequence T_1 containing all $t_1 \in T$ s.t. $I, t_1 \models \alpha_1$. Here, since $t \in T_1$, therefore T_1 is
885 non-empty and $t \in T_1 \cap N$. We know that $Keep(I, T_1, \alpha_1) \cup Keep(I, T_2, \alpha_2) \subseteq N$. Consequently
886 $Keep(I, T_1, \alpha_1) \subseteq N$. From the induction hypothesis, we have $I^N, t \models_{\neq} \alpha_1$. Therefore, we
887 have $I^N, t \models \alpha_1 \vee \alpha_2$.
- 888 ■ $\alpha = \diamond \alpha_1$. We have $I, t \models \diamond \alpha_1$ and we need to prove that $I^N, t \models_{\neq} \diamond \alpha_1$. $I, t \models \diamond \alpha_1$ means
889 that there exists $t' \in [t, \infty[$ such that $I, t' \models \alpha_1$, therefore $Anchors(I, T, \diamond \alpha_1)$ is non-empty
890 (see Definition 26). We know that $Anchors(I, T, \diamond \alpha_1) \subseteq Keep(I, T, \diamond \alpha_1) \subseteq N$, consequently
891 $Anchors(I, T, \diamond \alpha_1) \cap N$ is non-empty. Thanks to Definition 26 it is easy to see that for all $t_1 \in$
892 $Anchors(I, T, \diamond \alpha_1)$ we have $I, t_1 \models \alpha_1$. By the induction hypothesis on $Anchors(I, T, \diamond \alpha_1)$
893 and α_1 , since $Keep(I, Anchors(I, T, \diamond \alpha_1), \alpha_1) \subseteq N$, $t' \in Anchors(I, T, \diamond \alpha_1)$ (a non-empty
894 acceptable sequence w.r.t I) and $I, t' \models \alpha_1$, thus $I^N, t' \models \alpha_1$. Therefore, we have $I^N, t \models_{\neq}$
895 $\diamond \alpha_1$.
- 896 ■ $\alpha = \square \alpha_1$. We have $I, t \models \square \alpha_1$ and we need to prove that $I^N, t \models_{\neq} \square \alpha_1$. We know that for
897 all $t' \geq t$ we have $I, t' \models \alpha_1$. We can assert that for all $t' \in N \cap T$ such that $t' \geq t$, we have
898 $I^N, t' \models_{\neq} \alpha_1$. By the induction hypothesis on T and α_1 , $Keep(I, T, \alpha_1) = Keep(I, T, \square \alpha_1)$.
899 Consequently $Keep(I, T, \alpha_1) \subseteq N$ since for all $t' \in N \cap T$, we have $I^N, t' \models_{\neq} \alpha_1$. We
900 conclude that $I^N, t \models_{\neq} \square \alpha_1$.
- 901 ■ $\alpha = \heartsuit \alpha_1$. We have $I, t \models \heartsuit \alpha_1$ and we need to prove that $I^N, t \models_{\neq} \heartsuit \alpha_1$. $I, t \models \heartsuit \alpha_1$
902 means that there exists $t' \in \min_{\prec}(t)$ such that $I, t' \models \alpha_1$, therefore $Anchors(I, T, \heartsuit \alpha_1)$ is
903 non-empty (see Definition 26). We know that $Anchors(I, T, \heartsuit \alpha_1) \subseteq Keep(I, T, \heartsuit \alpha_1) \subseteq N$,
904 consequently $Anchors(I, T, \heartsuit \alpha_1) \cap N$ is non-empty. Thanks to Definition 26 it is easy to see
905 that for all $t_1 \in Anchors(I, T, \heartsuit \alpha_1)$ we have $I, t_1 \models \alpha_1$. By the induction hypothesis on
906 $Anchors(I, T, \heartsuit \alpha_1)$ and α_1 , since $Keep(I, T_1, \alpha_1) \subseteq N$ with $T_1 = Anchors(I, T, \heartsuit \alpha_1)$, and
907 T_1 is an acceptable sequence where $I, t' \models \alpha_1$ for all $t' \in T_1$, we conclude that $I^N, t' \models_{\neq} \alpha_1$
908 (I). Thanks to the construction of the pseudo-interpretation I^N , since $t' \in \min_{\prec^N}(t)$, therefore
909 $t' \in \min_{\prec}(t)$ (II). From (I) and (II), we conclude that $I^N, t \models_{\neq} \heartsuit \alpha_1$.
- 910 ■ $\alpha = \boxtimes \alpha_1$. We have $I, t \models \boxtimes \alpha_1$ and we need to prove that $I^N, t \models_{\neq} \boxtimes \alpha_1$. $I, t \models \boxtimes \alpha_1$ means
911 that for all $t' \in \min_{\prec}(t)$ we have $I, t' \models \alpha_1$, therefore for all $t' \in T' = \bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$
912 we have $I, t' \models \alpha_1$. In addition, thanks to the well-foundedness condition on \prec , T' is non-empty.
913 We know that $Anchors(I, T, \boxtimes \alpha_1) \subseteq Keep(I, T, \boxtimes \alpha_1) \subseteq N$ and that $Anchors(I, T, \boxtimes \alpha_1) =$
914 $DR(I, T')$ consequently $T' \cap N$ is non-empty. We use proof by contradiction. Suppose that
915 $I^N, t \not\models_{\neq} \boxtimes \alpha_1$, which means there exists $t' \in \min_{\prec^N}(t)$ s.t. $I^N, t' \not\models_{\neq} \alpha_1$. Thanks to
916 Proposition 29, if $t' \in \min_{\prec^N}(t)$, then $t' \in \min_{\prec}(t)$. Just a reminder, we have $T' =$
917 $\bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$ where for all $t'' \in T'$ we have $I, t'' \models \alpha_1$ (Note that T' is a non-empty
918 acceptable sequence w.r.t I). By the induction hypothesis on T' and α_1 , since $Keep(I, T', \alpha_1) \subseteq$
919 N , and $t' \in AS(I, \min_{\prec}(t)) \subseteq T'$, therefore $I^N, t' \models_{\neq} \alpha_1$. This conflicts with our supposition.
920 We conclude that there is no $t' \in \min_{\prec^N}(t)$ s.t. $I^N, t' \not\models_{\neq} \alpha_1$, and therefore $I^N, t \models_{\neq} \boxtimes \alpha_1$.
921 ◀

922 **C** Proof of results in Section 6

923 **NB:** The results marked (*) are introduced here, while they are omitted in the main text.

924 ► **Proposition 39.** Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI, $l(M) = (V, \prec)$ and $t, t', t_M, t'_M \in \mathbb{N}$ s.t.:

$$925 \quad t_M = \begin{cases} t, & \text{if } t < i; \\ i + (t - i) \bmod \pi, & \text{otherwise.} \end{cases} \quad t'_M = \begin{cases} t', & \text{if } t' < i; \\ i + (t' - i) \bmod \pi, & \text{otherwise.} \end{cases}$$

926 We have the following: $t' \in \min_{\prec}(t)$ iff $t'_M \in \min_{\prec_M}(t_M)$.

927 **Proof.** Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI, $l(M) = (V, \prec)$ and $t, t' \in \mathbb{N}$.

928 ■ For the only-if part, we assume that $t' \in \min_{\prec}(t)$. Following our assumption, there is no
929 $t'' \in [t, +\infty[$ s.t. $(t'', t') \in \prec$. We use a proof by contradiction. Suppose that $t'_M \notin \min_{\prec_M}(t_M)$,
930 which means there exists $t''_M \in [\min_{\prec}\{t_M, i\}, i + \pi[$ with $(V_M(t''_M), V_M(t'_M)) \in \prec_M$. Going
931 back to Definition 37, $V_M(t'_M) = V(t')$ and $V_M(t''_M) = V(t'')$. Consequently, $(V(t''_M), V(t'_M)) \in \prec$. Thanks
932 to Definition 37, (I) $(t''_M, t') \in \prec$. There are two possible cases for t, t' . If $t \in [0, i[$ then $t_M = t$
933 and (II) $t'_M \in [t, i + \pi[$. From (I) and (II), there exists $t''_M > t$ such that $(t''_M, t') \in \prec$. This
934 conflicts with our supposition. If $t \in [i, +\infty[$, then $t'_M \in [i, i + \pi[$ and t, t', t'' are in $\text{final}(I(M))$.
935 Thanks to proposition 10, there exists $t'' > t$ such that $V(t'') = V(t_M)$. Since $I(M) \in \mathcal{J}^{sd}$
936 and $(t''_M, t') \in \prec$ then $(t'', t) \in \prec$. Consequently, there exists $t'' > t$ such that $(t'', t) \in \prec$. This
937 conflicts with our supposition.

938 ■ For the if part, we assume that $t'_M \in \min_{\prec_M}(t_M)$. Following our assumption, there is no
939 $t''_M \in [\min_{\prec}\{t_M, i\}, i + \pi[$ with $(V_M(t''_M), V_M(t'_M)) \in \prec_M$. We use proof by contradiction.
940 Suppose that $t' \notin \min_{\prec}(t)$, which means there exists $t''' > t$ such that $(t''', t') \in \prec$. Let t'''_M be
941 defined as follows:

$$942 \quad t'''_M = \begin{cases} t''', & \text{if } t''' < i; \\ i + (t''' - i) \bmod \pi, & \text{otherwise.} \end{cases}$$

943 Thanks to definition 37, $V(t'''_M) = V_M(t'''_M)$, $V(t') = V_M(t'_M)$ and since $(t''', t') \in \prec$ then
944 $(V(t'''_M), V(t'_M)) \in \prec_M$. Consequently, (I) $(V(t'''_M), V(t'_M)) \in \prec_M$. . From (I) and (II), we have
945 $t'''_M \notin \min_{\prec_M}(t_M)$. This conflicts with our supposition. ◀

947 ► **Definition 49 (*)**. Given a UPI $I = (V, \prec)$, we define the UPPI $M(I) = (i, \pi, V_M, \prec_M)$ by:

- 948 ■ $i = \text{length}(\text{init}(I))$, $\pi = \text{card}(\text{range}(I))$;
- 949 ■ $V_M(t) = V(t)$ for all $t \in [0, i + \pi[$;
- 950 ■ for all $t, t' \in [0, i + \pi[$, $(V(t), V(t')) \in \prec$ iff $(t, t') \in \prec$.

951 ► **Proposition 42.** Let $\alpha \in \mathcal{L}^*$. We have that α is \mathcal{J}^{sd} -satisfiable iff there exists a UPPI M such
952 that $l(M), 0 \models \alpha$ and $\text{size}(l(M)) \leq |\alpha| \times 2^{|\mathcal{P}|}$.

953 **Proof.** Let $\alpha \in \mathcal{L}^*$.

954 ■ For the only if part, let α be \mathcal{J}^{sd} -satisfiable. Thanks to Theorem 21 and Proposition 35, there
955 exists a UPI $I = (V, \prec) \in \mathcal{J}^{sd}$ s.t. $I, 0 \models \alpha$ and $\text{size}(I) \leq |\alpha| \times 2^{|\mathcal{P}|}$. We define the UPPI $M(I)$
956 from I . It can be checked that $l(M(I)) = I$. Therefore, from \mathcal{J}^{sd} -satisfiable sentence α , we can
957 find a UPPI M such that $l(M), 0 \models \alpha$ and $\text{size}(l(M)) \leq |\alpha| \times 2^{|\mathcal{P}|}$.

958 ■ For the if part, let $M = (i, \pi, V_M, \prec_M)$ be a UPPI s.t. $l(M), 0 \models \alpha$. Since $l(M) \in \mathcal{J}^{sd}$, therefore
959 α is \mathcal{J}^{sd} -satisfiable. ◀

961 Lemma 41 is a particular case of the following Lemma.

962 ► **Lemma 50** (*). Let a UPPI $M = (i, \pi, V_M, \prec_M)$, $\alpha \in \mathcal{L}^*$, $\alpha_1 \in SF(\alpha)$ and $t, t' \in \mathbb{N}$ such that:

$$963 \quad t' = \begin{cases} t, & \text{if } t < i; \\ i + (t - i) \bmod \pi, & \text{otherwise.} \end{cases}$$

964 We have $\mathsf{l}(M), t' \models \alpha$ iff $\alpha_1 \in \mathit{lab}_\alpha^M(t)$.

965 **Proof.** Let a UPPI $M = (i, \pi, V_M, \prec_M)$, $\alpha \in \mathcal{L}^*$, $t \in \mathbb{N}$ and $\mathsf{l}(M) = (V, \prec)$. We use structural
966 induction to prove the Lemma. Let t' be a time point s.t. $t' = t$ if $t \in [0, i[$, and $t' = i + (t - i) \bmod \pi$ if
967 $t \in [i, +\infty[$.

968 ■ $\alpha = p$. If $t \in [0, i[$, then we have $V_M(t') = V(t)$, thus we have $p \in V_M(t)$ iff $p \in V(t)$, and
969 therefore $\mathsf{l}(M), t \models p$ iff $p \in \mathit{lab}_\alpha^M(t)$. If $t \in [i, +\infty[$, we have $V_M(t') = V(t)$. Following the
970 same reasoning as the previous case, $\mathsf{l}(M), t \models p$ iff $p \in \mathit{lab}_\alpha^M(t')$.

971 ■ $\alpha = \neg\alpha_1$. By the induction hypothesis, we have $\mathsf{l}(M), t \models \alpha_1$ iff $\alpha_1 \in \mathit{lab}_\alpha^M(t')$, and therefore
972 $\mathsf{l}(M), t \not\models \alpha_1$ iff $\alpha_1 \notin \mathit{lab}_\alpha^M(t')$. We conclude that $\mathsf{l}(M), t \models \neg\alpha_1$ iff $\neg\alpha_1 \in \mathit{lab}_\alpha^M(t')$.

973 ■ $\alpha = \alpha_1 \wedge \alpha_2$. By the induction hypothesis, we have $\mathsf{l}(M), t \models \alpha_1$ iff $\alpha_1 \in \mathit{lab}_\alpha^M(t')$ and
974 $\mathsf{l}(M), t \models \alpha_2$ iff $\alpha_2 \in \mathit{lab}_\alpha^M(t')$, and therefore $\mathsf{l}(M), t \models \alpha_1 \wedge \alpha_2$ iff $\alpha_1 \wedge \alpha_2 \in \mathit{lab}_\alpha^M(t')$.

975 ■ $\alpha = \diamond\alpha_1$.

976 ■ For the only-if part, let $\mathsf{l}(M), t \models \diamond\alpha_1$. There exists $t_1 \in [t, +\infty[$ s.t. $\mathsf{l}(M), t_1 \models \alpha_1$. For all
977 $t_1 \in \mathbb{N}$, there is a t'_1 s.t. $t'_1 = t_1$ if $t_1 \in [0, i[$ and $t'_1 = i + (t_1 - i) \bmod \pi$ if $t_1 \in [i, +\infty[$.
978 By the induction hypothesis, we have $\alpha_1 \in \mathit{lab}_\alpha^M(t'_1)$. If $t \in [0, i[$, there exists $t'_1 \geq t$ s.t.
979 $\alpha_1 \in \mathit{lab}_\alpha^M(t'_1)$, and therefore $\diamond\alpha_1 \in \mathit{lab}_\alpha^M(t)$. If $t \in [i, +\infty[$, there exists $t'_1 \in [i, i + \pi[$ s.t.
980 $\alpha_1 \in \mathit{lab}_\alpha^M(t'_1)$, and therefore $\diamond\alpha_1 \in \mathit{lab}_\alpha^M(t)$.

981 ■ For the if part, let $\mathsf{l}(M), t \not\models \diamond\alpha_1$. Following our assumption, $\mathsf{l}(M), t \models \neg\diamond\alpha_1$ for all
982 $t_1 \geq t$ we have $\mathsf{l}(M), t_1 \models \neg\alpha_1$. By the induction hypothesis, for all $t_1 \geq t$, we have
983 $\neg\alpha_1 \in \mathit{lab}_\alpha^M(t'_1)$ where $t'_1 = t_1$ if $t_1 \in [0, i[$ and $t'_1 = i + (t_1 - i) \bmod \pi$ if $t_1 \in [i, +\infty[$. It
984 is also worth to point out that for all t_1 , we have $\neg\alpha_1 \in \mathit{lab}_\alpha^M(t'_1)$. By Definition of $\mathit{lab}_\alpha^M(\cdot)$,
985 we have $\neg\diamond\alpha_1 \in \mathit{lab}_\alpha^M(t')$, and therefore $\diamond\alpha_1 \notin \mathit{lab}_\alpha^M(t')$.

986 ■ $\alpha = \diamond\alpha_1$.

987 ■ For the only-if part, let $\mathsf{l}(M), t \models \diamond\alpha_1$. There exists $t_1 \in \min_{\prec}(t)$ s.t. $\mathsf{l}(M), t_1 \models \alpha_1$. For
988 all $t_1 \in \mathbb{N}$, there is a t'_1 s.t. $t'_1 = t_1$ if $t_1 \in [0, i[$ and $t'_1 = i + (t_1 - i) \bmod \pi$ if $t_1 \in [i, +\infty[$.
989 By the induction hypothesis, we have (I) $\alpha_1 \in \mathit{lab}_\alpha^M(t'_1)$. From Proposition 39, we can see
990 that (II) $t_1 \in \min_{\prec}(t)$ iff $t'_1 \in \min_{\prec_M}(t')$. From (I) and (II), since there is $t'_1 \in \min_{\prec_M}(t')$
991 where $\alpha_1 \in \mathit{lab}_\alpha^M(t'_1)$, we conclude that $\diamond\alpha_1 \in \mathit{lab}_\alpha^M(t')$.

992 ■ For the if part, let $\mathsf{l}(M), t \not\models \diamond\alpha_1$. Following our assumption, $\mathsf{l}(M), t \models \neg\diamond\alpha_1$ for all
993 $t_1 \in \min_{\prec}(t)$ we have $\mathsf{l}(M), t_1 \models \neg\alpha_1$. By the induction hypothesis, for all $t_1 \in \min_{\prec}(t)$,
994 we have (I) $\neg\alpha_1 \in \mathit{lab}_\alpha^M(t'_1)$ where $t'_1 = t_1$ if $t_1 \in [0, i[$ and $t'_1 = i + (t_1 - i) \bmod \pi$ if
995 $t_1 \in [i, +\infty[$. From Proposition 39, we can see that (II) $t_1 \in \min_{\prec}(t)$ iff $t'_1 \in \min_{\prec_M}(t')$.
996 From (I) and (II), since there is no $t'_1 \in \min_{\prec_M}(t')$ s.t. $\alpha_1 \in \mathit{lab}_\alpha^M(t'_1)$, we conclude that
997 $\diamond\alpha_1 \notin \mathit{lab}_\alpha^M(t')$.

998 ◀

999 **D Proofs of results for Lemma 15**

1000 ► **Proposition 51** (*). Let $I = (V, \prec) \in \mathfrak{I}$ and $i \in \mathit{final}(I)$. For all $j \in \mathit{final}(I)$, there exists
1001 $j' \geq j$ such that $V(j') = V(i)$.

1002 **Proof.** Let $I = (V, \prec) \in \mathfrak{I}$ and $i, j \in \text{final}(I)$. Let E be the set defined by $E = \{i' \in \text{final}(I) :$
 1003 $V(i') = V(i)\}$. Since $i \in \text{final}(I)$, we have $E \neq \emptyset$. Suppose now that there does not exist $j' \geq j$
 1004 such that $V(j') = V(i)$. We have E is a non empty finite set of integers included in $[0, \dots, j - 1]$.
 1005 Let k be the integer defined by $k = \max\{k' \in E\}$. From the definitions of E and k , we have
 1006 $k \in \text{final}(I)$ and there does not exist $k' > k$ such that $V(k') = V(k)$. This contradicts Lemma 10.
 1007 We can conclude that there exists $j' \geq j$ such that $V(j') = V(i)$. ◀

1008 ▶ **Proposition 52 (*)**. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpreta-
 1009 tions over the same set of atomic propositions \mathcal{P} s.t. $\text{range}(I) = \text{range}(I')$. For all $i \in \text{final}(I)$ and
 1010 $i' \in \text{final}(I')$ such that $V(i) = V'(i')$, we have :

- 1011 (1) for all $j \in [i, +\infty[$ there exists $j' \in [i', +\infty[$ such that $V'(j') = V(j)$.
 1012 (2) for all $j \in \min_{\prec}(i)$ there exists $j' \in \min_{\prec'}(i')$ such that $V(j) = V'(j')$.

1013 **Proof.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over \mathcal{P} s.t.
 1014 $\text{range}(I) = \text{range}(I')$ and $i, i' \in \text{final}(I)$ such that $V(i) = V'(i')$.

- 1015 (1) Let j belonging to $[i, +\infty[$. Since $i \in \text{final}(I)$, we have $j \in \text{final}(I)$. Moreover, from the equality
 1016 $\text{range}(I) = \text{range}(I')$, we can assert that there exists $k \in \text{final}(I')$ such that $V'(k) = V(j)$.
 1017 Hence, from Proposition 51, there exists $j' \geq i'$ such that $V'(j') = V'(k) = V(j)$.
 1018 (2) Let $j \in \min_{\prec}(i)$. We have $j \in \text{final}(I)$. From Property (1), there exists $j' \geq i'$ such that
 1019 $V'(j') = V(j)$. Suppose that $j' \notin \min_{\prec'}(i')$. Since $j' \geq i'$, there exists $k' \geq i'$ such
 1020 that $(k', j') \in \prec'$. From Property (1), there exists $k \geq i$ such that $V(k) = V'(k')$. Since
 1021 $V(k) = V'(k')$, $V'(j') = V(j)$, $(k', j') \in \prec'$ and, I and I' are two faithful interpretations, we
 1022 can assert that $(k, j) \in \prec$. Consequently, since $k \geq i$ and $(k, j) \prec$, we have $j \notin \min_{\prec}(i)$. There
 1023 is a contradiction. We can conclude that $j' \in \min_{\prec'}(i')$.

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1025 ▶ **Proposition 53 (*)**. Let $\alpha \in \mathcal{L}^*$, $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful
 1026 interpretations over the same set of atomic propositions \mathcal{P} s.t. $\text{range}(I) = \text{range}(I')$. For $\alpha \in \mathcal{L}^*$
 1027 $i \in \text{final}(I)$ and $i' \in \text{final}(I')$ s.t. $V(i) = V'(i')$, we have :

$$1028 \quad I, i \models \alpha \text{ iff } I', i' \models \alpha.$$

1029 **Proof.** Let $I = (V, \prec)$, $I' = (V', \prec')$ be two faithful interpretations belonging to \mathfrak{I}^{sd} . over the same
 1030 set of atomic propositions \mathcal{P} s.t. $\text{range}(I) = \text{range}(I')$. Let $\alpha \in \mathcal{L}^*$, $i \in \text{final}(I)$ and $i' \in \text{final}(I')$
 1031 such that $V(i) = V'(i')$. Without loss of generality we suppose that α does not contain \vee , \square and \boxtimes .
 1032 This proposition can be proven by induction on the structure of the sentence α .

- 1033 ■ Base case : $\alpha = p$ with $p \in \mathcal{P}$. Since $V(i) = V'(i')$, we have $p \in V(i)$ iff $p \in V'(i')$, thus
 1034 $I, i \models p$ iff $I', i' \models p$.
 1035 ■ $\alpha = \diamond\alpha_1$. First we prove that $I, i \models \diamond\alpha_1$ implies $I', i' \models \diamond\alpha_1$. We assume that $I, i \models \diamond\alpha_1$.
 1036 Hence, there exists $j \in [i, +\infty[$ s.t. $j \in \min_{\prec}(i)$ and $I, j \models \alpha_1$. From Proposition 52 (2), there
 1037 exists $j' \in \min_{\prec'}(i')$ such that $V'(j') = V(j)$. By induction hypothesis, we have $I', j' \models \alpha_1$.
 1038 We can conclude that $I', i' \models \diamond\alpha_1$. The if part can be proved with a similar reasoning.

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1040 ▶ **Corollary 54 (*)**. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations
 1041 over the same set of atomic propositions \mathcal{P} s.t. $\text{range}(I) = \text{range}(I')$. For $i \in \text{final}(I)$ and $\alpha \in \mathcal{L}^*$,
 1042 we have : if $I, i \models \alpha$ then there exists $i' \in \text{final}(I')$ such that $I', i' \models \alpha$.

1043 ▶ **Proposition 55 (*)**. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpreta-
 1044 tions over \mathcal{P} such that $\text{init}(I) \doteq \text{init}(I')$ and $\text{range}(I) = \text{range}(I')$. Then we have :

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For all $t, t' \in \text{init}(I)$, $t' \in \text{min}_{\prec}(t)$ iff $t' \in \text{min}_{\prec'}(t)$.

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Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over \mathcal{P} such that $\text{init}(I) \doteq \text{init}(I')$ and $\text{range}(I) = \text{range}(I')$ and, $t, t' \in \text{init}(I)$ such that $t' \in \text{min}_{\prec}(t)$. Suppose that $t' \notin \text{min}_{\prec'}(t)$. Since $t' \geq t$, there exists $t'' \geq t$ such that $(t'', t') \in \prec'$. There are two possible cases.

■ $t'' \in \text{init}(I')$. Since $\text{init}(I) \doteq \text{init}(I')$, we have $V'(t'') = V(t'')$. Moreover, since I and I' are two faithful interpretations and $V'(t') = V(t')$, we have $(t'', t') \in \prec$. Since $t'' \geq t$, it follows that $t' \notin \text{min}_{\prec}(t)$. There is a contradiction. We can conclude that $t' \in \text{min}_{\prec'}(t)$.

■ $t'' \in \text{final}(I')$. Since $\text{range}(I) = \text{range}(I')$, there exists $t''' \in \text{final}(I)$ such that $V'(t'') = V(t''')$. Moreover, since I and I' are two faithful interpretations and $V'(t') = V(t')$, we have $(t''', t') \in \prec$. Since $t''' \geq t$, It follows that $t' \notin \text{min}_{\prec}(t)$. There is a contradiction. We can conclude that $t' \in \text{min}_{\prec'}(t)$.

Same reasoning can be applied to prove the if part. ◀

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► **Proposition 56** (*). Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over \mathcal{P} such that $\text{init}(I) \doteq \text{init}(I')$ and $\text{range}(I) = \text{range}(I')$. For all $t \in \text{init}(I)$ and $t' \in \text{final}(I)$ such that $t' \in \text{min}_{\prec}(t)$ we have $\{t'' \in \text{final}(I') : V'(t'') = V(t')\} \subseteq \text{min}_{\prec'}(t)$.

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Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over \mathcal{P} such that $\text{init}(I) \doteq \text{init}(I')$ and $\text{range}(I) = \text{range}(I')$ and, $t \in \text{init}(I)$, $t' \in \text{final}(I)$, $t'' \in \text{final}(I')$ such that $t' \in \text{min}_{\prec}(t)$ and $V'(t'') = V(t')$. We will prove that $t'' \in \text{min}_{\prec'}(t)$.

Suppose that $t'' \notin \text{min}_{\prec'}(t)$. Since $t'' \geq t$, there exists $t''' \geq t$ such that $(t''', t'') \in \prec'$. There are two possible cases.

■ $t''' \in \text{init}(I')$. Since $\text{init}(I) \doteq \text{init}(I')$, we have $V'(t''') = V(t''')$. Moreover, since I and I' are two faithful interpretations and $V'(t'') = V(t'')$, we have $(t''', t'') \in \prec$. Since $t''' \geq t$, it follows that $t' \notin \text{min}_{\prec}(t)$. There is a contradiction. We can conclude that $t'' \in \text{min}_{\prec'}(t)$.

■ $t''' \in \text{final}(I')$. Since $\text{range}(I) = \text{range}(I')$, there exists $u \in \text{final}(I)$ such that $V'(t''') = V(u)$. Moreover, since I and I' are two faithful interpretations and $V'(t'') = V(t'')$, we have $(u, t'') \in \prec$. Since $u \geq t$, it follows that $t' \notin \text{min}_{\prec}(t)$. There is a contradiction. We can conclude that $t'' \in \text{min}_{\prec'}(t)$.

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► **Lemma 57** (*). Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over \mathcal{P} such that $V'(0) = V(0)$, $\text{init}(I) \doteq \text{init}(I')$, and $\text{range}(I) = \text{range}(I')$. Then for all $\alpha \in \mathcal{L}^*$, we have :

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For all $t \in \text{init}(I) \cup \{0\}$, $I, t \models \alpha$ iff $I', t \models \alpha$.

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Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over \mathcal{P} such that $V'(0) = V(0)$, $\text{init}(I) \doteq \text{init}(I')$, and $\text{range}(I) = \text{range}(I')$. Let $\alpha \in \mathcal{L}^*$ and $t \in \text{init}(I) \cup \{0\}$. Without loss of generality we suppose that α does not contain \vee , \square and \boxminus .

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First, notice that in the case where $\text{init}(I)$ and $\text{init}(I')$ are empty intervals, we necessarily have $t = 0$. Moreover, since $t \in \text{final}(I)$ and $t \in \text{final}(I')$ and $V(0) = V'(0)$, from Proposition 53, we can assert that $I, t \models \alpha$ iff $I', t \models \alpha$. Consequently, the property to be proved is true. Now, we will suppose that $\text{init}(I)$ and $\text{init}(I')$ are non empty intervals. Hence, we have $t \in \text{init}(I)$ and $t \in \text{init}(I')$. We will prove that $I, t \models \alpha$ iff $I', t \models \alpha$ by structural induction on α .

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■ Base case : $\alpha = p$. Since $t \in \text{init}(I)$, we have $V(t) = V'(t)$. Hence, $p \in V(t)$ iff $p \in V'(t)$. Thus $I, t \models p$ iff $I', t \models p$.

■ $\alpha = \neg\alpha_1$. By induction hypothesis, we have $I, t \models \alpha_1$ iff $I', t \models \alpha_1$. Hence, it is not the case that $I, t \models \alpha_1$ iff it is not the case that $I', t \models \alpha_1$. We can conclude that, $I, t \models \neg\alpha_1$ iff $I', t \models \neg\alpha_1$.

- 1090 ■ $\alpha = \alpha_1 \wedge \alpha_2$. We have $I, t \models \alpha_1 \wedge \alpha_2$ iff $I, t \models \alpha_1$ and $I, t \models \alpha_2$. Using the induction hypothesis,
 1091 it follows that $I, t \models \alpha_1$ and $I, t \models \alpha_2$ iff $I', t \models \alpha_1$ and $I', t \models \alpha_2$. We can conclude that
 1092 $I, t \models \alpha_1 \wedge \alpha_2$ iff $I', t \models \alpha_1 \wedge \alpha_2$.
- 1093 ■ $\alpha = \diamond \alpha_1$. Suppose that $I, t \models \diamond \alpha_1$. There exists a $t' \in [t, +\infty[$ s.t. $I, t' \models \alpha_1$. Two cases are
 1094 possible w.r.t. t' .
- 1095 ■ $t' \in \text{init}(I)$. By induction hypothesis, we have $I', t' \models \alpha_1$. Hence, we can conclude that
 1096 $I', t \models \diamond \alpha_1$.
 - 1097 ■ $t' \in \text{final}(I)$. Since $\text{range}(I) = \text{range}(I')$, there exists $t'' \in \text{final}(I')$ such that $V'(t'') =$
 1098 $V(t')$. From Proposition 53, we have $I', t'' \models \alpha_1$. Since, $t'' > t$ we have $I', t \models \diamond \alpha_1$.
- 1099 Same reasoning can be applied to prove the if part.
- 1100 ■ $\alpha = \heartsuit \alpha_1$. Suppose that $I, t \models \heartsuit \alpha_1$. There exists $t' \in \text{min}_{\prec}(t)$ s.t. $I, t' \models \alpha_1$. Two cases are
 1101 possible w.r.t. t' .
- 1102 ■ $t' \in \text{init}(I)$. By induction hypothesis, we have $I', t' \models \alpha_1$. Moreover, from Proposition 55,
 1103 we have $t' \in \text{min}_{\prec}(t)$. Hence, we can conclude that $I', t \models \heartsuit \alpha_1$.
 - 1104 ■ $t' \in \text{final}(I)$. Since $\text{range}(I) = \text{range}(I')$, there exists $t'' \in \text{final}(I')$ such that $V'(t'') =$
 1105 $V(t')$. From Proposition 53, we have $I', t'' \models \alpha_1$. From Proposition 56, we have $t'' \in$
 1106 $\text{min}_{\prec}(t)$. Hence, we can conclude that $I', t \models \heartsuit \alpha_1$.
- 1107 Same reasoning can be applied to prove the if part.

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1109 Lemma 15 is a direct result of result of Lemma 57.