Qualitative conditioning in an interval-based possibilistic setting

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Abstract

Possibility theory and possibilistic logic are well-known uncertainty frameworks particularly suited for representing and reasoning with uncertain, partial and qualitative information. Belief update plays a crucial role when updating beliefs and uncertain pieces of information in the light of new evidence. This paper deals with conditioning uncertain information in a qualitative interval-valued possibilistic setting. The first important contribution concerns a set of three natural postulates for conditioning interval-based possibility distributions. We show that any interval-based conditioning satisfying these three postulates is necessarily based on the set of compatible standard possibility distributions. The second contribution consists in a proposal of efficient procedures to compute the lower and upper endpoints of the conditional interval-based possibility distribution while the third important contribution provides a syntactic counterpart of conditioning interval-based possibility distributions in case where these latter are compactly encoded in the form of possibilistic knowledge bases.

Interval-based possibilistic logic, conditioning, possibility theory

1 Introduction

Many problems and applications need efficient formalisms for encoding and reasoning with uncertain, partial information or knowledge. Possibility theory and possibilistic logic [1, 2, 3, 4, 5] are uncertainty frameworks particularly suited for representing and reasoning with uncertain, incomplete, prioritized and qualitative information. In the literature, many extensions have been proposed for possibilistic logic to deal for instance with imprecise certainty degrees [6, 7], symbolic certainty weights [8, 9], multi-agent beliefs [10], temporal and uncertain information [11], uncertain conditional events [12, 13, 14], generalized possibilistic logic [1, 4, 15], justified beliefs [16], etc.

Interval-based uncertainty representations extend the underlying uncertainty settings in order to encode uncertainty by means of intervals of possible degrees instead of single values. Such extensions allow more flexible representations especially to deal with poor information, imprecise or ill-known beliefs, confidence intervals and multi-source information [17, 18]. Such representations are very widely used in some applications such as sensitivity analysis. In this paper, we are interested in interval-based possibilistic logic [6] which extends the standard possibilistic logic setting to allow intervals of possible degrees instead of single values attached to the formulas of the knowledge base.

Conditioning is an important task for updating the current uncertain information when a new sure piece of information is received. A conditioning operator is designed to satisfy some desirable properties such as giving priority to the new information and ensuring minimal change while transforming an initial distribution into a conditional one. Conditioning in standard (single-valued) possibility theory has been addressed in many works [19, 20, 21, 22, 23, 24, 25, 26, 27]. There are two major definitions of a possibility theory: min-based (or qualitative) possibility theory and product-based (or quantitative) possibility theory. At the semantic level, these two theories share the same definitions, including the concepts of possibility distributions, necessity measures, possibility measures and the definition of normalization conditions. However, they differ in the way they define possibilistic conditioning. Indeed, in possibility theory, there are two main definitions of possibilistic conditioning. The first one is called min-based conditioning [19, 28] (or qualitative-based conditioning) which is appropriate in situations where only the ordering between events is important. In this case, the unit interval $[0, 1]$ is viewed as an ordinal scale where only the minimum and the maximum operations are used for propagating uncertainty degrees. The second definition of conditioning is called product-based conditioning (or quantitative-based conditioning) where the unit interval is used in a general sense. In this case, the product operation can also be used in the propagation of uncertainty degrees. In the context of interval-based possibility theory [6, 7, 29], an extension of a conditioning operator is proposed for the interval-based setting. This is only done for the *product-based conditioning*. This extension is based on conditioning compatible possibility distributions and a syntactic counterpart for conditioning possibilistic logic bases is also proposed. In [12, 13], the authors dealt with some issues regarding inference (propagating possibility and necessity bounds) and independence where the beliefs are encoded using the concept of uncertain conditionals in a possibilistic setting.

This paper is primarily oriented to the study of min-based conditioning in an interval-based possibilistic setting and contains three major contributions:

- The first contribution (Section 4, Theorem 1) deals with conditioning in an interval-based possibility theory setting. We first propose three natural postulates for an interval-based conditioning. We show that any interval-based conditioning satisfying these postulates is necessarily based on applying min-based conditioning on each compatible standard possibility distribution.

- The second contribution (Section 4) consists in providing the exact lower and upper endpoints of min-based conditioning an interval-based distribution and a proposal of efficient procedures to compute the lower and upper endpoints of the conditional interval-based possibility distribution.

- The third contribution (Section 5) concerns syntactic computations of conditioning where interval-based possibility distributions are compactly represented by interval-based knowledge bases. We show that interval-based conditioning has the same computational complexity as the standard min-based conditioning.

Before presenting these contributions, let us first provide a brief refresher on possibility theory and possibilistic logic.

2 Brief reminder on possibility theory and possibilistic logics

Possibility theory [30, 31] is a well-known alternative uncertainty theory. This framework was coined by L. Zadeh [31] and it is developed by several researchers (eg. Dubois and Prade [32], Yager [33] and Borgelt and Kruse [34]). Possibility theory is based on a pair of dual measures allowing to evaluate the knowledge/ignorance relative to the event in hand. Among the main concepts of this framework are the ones of possibility distribution and possibilistic knowlegde base.

2.1 Possibility distributions

A possibility distribution, denoted π, is a mapping that attaches to every state ω of the world Ω (the universe of discourse or the set of states of the world) a degree in the unit interval $[0,1]$ expressing a partial knowledge over the world. The degree $\pi(\omega)$ associated with a state ω represents the degree of compatibility (or consistency) of the state ω with the available knowledge. By convention, $\pi(\omega)$=1 means that ω is fully consistent with the available knowledge, while $\pi(\omega)$=0 means that ω is impossible. $\pi(\omega)>\pi(\omega')$ simply means that ω is more compatible than ω'. A possibility distribution π is said to be normalized if there exists an interpretation ω such that $\pi(\omega)$=1, it is said to be subnormalized otherwise (subnormalized possibility distributions encode inconsistent sets of beliefs).

A possibility distribution allows to define two dual functions from 2^Ω to $[0,1]$ called possibility and necessity measures and denoted by Π and N respectively. They are defined as follows:

$$\Pi(\phi) = \max\{\pi(\omega) : \omega \in \phi\}, \text{ and}$$
$$N(\phi) = 1 - \Pi(\overline{\phi}).$$

Note that $\overline{\phi}$ denotes the complement of ϕ in Ω (namely, $\overline{\phi}$=$\Omega\backslash\phi$). $\Pi(\phi)$ measures to what extent the event ϕ is compatible with the available knowledge encoded by π while $N(\phi)$ measures to what extent it is entailed from π with certainty.

As already mentioned, possibility degrees are interpreted either i) *qualitatively* (in min-based possibility theory) where only the *ordering* of the values matters, or *quantitatively* (in product-based possibility theory) where the possibilistic scale $[0,1]$ is quantitative as in probability theory.

2.2 Conditioning a possibility distribution

In the standard possibilistic setting, conditioning comes down to updating a possibility distribution π encoding the current knowledge when a completely sure event called *evidence* or *observation*, denoted by $\phi\subseteq\Omega$, is received. This results in a conditional possibility distribution denoted by $\pi(.|\phi)$. There are many definitions of conditioning operators in the standard possibilistic setting [19, 20, 21, 22, 23].
Hisdal [19] proposed that the definition of a conditioning operator in the qualitative setting should satisfy the condition:

$$\forall \omega \in \phi, \pi(\omega) = \min(\pi(\omega|\phi), \Pi(\phi)).$$

Dubois and Prade [28] proposed to select the largest conditional possibility distribution satisfying this condition, leading to the following conditioning operator.

Definition 1 (min-based conditioning)**.** *Let π be a possibility distribution, $\phi \subseteq \Omega$ be a sure event. We define* min*-based conditioning of π by ϕ, simply denoted $\pi(.|_m\phi)$ as:*

$$\forall\omega\in\Omega, \pi(\omega|_m\phi) = \begin{cases} 1 & \text{if } \pi(\omega)=\Pi(\phi) \text{ and } \omega\in\phi; \\ \pi(\omega) & \text{if } \pi(\omega)< \Pi(\phi) \text{ and } \omega\in\phi; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

When $\Pi(\phi)$=0, then by convention $\forall\omega\in\Omega$, $\pi(\omega|_m\phi)$=1.
Conditioning a possibility distribution π with a completely sure piece of information $\phi\subseteq\Omega$ using Equation (1) results in a new and updated possibility distribution π'=$\pi(.|_m\phi)$ that is normalized (namely, $\max_{\omega\in\Omega}(\pi'(\omega)$=1) and fully accepting ϕ (namely, $\forall\omega\notin\phi$, $\pi'(\omega)$=0).

In case where possibility degrees are interpreted quantitatively like in the probabilistic setting, conditioning a possibility distribution is done using the so-called product-based conditioning [35], defined as follows (for $\Pi(\phi)\neq 0$):

$$\pi(\omega_i|_*\phi) = \begin{cases} \frac{\pi(\omega_i)}{\Pi(\phi)} & \text{if } \omega_i \in \phi; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

A possibility distribution can be compactly encoded in the form of possibilistic logic knowledge bases [3] or by means of possibilistic graphical models [34]. In this paper, we deal only with possibility distributions compactly encoded in the form of possibilistic logic knowledge bases.

2.3 Possibilistic logic knowledge bases

Possibilistic knowledge bases [1, 36, 3, 4, 5] are one of the well-known compact representations of possibility distributions. In possibilistic logic, weights are attached to formulas instead of elementary worlds. A possibilistic formula is a pair (φ, α) where φ is a propositional logic formula and $\alpha \in [0, 1]$ is a certainty degree associated with φ. The higher the certainty degree α is, the more important is the formula φ . A possibilistic base $K = \{(\varphi_i, \alpha_i), 1 \leq i \leq n\}$ is simply a set of possibilistic formulas as shown in the following example.

Example 1. *In this example, we consider a toy example from the medical area. The knowledge base K is given as follows:*

Formulas	Weights
$Flu \vee Cold$	1
$\neg Fever$	1
$Cold \Rightarrow Sneezing$	.9
$Flu \Rightarrow Cough$	.7
Flu	.6

Given a possibilistic base K, we can generate a unique possibility distribution where interpretations ω satisfying all propositional formulas in K have the highest possible degree $\pi(\omega)=1$ (since they are fully consistent), whereas the others are pre-ordered with respect to the highest formulas they falsify. More formally:

Definition 2. *Let K be a possibilistic knowledge base. Then, the corresponding possibility distribution π_K is given by: $\forall \omega \in \Omega$,*

$$\pi_K(\omega) = \begin{cases} 1 & \text{if } \forall (\varphi, \alpha) \in K,\ \omega \models \varphi \\ 1 - \max\{\alpha_i : (\varphi_i, \alpha_i) \in K, \omega \not\models \varphi_i\} & \text{otherwise.} \end{cases} \tag{3}$$

Ω here denotes the set of propositional interpretations. $\omega \models \varphi$ means that ω is a model of (or satisfies) φ in the sense of propositional logic.

Example 2. *(Example 1 continued) The possibility distribution induced by the knowledge base of Example 1 is as follows:*

State (ω)	$\pi_K(\omega)$
Flu Cold Fever Sneezing Cough	*0.0*
Flu Cold Fever Sneezing ¬Cough	*0.0*
Flu Cold Fever ¬Sneezing Cough	*0.0*
Flu Cold Fever ¬Sneezing ¬Cough	*0.0*
Flu Cold ¬Fever Sneezing Cough	*1.0*
Flu Cold ¬Fever Sneezing ¬Cough	*0.3*
Flu Cold ¬Fever ¬Sneezing Cough	*0.1*
Flu Cold ¬Fever ¬Sneezing ¬Cough	*0.1*
Flu ¬Cold Fever Sneezing Cough	*0.0*
... ...	...
... ...	...
¬Flu ¬Cold ¬Fever ¬Sneezing ¬Cough	*0.0*

In this example, we have five propositional variables in the knowledge base K. Hence, the possibility distribution π_K is over 2^5 states. Note that all the missing ones are associated with a zero possibility degree.

An important notion that plays a central role in the inference process and conditioning is the one of strict α-cut. Let α be a positive real number. A strict α-cut, denoted by K_α, is a set of propositional formulas defined by $K_\alpha = \{\varphi : (\varphi, \beta) \in K \text{ and } \beta > \alpha\}$. The strict α-cut is useful to measure the inconsistency degree of K denoted by $Inc(K)$ and defined by:

$$Inc(K) = \begin{cases} 0 & \text{if } K_0 \text{ is consistent} \\ \max\{\alpha : K_\alpha \text{ is inconsistent}\} & \text{otherwise} \end{cases} \tag{4}$$

If $Inc(K)$=0 then K is said to be completely consistent. If a possibilistic base K is partially inconsistent, then $Inc(K)$ can be seen as a threshold below which every formula is considered as not enough entrenched to be taken into account in the inference process.

The concept of α-cut can be used to provide the syntactic counterpart of conditioning a possibilistic knowledge base with a propositional formula:

Definition 3. *Let K be a possibilistic knowledge base and ϕ be a sure piece of information. The result of conditioning K by ϕ, denoted K_ϕ is defined as follows:*

$$K_\phi = \{(\phi, 1)\} \cup \{(\varphi, \alpha) : (\varphi, \alpha) \in K \textit{ and } K_{\geq\alpha} \wedge \phi \textit{ is consistent.}\}$$

Namely, K_ϕ is obtained by considering ϕ with a certainty degree '1', plus weighted formulas (φ, α) of K such that their α-cut is consistent with ϕ (the notation $K_{\geq\alpha}$ means the formulas of K associated with degrees greater or equal to α). It can be checked that:

$$\forall \omega \in \Omega,\ \pi_{K_\phi}(\omega) = \pi_K(\omega|\phi).$$

Next section briefly recalls main concepts of interval-based possibility theory and interval-based possibilistic logic.

3 Interval-based extension of possibilistic logic

Before giving a refresher on interval-based possibility theory [6], let us first start with an example to motivate the current work.

Example 3. *Let us choose an example from the football competitions area. Suppose we are interested in betting on football competitions. To increase our chances, we decided to do a survey on the Internet concerning the upcoming Euro championship. This survey is concerned with the views of football fans and supporters about the chances of two national teams we are interested in to reach the final. For the sake of simplicity, assume that we are interested only in the chances that the french and german teams have to get into the final match. We denote in the remainder of this example by f (resp. g) the statement that the french (resp. german) team will get into the final. To collect people's opinions, we posted a question on a specialized Internet survey platform. In addition, we also asked people to specify how much they believe in their answers using a unit scale $[0, 1]$ with the convention that the degree 1 should be attached to the most plausible scenario*[1] *while the degree 0 totally excludes the corresponding scenario. Values in between allow to rank order the other scenarios. Assume that we've got answers from five people p_1,..,p_5. We've got the plausibility levels of these people with respect to the different scenarios summarized as follows:*

In this example, the confidence degrees provided by the responders can be viewed as possibility degrees. Now, suppose that we got hundreds or thousands of answers or suppose that there is a large number of variables, then it makes sense to find a compact way to encode the obtained answers and more importantly to reason with them (answer any request of interest and update the available information when new sure information is obtained). Interval-valued possibility theory is suited for encoding and reasoning with this kind

[1]In this example, the scenario fg means both **French** and **German** teams will attend the final match while the scenario $f\neg g$ means that the **french team will attend the final** contrary to the **german team** .

	p_1	p_2	p_3	p_4	p_5
$f\ g$	1	1	1	.8	.5
$\neg f\ g$	.7	1	.9	1	1
$f\ \neg g$	.3	.2	.4	.6	.2
$\neg f\ \neg g$	.4	.5	.3	.6	5

Table 1: Example of multiple sources information

of information. For instance, the available knowledge from Example 3 could be summarized as an interval-valued possibility distribution given in Table 2.

	$I\pi$
$f\ g$	$[.5, 1]$
$\neg f\ g$	$[.7, 1]$
$f\ \neg g$	$[.2, .6]$
$\neg f\ \neg g$	$[.3, .6]$

Table 2: Interval-based distribution corresponding to the multiple source information of Table 1.

Let us now formally introduce the concept of interval-based possibility distributions and interval-based possibilistic knowledge bases.

3.1 Interval-based possibility distributions

As illustrated in Table 2, in the interval-based possibilistic setting, the available knowledge is encoded by an interval-based possibility distribution $I\pi$ where each state ω is associated with an interval $I\pi(\omega) = [\underline{I\pi}(\omega),\overline{I\pi}(\omega)]$ of possible values of the possibility degree $\pi(\omega)$ [6]. If I is an interval, then we denote by $\overline{I}$ and $\underline{I}$ its upper and lower endpoints respectively. When all I's associated with interpretations (or formulas) are singletons (meaning that $\overline{I} = \underline{I}$), we refer to standard distributions (resp. standard possibilistic bases). Here, $\underline{I\pi}(\omega)$ (resp. $\overline{I\pi}(\omega)$) denotes the lower (resp. upper) endpoints of the possibility degree of ω.

Definition 4 (Interval-based possibility distribution). *An interval-valued possibility distribution $I\pi$ is a mapping $I\pi : \Omega \to I$ from the universe of discourse Ω to the set I of all subintervals of the interval $[0, 1]$, with the normalization property requiring that $\max_{\omega \in \Omega} \overline{I\pi}(\omega)=1$.*

As in [6], we also interpret an interval-based possibility distribution as a family of compatible standard possibility distributions defined by:

Definition 5. *Let $I\pi$ be an interval-based possibility distribution. A normalized possibility distribution π is said to be compatible with $I\pi$ if and only if $\forall \omega \in \Omega,\ \pi(\omega) \in I\pi(\omega)$.*

An interval-based possibility distribution accepts at least one compatible distribution. We denote by $\mathcal{C}(I\pi)$ the set of all compatible possibility distributions with $I\pi$.

Example 4. *Let $I\pi$ be a possibility distribution described in the Table 3. Then following Definition 5, the possibility distribution π_1 and π_2 (from Table 3) are compatible with $I\pi$. However, π_3 is not compatible with $I\pi$ since $\pi_3(fg)$=.2 $\notin [.5, 1]$=$I\pi(fg)$.*

The following defines the concept of interval closure of a set of uncertainty degrees

Definition 6 (Interval closure). *Let A be a set of degrees between 0 and 1. We define the interval closure of A, denoted by* $\mathrm{IntCl}(A)$*, as the smallest (narrowest) interval that contains all the elements of A.*

$\omega\in\Omega$	$I\pi(\omega)$
fg	$[.5, 1]$
$f\neg g$	$[.7, 1]$
$\neg fg$	$[.2, .6]$
$\neg f\neg g$	$[.3, .6]$

$\omega\in\Omega$	$\pi_1(\omega)$	$\pi_2(\omega)$	$\pi_3(\omega)$
fg	.8	1	.2
$f\neg g$	1	.8	1
$\neg fg$	.5	.3	.5
$\neg f\neg g$	.5	.6	.6

Table 3: Example of compatible and non compatible possibility distributions.

Example 5. *Assume that A is a set defined as follows: A=$[.8, .9]\cup\{1\}$ then the closure of A is* $\mathrm{IntCl}(A)$=$[.8, 1]$. *Clearly, the interval $[.8, 1]$ is the narrowest sub-interval of $[0, 1]$ containing all the values of A.*

Let us now first provide a brief reminder of interval-based possibilistic logic bases [27] which allow to compactly represent interval-based possibility distributions.

3.2 Interval-based possibilistic logic

Contrary to the standard possibilistic logic where the uncertainty is described with single values, interval-based possibilistic logic uses intervals. We use closed sub-intervals $I\subseteq[0, 1]$ to encode the uncertainty associated with formulas or interpretations.

The syntactic representation of interval-based possibilistic logic generalizes the notion of a possibilistic base to an interval-based possibilistic knowledge base. Formally,

Definition 7. *An interval-based possibilistic knowledge base, denoted by IK, is a set of propositional formulas associated with intervals of certainty degrees:*

$$IK = \{(\varphi, I), \varphi \in \mathcal{L} \text{ and } I \text{ is a closed sub-interval of } [0, 1]\}$$

In Definition 7, $\varphi\in\mathcal{L}$ denotes a formula of a propositional language $\mathcal{L}$. As in standard possibilistic logic, an interval-based knowledge base IK is also a compact representation of an interval-based possibility distribution $I\pi_{IK}$ [6].

Definition 8. *Let IK be an interval-based possibilistic base. Then :*

$$I\pi_{IK}(\omega) = \left[\underline{I\pi}_{IK}(\omega), \overline{I\pi}_{IK}(\omega)\right]$$

where:

$$\underline{I\pi}_{IK}(\omega) = \begin{cases} 1 & \text{if } \forall(\varphi, I)\in IK,\ \omega \models \varphi \\ 1 - \max\{\overline{I} : (\varphi, I)\in K, \omega \not\models \varphi\} & \text{otherwise.} \end{cases}$$

and

$$\overline{I\pi}_{IK}(\omega) = \begin{cases} 1 & \text{if } \forall(\varphi, I)\in IK,\ \omega \models \varphi \\ 1 - \max\{\underline{I} : (\varphi, I)\in K, \omega \not\models \varphi\} & \text{otherwise.} \end{cases}$$

Given an interval-based possibilistic base IK, we define two particular compatible possibilistic bases $\underline{IK}$ and $\overline{IK}$ by selecting either lower endpoints of intervals or upper endpoints of intervals:

- $\underline{IK} = \{(\varphi, \alpha) : (\varphi, [\alpha, \beta])\in IK\}$
- $\overline{IK} = \{(\varphi, \beta) : (\varphi, [\alpha, \beta])\in IK\}$

Example 6. *Let IK=$\{(a\wedge b, [.4, .5]), (\neg a\vee b, [.3, .9])\}$ be an interval-based possibilistic base. The interval-based possibility distribution corresponding to IK according to Definition 8 is given in Table 4.*

From the knowledge base IK, we can compute Inc($\underline{IK}$) (with $\underline{IK}$=$\{(a\wedge b, .4), (\neg a\vee b, .3)\}$) which is 0 since $\underline{IK}$ is fully consistent.

ω	$I\pi_{IK}(\omega)$
ab	[1, 1]
$a\neg b$	[.1, .6]
$\neg ab$	[.5, .6]
$\neg a\neg b$	[.5, .6]

Table 4: Example of an interval-based distribution induced by an interval-based knowledge base.

This is for the representation part of the interval-based possibilistic setting. The question now is *how to extend the* min*-based conditioning operator for conditioning interval-valued possibility distributions and interval-valued possibilistic logic knowledge bases?*

4 Conditioning interval-valued possibility distributions

Before presenting our interval-based extension to the min-based possibilistic conditioning, let us first focus on some natural properties that an interval-based conditioning should satisfy in a possibilistic setting.

4.1 Three natural requirements for the interval-based conditioning

The first natural requirement concerns the *degenerate* case, namely when each interval $I\pi(\omega)$ contains exactly one single degree $\pi(\omega)$. The result of the new conditioning procedure should coincide with the result $\pi(.|_m\phi)$ of the original conditioning procedure (Definition 1). For each possibility distribution π, by $[\pi,\pi]$ we denote its interval-valued representation, i.e., an interval-valued possibility distribution for which, for every $\omega\in\Omega$, we have $I\pi(\omega)=[\pi(\omega),\pi(\omega)]$. In these terms, the above requirement takes the following form:

P1. For all π, $\phi\subseteq\Omega$ and $\omega\in\Omega$, $([\pi,\pi])(\omega|\phi)= [\pi(\omega|_m\phi),\pi(\omega|_m\phi)]$.
In other terms, let π be any possibility distribution and $I\pi$ such that $\forall\omega$, $I\pi(\omega)=[\pi(\omega),\pi(\omega)]$. Then $\forall\phi$, $I\pi(\omega|\phi)=[\pi(\omega|_m\phi),\pi(\omega|_m\phi)]$.

The second requirement is related to the fact that we do not know the exact values $\pi(\omega)$ since we only have partial information about them. In principle, if we can get some additional information about these values, then this would lead, in general, to narrower intervals (indeed, the width of an interval captures the ignorance regarding the exact value of $\pi(\omega)$). Let us define the concepts of *meta-specificity* between interval-based possibility distributions:

Definition 9. *Let $I\pi$ and $I\pi'$ be two interval-based possibility distributions. Then $I\pi$ is said to be more meta-specific than $I\pi'$, denoted $I\pi\subseteq I\pi'$, if $I\pi(\omega)\subseteq I\pi'(\omega)$ holds for all $\omega\in\Omega$*

It is reasonable to require that if we have new information about the original values $\pi(\omega)$, this should help us also to narrow down the corresponding values of conditional distributions:

P2. If $I\pi$ is more meta-specific than $I\pi'$ (namely, $I\pi\subseteq I\pi'$) then $I\pi(.|\phi)$ is more meta-specific than $I\pi'(.|\phi)$ (namely, $I\pi(.|\phi)\subseteq I\pi'(.|\phi)$).

It is obvious that postulates **P1** and **P2** are not sufficient to fully characterize the new extension. For example, we can take $([\pi,\pi])(.|\phi)=[\pi(.|_m\phi),\pi(.|_m\phi)]$ for degenerate interval-valued possibility distributions and $I\pi(\omega|\phi)=[0,1]$ for all other $I\pi$. To avoid such extensions, it is reasonable to impose the following minimality condition:

P3. $I\pi(.|\phi)$ is the narrowest interval-based distribution satisfying **P1**–**P2**. Namely, there exist no operation $I\pi(.|_1\phi)$ that satisfies both properties **P1**–**P2** and for which:

- $I\pi(\omega|_1\phi)\subseteq I\pi(\omega|\phi)$ for all $I\pi$, ω, and ϕ,

- $I\pi(\omega|_1\phi) \neq I\pi(\omega|\phi)$ for some $I\pi$, ω, and ϕ.

P3. ensures that $I\pi(.|\phi)$ computed applying the min-based conditoning on the compatible distributions cannot be narrowed further using another conditioning operator while satisfying properties **P1, P2**.

The following theorem provides our first main result where we show that there is only one interval-based conditioning satisfying **P1-P3** and where the interval conditional possibility degree $I\pi(\omega|\phi)$ is defined as the interval closure of the set of all $\pi(.|_m\phi)$, where π is compatible with $I\pi$.

Theorem 1. *There exists exactly one interval-based conditioning, denoted by $I\pi(.|_m\phi)$, that satisfies the properties* **P1**–**P3**, *and which is defined by:* $\forall\omega \in \Omega$,

$$I\pi(\omega|_m\phi) = \mathrm{IntCl}(\{\pi(\omega|_m\phi) : \pi \in \mathcal{C}(I\pi)\}) \tag{5}$$

where IntCl *is the interval closure given in Definition 6.*

Proof.
1°. We need to prove:

- that this closure $I\pi(.|_m\phi)$ satisfies the properties **P1**–**P3**, and
- that every operation $I\pi(.|\phi)$ that satisfies the properties **P1**–**P3** coincides with the interval closure of $I\pi(.|_m\phi)$.

2°. One can easily check that $I\pi(.|_m\phi)$ satisfies the properties **P1**–**P2**.

3°. Let us now prove that if an operation $I\pi(.|\phi)$ satisfies the properties **P1**–**P2**, then for every $I\pi$ and ϕ, we have $I\pi(.|_m\phi) \subseteq I\pi(.|\phi)$.

Then, for every distribution $\pi\in\mathcal{C}(I\pi)$, we have $([\pi,\pi]) \subseteq I\pi$ and thus, due to the postulate **P2**, we have $([\pi,\pi])(.|\phi) \subseteq I\pi(.|\phi)$. By the property **P1**, we have $([\pi,\pi])(\omega|\phi) = [\pi(\omega|\phi), \pi(\omega|\phi)]$. Thus, the above inclusion means that $\pi(.|\phi) \in I\pi(.|\phi)$.

The interval $I\pi(\omega|\phi)$ therefore contains all the values $\pi(\omega|\phi)$ corresponding to all possible $\pi\in\mathcal{C}(I\pi)$:

$$\{\pi(\omega|\phi) : \pi \in \mathcal{C}(I\pi)\} \subseteq I\pi(\omega|\phi).$$

Since the set $I\pi(\omega|\phi)$ is an interval, it therefore contains, with the set $\{\pi(\omega|\phi) : \pi\in\mathcal{C}(I\pi)\}$, its interval closure, i.e., the set $I\pi(\omega|_m\phi)$. Thus, we conclude that $I\pi(\omega|_m\phi) \subseteq I\pi(\omega|\phi)$ for all ω.

The statement is proven.

4°. We can now prove that $I\pi(.|_m\phi)$ also satisfies the property **P3**.

Indeed, if there is some other operation $|_1$ that satisfies **P1** and **P2**, and for which $I\pi(\omega|_1\phi) \subseteq I\pi(\omega|_m\phi)$ for all ω, then, since we have already proven the opposite enclosure in Part 3 of this proof, we conclude that $I\pi(\omega|_1\phi) = I\pi(\omega|_m\phi)$ for all ω, so indeed no narrower conditioning operation is possible.

5°. To complete the proof and show that there is only one solution, let us show that if some $I\pi(.|\phi)$ satisfies the properties **P1**–**P3**, then it coincides with $I\pi(.|_m\phi)$.

Indeed, by Part 3 of this proof, we have $I\pi(\omega|_m\phi) \subseteq I\pi(\omega|\phi)$ for all ω. If we had $I\pi(\omega|_m\phi) \neq I\pi(\omega|\phi)$ for some ω and ϕ, this would contradict the minimality property **P3**. Thus, indeed, $I\pi(.|_m\phi) = I\pi(.|\phi)$. Uniqueness is proven, and so is for the proposition. □

We can now go one step beyond Theorem 1 and provide the exact bounds of intervals associated with $I\pi(.|_m\phi)$.

4.2 Computing lower and upper endpoints of conditional interval-based possibility distributions

The aim of this section is to compute the lower and upper endpoints of the conditional interval-based possibility distribution.

Proposition 1. *Let $I\pi$ be an interval-based distribution. Then the interval-based conditional distribution, satisfying **P1–P3**, is described by* $I\pi(\omega|_m\phi)$=[$\underline{I\pi}(\omega|_m\phi)$, $\overline{I\pi}(\omega|_m\phi)$], *such that* $\forall\omega\in\Omega$:

$$\underline{I\pi}(\omega|_m\phi)=\begin{cases} 0 & \text{if } \omega \notin \phi \\ 1 & \text{if } \forall\omega'\neq\omega,\ \omega'\in\phi \text{ and } \underline{I\pi}(\omega)\geq\overline{I\pi}(\omega') \\ \underline{I\pi}(\omega) & \text{otherwise} \end{cases}$$

and

$$\overline{I\pi}(\omega|_m\phi)=\begin{cases} 0 & \text{if } \omega \notin \phi \\ 1 & \text{if } \overline{I\pi}(\omega)\geq\underline{I\Pi}(\phi), \\ \overline{I\pi}(\omega) & \text{otherwise} \end{cases}$$

Let us briefly comment Proposition 1. Let $\omega\in\Omega$ be an interpretation. First, for $\omega\notin\phi$, whatever the considered compatible possibility distribution π, we have $\pi(\omega|_m\phi)$=0. Hence, $I\pi(\omega|_m\phi)=[0,0]$. Assume now that $\omega\in\phi$ and $\forall\omega'\in\phi$, $\underline{I\pi}(\omega)\geq\overline{I\pi}(\omega')$. This means that whatever is the considered compatible possibility distribution π, we have $\pi(\omega)\geq\max\{\pi(\omega):\ \omega\in\phi\}=\Pi(\phi)$. Hence, $\pi(\omega|_m\phi)$=1 and $I\pi(\omega|_m\phi)=[1,1]$. Now, the last case for determining lower endpoint concerns the case where $\exists\omega'\in\phi$ such that $\underline{I\pi}(\omega)<\overline{I\pi}(\omega')$. This means that there exists a compatible possibility distribution π such that $\pi(\omega)=\underline{I\pi}(\omega)<\Pi(\phi)$, hence $\pi(\omega|_m\phi)=\underline{I\pi}(\omega)$ which is the smallest possible value. Similar reasoning goes for upper endpoints.

Example 7. *Let $I\pi$ of Table 5 (left side table) be the interval-based possibility distribution that we want to condition with the new piece of information ϕ=$\neg c$.* min-*based conditional distribution $I\pi(.|_m\phi)$ given in Table 5 (right side table) is obtained using either Proposition 1 or Theorem 1. For instance, for ω=$ab\neg c$, whatever the considered compatible possibility distribution π, we have $\pi(ab\neg c|\phi)$ between .1 and 1. Thus, the interval closure of $I\pi(ab\neg c|_m\phi)$=[.1, 1].*

$\omega\in\Omega$	$I\pi(\omega)$
abc	[1, 1]
$a\neg bc$	[.4,.6]
$\neg abc$	[.3,.6]
$\neg a\neg bc$	[.3,.6]
$ab\neg c$	[.1,.7]
$a\neg b\neg c$	[.4,.6]
$\neg ab\neg c$	[.1,.6]
$\neg a\neg b\neg c$	[.3,.6]

$\omega\in\Omega$	$I\pi(\omega|\phi)$
abc	[0, 0]
$a\neg bc$	[0, 0]
$\neg abc$	[0, 0]
$\neg a\neg bc$	[0, 0]
$ab\neg c$	[.1, 1]
$a\neg b\neg c$	[.4, 1]
$\neg ab\neg c$	[.1, 1]
$\neg a\neg b\neg c$	[.3, 1]

Table 5: Interval-based distribution $I\pi$ and its conditioned distribution $I\pi(.|\phi)$

5 Syntactic computations of interval-based conditioning

In this section, we provide the syntactic counterpart of the interval-based conditioning presented above.

5.1 Computing conditioned knowledge bases

Given an interval-based possibilistic knowledge base IK and a new evidence ϕ, our aim is to compute the conditional base IK_ϕ corresponding to conditioning the information encoded in IK with ϕ. As illustrated in Figure 1, the aim of this subsection is therefore to propose the syntactic characterization of conditioning such that:

$$\forall \omega \in \Omega,\ \ I\pi_{IK}(\omega|_m\phi) = I\pi_{IK_\phi}(\omega),$$

where $I\pi_{IK_\phi}$ is the interval-based distribution associated with IK_ϕ, and $I\pi_{IK}(.|_m\phi)$ is the result of conditioning $I\pi_{IK}$ using the conditioning operator presented in the previous subsection (Proposition 1 or Theorem 1), and $[\phi]$ is the set of models of ϕ.

$$\begin{tikzcd}
\text{input=}\phi\, IK \arrow[rr, "\text{Definition 8}"] \arrow[d, "\text{aim of this section}"'] & & I\pi_{IK}^{\text{input=}\phi} \arrow[d, "\text{Theorem 1}"] \\
IK_\phi \arrow[r, "\text{Definition 8}"] & I\pi_{IK_\phi}(.) \arrow[r, leftrightarrow] & I\pi_{IK}(.|\phi)
\end{tikzcd}$$

Figure 1: Conditioning interval-valued possibilistic information at the semantic and syntactic levels

We first need to introduce some notations.

- $\alpha = Inc(\overline{IK} \cup \{(\phi, 1)\})$ and $\beta = Inc(\underline{IK} \cup \{(\phi, 1)\})$, Intuitively, α and β compute inconsistency degree intervals resulting from assuming that ϕ is fully true. This offers a characterization of $\overline{I\Pi}(\phi)$=$1-\beta$ and $\underline{I\Pi}(\phi)$=$1-\alpha$.
- Let ω^* be a model of $\{\psi : (\psi, I) \in IK$ and $\underline{I}>\beta\} \cup \{\phi\}$. Let $IK_{\neg\omega^*}$=$IK \cup \{(\neg\omega^*, [1, 1])\}$ be a base obtained by adding the negation of ω^*, then we compute $\gamma = Inc(\underline{IK}_{\neg\omega^*} \cup \{(\phi, 1)\})$. Models ω of $\{\psi : (\psi, I) \in IK$ and $\underline{I} > \beta\}$ are exactly those having $\overline{I\pi}(\omega)$=$\overline{I\Pi}(\phi)$. γ computes the second best value of models of ϕ (since a model ω^* is excluded from IK) which is very useful for characterizing $\underline{I\pi}(\omega|_m\phi)$.

With the help of these notations, we are now ready to present the third contribution of this paper.

Theorem 2. *Let IK be an interval-based knowledge base. Let $I\pi_{IK}$ be its associated possibility distribution. Let IK_ϕ=$\{(\phi, [1, 1])\} \cup \{(\varphi, I) : (\varphi, I) \in IK,$ and $\underline{I}>\alpha\} \cup \{(\varphi, [0, \overline{I}]) : (\varphi, I) \in IK,$ and $\underline{I}<\alpha$ and $\overline{I}>\gamma\}$. Then:*

$$\forall \omega, I\pi_{IK}(\omega|_m\phi) = I\pi_{IK_\phi}(\omega);$$

where $I\pi_{IK}(.|_m\phi)$ is the result of applying min*-based interval conditioning on $I\pi_{IK}$ (see Proposition 1 and Theorem 1), and $I\pi_{IK_\phi}$ is the interval-based distribution associated with IK_ϕ using the Definition 8.*

The knowledge base IK_ϕ resulting from conditioning IK with ϕ is composed of three parts:

- The first consists in adding ϕ as a fully certain information, $\{(\phi, [1, 1])\}$. From Definition 8, all worlds that are outside ϕ (not satisfying ϕ) are excluded. This is in accordance with Proposition 1.
- The second part, $\{(\varphi, I) : (\varphi, I) \in IK,$ and $\underline{I}>\alpha\}$, contains a subbase of IK where the intervals are unchanged. This encodes the third item of definition of $\underline{I\pi}(\omega)$ and $\overline{I\pi}(\omega)$ in Proposition 1 (recall that $1-\alpha$=$\underline{I\Pi}(\phi)$).
- The last part encodes exactly the situation where some possibility degrees (in Proposition 1) are shifted up to 1. This is reflected in possibilistic knowledge bases by shifting down some certainty degrees to 0.

Indeed, the knowledge base IK_ϕ is composed of three parts: i) the new sure piece of evidence ϕ associated with the interval $[1, 1]$, ii) the formulas in IK that belong to the consistent part of IK with ϕ (the intervals associated with this second part are kept unchanged) while iii) the last part, namely formulas in the inconsistent part of IK with ϕ will see their intervals changed to allow satisfying the normalization condition at semantic level. Recall that to ensure a possibility degree of 1 at the semantic level, the insconsistency degree of at least one compatible base should be 0.

Proof. In order now to prove the theorem we have to show that $\forall\omega\in\Omega$, $I\pi_{IK_\phi}=I\pi_{IK}(.|_m[\phi])$. First note that if an interpretation ω is not a model of ϕ, then by definition we have:

$$I\pi_{IK_\phi}(\omega) = I\pi_{IK}(\omega|_m[\phi]) = [0,0]$$

This is explained by the presence of $(\phi,[1,1])$ in IK_ϕ.
Now, for $\omega\models\phi$, we have two distinct cases:

- The case where ω falsifies a formula from: $\{(\varphi,I):(\varphi,I)\in IK,\ \text{and}\ \underline{I}>\alpha\}$ then:
$$\begin{aligned}
\underline{I\pi}_{IK_\phi}(\omega) &= 1-\max\{\overline{I}:(\varphi,I)\in IK \text{ and } \underline{I}>\alpha\}.\\
&= 1-\max\{\overline{I}:(\varphi,I)\in IK \text{ and } \underline{I}>Inc(\overline{IK}\cup\{(\phi,1)\})\}\\
&= 1-\max\{\overline{I}:(\varphi,I)\in IK \text{ and } \underline{I}>1-\underline{I\Pi}([\phi])\}\\
&= 1-\max\{\overline{I}:(\varphi,I)\in IK \text{ and } 1-\underline{I}<\underline{I\Pi}([\phi])\}\\
&= \underline{I\pi}(\omega) \quad \text{ if } \overline{I\pi}(\omega)<\underline{I\Pi}([\phi])\\
&= \underline{I\pi}_{IK}(\omega|_m[\phi]).
\end{aligned}$$
$$\begin{aligned}
\overline{I\pi}_{IK_\phi}(\omega) &= 1-\max\{\underline{I}:(\varphi,I)\in IK \text{ and } \underline{I}>\alpha\}.\\
&= 1-\max\{\underline{I}:(\varphi,I)\in IK \text{ and } \underline{I}>Inc(\overline{IK}\cup\{(\phi,1)\})\}\\
&= 1-\max\{\underline{I}:(\varphi,I)\in IK \text{ and } \underline{I}>1-\underline{I\Pi}([\phi])\}\\
&= 1-\max\{\underline{I}:(\varphi,I)\in IK \text{ and } 1-\underline{I}<\underline{I\Pi}([\phi])\}\\
&= \overline{I\pi}(\omega) \quad \text{ if } \overline{I\pi}(\omega)<\underline{I\Pi}([\phi])\\
&= \overline{I\pi}_{IK}(\omega|_m[\phi]).
\end{aligned}$$

- The case where ω falsifies a formula from: $\{(\varphi,[0,\overline{I}]):(\varphi,I)\in IK,\ \text{and}\ \underline{I}\leq\alpha \text{ and } \overline{I}>\gamma\}$ then:
As γ computes the second best value of models of ϕ, for this proof, we use $secondbest(IK)$ to determine γ and $secondbest(IK)=1-secondbest(I\pi_{IK})$.
$$\begin{aligned}
\underline{I\pi}_{IK_\phi}(\omega) &= 1-\max\{\overline{I}:(\varphi,I)\in IK \text{ and } \underline{I}\leq\alpha \text{ and } \overline{I}>\gamma\}\\
&= 1-\max\{\overline{I}:(\varphi,I)\in IK \text{ and } \underline{I}\leq Inc(\overline{IK}\cup\{(\phi,1)\}) \text{ and } \overline{I}>secondbest(IK)\}\\
&= 1-\max\{\overline{I}:(\varphi,I)\in IK \text{ and } \underline{I}\leq 1-\underline{I\Pi}([\phi]) \text{ and } \overline{I}>1-secondbest(I\pi_{IK})\}\\
&= 1-\max\{\overline{I}:(\varphi,I)\in IK \text{ and } 1-\underline{I}\geq\underline{I\Pi}([\phi]) \text{ and } 1-\overline{I}<secondbest(I\pi_{IK})\}\\
&= \underline{I\pi}(\omega) \quad \text{ if } \overline{I\pi}(\omega)\geq\underline{I\Pi}([\phi]) \text{ and } \underline{I\pi}(\omega)<secondbest(I\pi_{IK})\\
&= \underline{I\pi}_{IK}(\omega|_m[\phi]).
\end{aligned}$$
$$\begin{aligned}
\overline{I\pi}_{IK_\phi}(\omega) &= 1-\max\{0:(\varphi,I)\in IK \text{ and } \underline{I}\leq\alpha \text{ and } \overline{I}>\gamma\}.\\
&= 1-\max\{0:(\varphi,I)\in IK \text{ and } \underline{I}\leq Inc(\overline{IK}\cup\{(\phi,1)\}) \text{ and } \overline{I}>secondbest(IK)\}\\
&= 1-\max\{0:(\varphi,I)\in IK \text{ and } \underline{I}\leq 1-\underline{I\Pi}([\phi]) \text{ and } \overline{I}>1-secondbest(I\pi_{IK})\}\\
&= 1-\max\{0:(\varphi,I)\in IK \text{ and } 1-\underline{I}\geq\underline{I\Pi}([\phi]) \text{ and } 1-\overline{I}>secondbest(I\pi_{IK})\}\\
&= 1 \quad \text{ if } \underline{I\pi}(\omega)\geq\underline{I\Pi}([\phi]) \text{ and } \underline{I\pi}(\omega)<secondbest(I\pi_{IK})\\
&= \overline{I\pi}_{IK}(\omega|_m[\phi]).
\end{aligned}$$

□

Let us see an example to illustrate Theorem 2.

Example 8. *Let IK be an interval-based possibilistic knowledge base such that* $IK=\{(a\wedge b,[.4,.6]),(a,[0,.7]),(c\vee\neg b,[.3,.9])\}$. *The associated interval-based possibility distribution* π_{IK} *(using Definition 8) is the same as the one given in Table 5. Let* $\phi=\neg c$ *(and* $\phi=[\phi]$ *the set of models of* $\neg c$*) be the new evidence. For the computation of* IK_ϕ*, let us first compute the values of* α, β *and* γ*. Then, we have:* $\alpha=Inc(\{(a\wedge b,.6),(a,.7),(c\vee\neg b,.9),(\neg c,1)\})=.6$, $\beta=Inc(\{(a\wedge b,.4),(a,0),(c\vee\neg b,.3),(\neg c,1)\})=.3$ *and* $\gamma=.4$.

Hence, according to Theorem 2, the result of conditioning IK by ϕ *is given by:* $IK_\phi=\{(a\wedge b,[0,.6]),(a,[0,.7]),(c\vee\neg b,[0,.9]),(\neg c,[1,1])\}$. *And if we compare with Example 7, where the distribution* $I\pi(.|_m\phi)$ *is conditioned according to Proposition 1 then the associated interval-based distribution to* IK_ϕ *is exactly the same. Hence, Theorem 2 indeed provides a compact encoding of the conditioning procedure.*

The following proposition gives the computational complexity of conditioning an interval-based possibilistic knowledge base IK according to Theorem 2.

Proposition 2. *Let IK be an interval-based possibilistic knowledge base and* ϕ *be the new evidence. Let* IK_ϕ *be an interval-based possibilistic knowledge base computed according to Theorem 2. Then* IK_ϕ *have the same size as IK and computing* IK_ϕ *is in* $O(\log_2(m).SAT)$ *where SAT is a satisfiability test of a set of propositional clauses and m is the number of different weights in* $\overline{IK}$ *and* $\underline{IK}$.

Clearly, once the parameters α, β, γ are computed, computing IK_ϕ from $\{IK, \phi, \alpha, \beta, \gamma\}$ is straightforward and it is done in linear time. Indeed, computing α, β, γ mainly comes down to compute the inconsistency degrees of $\overline{IK}$ and $\underline{IK}$. This needs $\log_2(m)$ calls to a SAT solver exactly as in standard possibilistic logic [3]. Hence, the syntactic counterpart of conditioning an interval-based possibilistic base has exactly the same computational complexity as computing the min-based conditioning of a standard possibilistic base.

6 Concluding remarks and discussions

This paper addressed the issue of conditioning in a qualitative interval-based possibilistic setting. The interval-based extension of the standard possibilistic setting offers a flexible model for encoding multiple source information. However, no form of qualitative conditioning has been proposed in this framework. This work fills this gap by proposing an efficient extension of the min-based conditioning to the interval-based setting. Three main contributions are presented:

i) A set of three natural postulates **P1-P3** ensuring that any interval-based conditioning satisfying these three postulates is necessarily based on min-based conditioning the set of compatible standard possibility distributions. The first postulate **P1** aims to recover the standard min-based conditioning in case where all the intervals contain singleton values (all lower endpoints coincide with upper endpoints). The second postulate **P2** captures a kind of meta-specificity regarding conditioning interval-based sets of beliefs while the third postulate **P3** aims to ensure a minimality condition.

ii) Efficient procedures to compute the lower and upper endpoints of the conditional interval-based possibility distribution. Such procedures exclude any state of the world that is inconsistent with the new evidence in hand and perform some kind of normalization based on the concept of compatible possibility distribution without generating the whole set of compatible distributions.

iii) A syntactic counterpart of conditioning interval-based possibilistic bases. This counterpart performs some tests and does some modifications on the formulas of the original knowledge base such that the new evidence is integrated with a certainty degree of 1. This ensures the same result as if the knowledge base were conditioned at the semantic level.

Interestingly enough, the syntactic counterpart has the same complexity as conditioning standard possibilistic knowledge bases. More precisely, conditioning an interval-based possibilistic knowledge base does not require extra computational cost compared with conditioning a standard possibilistic base.

Our approach can be applied in many applications, especially when dealing with multiple sources or imprecise qualitative information. The approach is appealing since it allows a compact representation of knowledge while the conditioning operation is performed in time complexity equivalent to conditioning a standard possibilistic knowledge base. Indeed, the proposed approach generalizes the standard qualitative possibilistic conditioning and allows to update the current knowledge a syntactic way without any extra cost compared to conditioning standard possibilistic knowledge bases.

In [7], a set of seven postulates **IC1–IC7** have been proposed for product-based conditioning (which is another form of conditioning in a standard but quantitative possibility theory, see Equation 2). To relate our postulates **P1**–**P3** to postulates **IC1**–**IC7**, note that **IC1**–**IC7** use the product-based operator while **P1**–**P3** use the min-based conditioning. Now, if **P1** is replaced by **P'1** stating that:

$$\mathbf{P'1}\ \forall \pi, \phi \subseteq \Omega \text{ and } \omega \in \Omega, ([\pi, \pi])(\omega|\phi) = [\pi(\omega|_*\phi), \pi(\omega|_*\phi)]. \tag{6}$$

Then we can show that an interval-based conditioning that satisfies **P'1**, **P2**, **P3** necessarily satisfies **IC1**–**IC7** but the converse is false. Indeed, as pointed out in [7], using the unique conditioning satisfying properties **IC1**–**IC7** may lead in some situations to discontinuous intervals if we rely on the min-based conditioning.

In [12, 13, 14], the authors deal with some issues like inference and conditioning in a possibilistic setting where the available knowledge consists in a set of conditional events. In particular, the authors deal with such issues using different t-norms including the *min*-based operator. While they refer to classes of conditional possibility measures, there is no reference to interval-based possibility distributions or interval-based knowledge bases as representations encoding the available knowledge. The current paper proposes the first *min*-based conditioning operator for interval-based possibilistic knowledge bases.

Recently, in [37], we proposed a set-valued extension to possibility theory. We provide a characterization of set-valued possibilistic logic bases and set-valued possibility distributions in terms of compatible possibilistic logic bases and compatible possibility distributions respectively. The main difference between the interval-based representation proposed in this paper and the set-based one in [37] is that summarizing a set of knowledge bases (resp. a set of possibility distributions) within an interval-based knowledge base (resp. interval-based possibility distribution) incurs more information loss. However, both representations are non trivial and guarantee that the semantics given to the interval-based/set-based representation in terms of compatible bases/distributions contains the initial knowledge bases or possibility distributions. Moreover, the interval-based representation requires less computations when it comes to reasoning and conditioning.

As a future work, we will address conditioning in another form of compact representations of interval-based possibility distributions which are interval-based possibilistic networks [38].

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