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Compactification and decompactification by weights on Bergman spaces

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Abstract. We characterize the symbols φ for which there exists a weight w such that the weighted composition operator $M_w C_\varphi$ is compact on the weighted Bergman space \mathfrak{B}_α^2 . We also characterize the symbols for which there exists a weight w such that $M_w C_\varphi$ is bounded but not compact. We also investigate when there exists w such that $M_w C_\varphi$ is Hilbert-Schmidt on \mathfrak{B}_α^2 .

MSC 2010 primary: 47B33 ; secondary: 30H20

Key-words Bergman space – compactification – composition operator – decompactification – Hilbert-Schmidt operator – weighted Bergman space – weighted composition operator

1 Introduction

It is known (see [4] for instance) that “weightening” a composition operator C_φ on the Hardy space H^2 by some weight w , we can improve its compactness properties, and even its membership in Schatten classes S_p , or the decay of its approximation numbers ([9, Theorem 2.3], [12]), or at the opposite make a compact composition operator non compact ([12]).

In this paper, we consider weighted composition operators $M_w C_\varphi$ on the weighted Bergman spaces \mathfrak{B}_α^2 , with $\alpha > -1$. Note that for such an operator to be bounded from \mathfrak{B}_α^2 into itself, it is necessary that $w \in \mathfrak{B}_\alpha^2$ (since $w = (M_w C_\varphi)(\mathbb{1})$).

We show in Section 3 that C_φ can be weighted to become compact on \mathfrak{B}_α^2 if and only if the set where φ has an angular derivative has null measure.

In Section 4, we show that there exists a weight w such that $M_w C_\varphi$ is bounded but not compact on \mathfrak{B}_α^2 if and only if $\|\varphi\|_\infty = 1$.

In Section 5, we study when $M_w C_\varphi$ can be Hilbert-Schmidt on \mathfrak{B}_α^2 for some weight w .

2 Notation and background

The weighted Bergman space \mathfrak{B}_α^2 , with $\alpha > -1$, is the space of all analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ on the unit disk \mathbb{D} such that

$$\|f\|_{\mathfrak{B}_\alpha^2}^2 = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty,$$

where A is the normalized area measure on \mathbb{D} . When $\alpha = 0$, we write simply \mathfrak{B}^2 instead of \mathfrak{B}_0^2 and call it the Bergman space.

Every analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ defines a bounded composition operator $C_\varphi: f \mapsto f \circ \varphi$ from \mathfrak{B}_α^2 into itself ([16, Proposition 3.4]).

The pull-back measure A_φ of φ is defined as:

$$A_\varphi(B) = A[\varphi^{-1}(B)] \quad \text{for all Borel sets } B \subseteq \mathbb{D}.$$

Let μ be a finite Borel measure on \mathbb{D} . For $\beta > 1$, the measure μ is said a β -Carleson measure if:

$$(2.1) \quad \sup_{|\xi|=1} \mu[S(\xi, h)] = O(h^\beta),$$

where

$$S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| < h\}$$

is the Carleson box of size h centered at $\xi \in \mathbb{T} = \partial\mathbb{D}$. The measure μ is said a *vanishing β -Carleson measure* if:

$$(2.2) \quad \sup_{|\xi|=1} \mu[S(\xi, h)] = o(h^\beta) \quad \text{as } h \rightarrow 0.$$

Recall the following result (see [7] and [16, Theorem 4.3]).

Theorem 2.1. *Let μ be a finite Borel measure on \mathbb{D} . Then:*

- (a) $\mathfrak{B}_\alpha^2 \subseteq L^2(\mu)$ if and only if μ is an $(\alpha + 2)$ -Carleson measure.
Moreover, when this happens, the canonical inclusion $J_\mu: \mathfrak{B}_\alpha^2 \rightarrow L^2(\mu)$ is bounded.
- (b) The canonical inclusion $J_\mu: \mathfrak{B}_\alpha^2 \rightarrow L^2(\mu)$ is compact if and only if μ is a vanishing $(\alpha + 2)$ -Carleson measure.

Corollary 2.2. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map and $w \in \mathfrak{B}_\alpha^2$. Set, for every Borel set B in \mathbb{D} :*

$$(2.3) \quad \mu_{w, \varphi}(B) = \int_{\varphi^{-1}(B)} |w(z)|^2 (1 - |z|^2)^\alpha dA(z).$$

Then:

(a) The weighted composition operator $M_w C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$, defined as:

$$(2.4) \quad (M_w C_\varphi)f = w(f \circ \varphi),$$

is bounded if and only if $\mu_{w,\varphi}$ is an $(\alpha + 2)$ -Carleson measure.

(b) The weighted composition operator $M_w C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is compact if and only if $\mu_{w,\varphi}$ is a vanishing $(\alpha + 2)$ -Carleson measure.

Proof. Observe that, for all $f \in \mathfrak{B}_\alpha^2$, we have

$$\|(M_w C_\varphi)f\|_{\mathfrak{B}_\alpha^2}^2 = \int_{\mathbb{D}} |f[\varphi(z)]|^2 |w(z)|^2 (1 - |z|^2)^\alpha dA(z) = \|f\|_{L^2(\mu_{w,\varphi})}^2. \quad \square$$

3 Compactification

Recall the following definitions (see [20, Section 4.1]).

Definition 3.1. A holomorphic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ has an angular limit (or a non-tangential limit) l at $\xi \in \mathbb{T}$ if $\varphi(z)$ converges to l whenever z tends to ξ inside any angular sector in \mathbb{D} whose vertex is ξ . Then l is called the angular limit of φ at ξ and is denoted:

$$l = \angle \lim_{z \rightarrow \xi} \varphi(z).$$

Definition 3.2. A holomorphic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ has an angular derivative at $\xi \in \mathbb{T}$ if it has an angular limit ζ at ξ , with $|\zeta| = 1$ and:

$$\angle \lim_{z \rightarrow \xi} \frac{\varphi(z) - \zeta}{z - \xi}$$

exists and is finite. This limit is called the angular derivative of φ at ξ and is denoted by $\varphi'(\xi)$.

Let us also recall that the Julia-Carathéodory theorem (see [20, Section 4.2]), says that φ has an angular derivative at $\xi \in \mathbb{T}$ if and only if:

$$(3.1) \quad \delta := \liminf_{z \rightarrow \xi} \frac{1 - |\varphi(z)|}{1 - |z|} < +\infty,$$

or, equivalently:

$$(3.2) \quad \limsup_{z \rightarrow \xi} \frac{1 - |z|}{1 - |\varphi(z)|} > 0,$$

and, when this happens, we have $\delta > 0$ and $\varphi'(\xi) = \xi \bar{\zeta} \delta$, so $|\varphi'(\xi)| = \delta$.

We define

$$(3.3) \quad \mathcal{AD}(\varphi) = \{\xi \in \mathbb{T}; \varphi \text{ has an angular derivative at } \xi\},$$

and we call it the *angular derivative set* of φ .

B. MacCluer and J. Shapiro proved ([16, Theorem 3.5]) that, for $\alpha > -1$, the composition operator $C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is compact if and only if:

$$(3.4) \quad \mathcal{AD}(\varphi) = \emptyset.$$

Asking for a compactification, we have the following result.

Theorem 3.3. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map. Then the following assertions are equivalent, for the weighted Bergman space \mathfrak{B}_α^2 , with $\alpha > -1$:*

- 1) *there exists a holomorphic function w , with $w \not\equiv 0$, such that the weighted composition operator $M_w C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is compact;*
- 2) *there exists a weight $w \in H^\infty$, with $w \not\equiv 0$, such that the weighted composition operator $M_w C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is compact;*
- 3) *the angular derivative set of φ has null measure:*

$$(3.5) \quad m[\mathcal{AD}(\varphi)] = 0,$$

where m is the normalized Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$;

$$4) \quad \lim_{z \rightarrow \xi} \frac{1 - |z|}{1 - |\varphi(z)|} = 0 \text{ for almost all } \xi \in \mathbb{T}.$$

For example, if $\varphi(z) = \frac{1+z}{2}$, then C_φ is not compact on \mathfrak{B}_α^2 , but it is compactifiable by a weight in H^∞ .

When the equivalent conditions of Theorem 3.3 are satisfied, we say that composition operator C_φ is *compactifiable*.

The proof will be based on the following result of Moorhouse ([17, Corollary 1]; see also [2, Proposition 1]).

Proposition 3.4 (Moorhouse). *Let $\alpha > -1$. Let φ and w be analytic functions on \mathbb{D} . Then:*

- 1) *If the weighted composition operator $M_w C_\varphi$ is compact on \mathfrak{B}_α^2 , we have:*

$$(3.6) \quad \lim_{|z| \rightarrow 1} |w(z)|^2 \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} = 0.$$

- 2) *When w is bounded, $M_w C_\varphi$ is compact on \mathfrak{B}_α^2 if and only if*

$$(3.7) \quad \lim_{|z| \rightarrow 1} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

For 1), we compute:

$$\|(M_w C_\varphi)^*(k_z)\|_{(\mathfrak{B}_\alpha^2)^*}^2 = |w(z)|^2 \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2},$$

where k_z is the normalized reproducing kernel of \mathfrak{B}_α^2 , and, using that k_z weakly converges to 0 as $|z| \rightarrow 1$, we obtain:

$$\lim_{|z| \rightarrow 1} |w(z)|^2 \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} = 0.$$

To obtain the necessary condition in 2), we use the following, easily checked, fact which shows that (3.7) is equivalent to (3.6), since w is bounded in 2).

Lemma 3.5. *Let $f, g: \mathbb{D} \rightarrow [0, \infty)$ be two bounded functions. Then the following assertions are equivalent:*

- a) $\lim_{|z| \rightarrow 1} f(z)g(z) = 0$;
- b) $\lim_{|z| \rightarrow 1} \min[f(z), g(z)] = 0$;
- c) $\lim_{|z| \rightarrow 1} [f(z)]^a [g(z)]^b = 0$, for all $a, b > 0$.

The sufficient condition in 2) is proved by [17, Lemma 1].

Proof of Theorem 3.3. The implication 2) \Rightarrow 1) needs no comment.

3) \Rightarrow 2) Assume that $m[\mathcal{AD}(\varphi)] = 0$. A theorem of Privalov (see [21, Vol. I, bottom of page 276]), asserts the existence a function $w \not\equiv 0$ in H^∞ such that

$$(3.8) \quad \lim_{z \rightarrow \xi} w(z) = 0 \quad \text{for all } \xi \in \mathcal{AD}(\varphi).$$

The Schwarz-Pick lemma (see [1, Corollary 2.40]) tells that

$$\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \leq 2 \frac{1 - |z|}{1 - |\varphi(z)|} \leq 2 \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}.$$

Hence for $\xi \in \mathcal{AD}(\varphi)$, we have

$$\lim_{z \rightarrow \xi} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

For $\xi \notin \mathcal{AD}(\varphi)$, thanks to the Julia-Carathéodory theorem and (3.2), we also have

$$\lim_{z \rightarrow \xi} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0.$$

Hence

$$(3.9) \quad \lim_{z \rightarrow \xi} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0 \quad \text{for all } \xi \in \mathbb{T}.$$

By a compactness argument, we obtain that

$$(3.10) \quad \lim_{|z| \rightarrow 1} |w(z)|^2 \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0;$$

in fact, if (3.10) failed, there would be a sequence (z_n) such that $|z_n| \xrightarrow{n \rightarrow \infty} 1$ and for which

$$\limsup_{n \rightarrow \infty} |w(z_n)|^2 \frac{1 - |z_n|^2}{1 - |\varphi(z_n)|^2} > 0;$$

by compactness a subsequence converges to some $\xi \in \partial\mathbb{D}$, and that would contradict (3.9).

Since w is bounded, it follows from Proposition 3.4, that $M_w C_\varphi$ is compact on \mathfrak{B}_α^2 .

1) \Rightarrow 3) Assume that $M_w C_\varphi$ is compact with w analytic and $w \not\equiv 0$. By Proposition 3.4, 1), we have:

$$(3.11) \quad \lim_{|z| \rightarrow 1} |w(z)|^2 \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} = 0.$$

In particular, for every $\xi \in \mathbb{T}$:

$$(3.12) \quad \lim_{z \rightarrow \xi} |w(z)|^2 \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} = 0.$$

Now, for every $\xi \in \mathcal{AD}(\varphi)$, we have, if ζ is the angular limit of φ at ξ :

$$\lim_{z \rightarrow \xi} \left| \frac{\varphi(z) - \zeta}{z - \xi} \right| = |\varphi'(\xi)| < \infty.$$

If z belongs to an angular sector S_ξ of vertex ξ , there is a positive constant C , depending only on this sector, such that $|z - \xi| \leq C(1 - |z|)$; hence

$$\liminf_{z \rightarrow \xi, z \in S_\xi} \frac{1 - |z|}{1 - |\varphi(z)|} \geq \liminf_{z \rightarrow \xi, z \in S_\xi} C \frac{|z - \xi|}{|\varphi(z) - \zeta|} = \frac{C}{|\varphi'(\xi)|} > 0.$$

Then, it follows, with (3.12), that $\lim_{z \rightarrow \xi, z \in S_\xi} w(z) = 0$. Since the angular sector S_ξ is arbitrary, we get that $\angle \lim_{z \rightarrow \xi} w(z) = 0$.

By another theorem of Privalov (see [21, Chapter XIV, Theorem (1.1) and Theorem (1.9)], [6, Chapter VI, Theorem 2.3], or [5, Chapter II, Exercise 10], where it is called ‘‘local Fatou theorem’’), it follows, since $w \not\equiv 0$, that $m[\mathcal{AD}(\varphi)] = 0$.

3) \iff 4) follows from the Julia-Caratheodory theorem, as stated in (3.2). \square

Remark 1. The implication 2) \Rightarrow 3) can be proved using the classical F. and M. Riesz theorem (see [3, Theorem 2.2]) instead of Privalov’s theorem.

Remark 2. Condition (3.11) is necessary for the compactness of $M_w C_\varphi$; however, it is not sufficient in general without this assumption that $w \in H^\infty$. An example is given in [2, Section 5, Corollary 4] for which $M_w C_\varphi$, with $w = \varphi'$, is not even bounded on \mathfrak{B}^2 .

4 Decompactification

4.1 The main result

In the sequel, as usual, $\alpha > -1$.

Definition 4.1. We say that the composition operator $C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is decompactifiable if there exists a weight $w \in \mathfrak{B}^2$ such that the weighted composition operator $M_w C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is bounded but not compact.

Our main result is the following.

Theorem 4.2. Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map. Then the composition operator $C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is decompactifiable if and only if $\|\varphi\|_\infty = 1$.

It is a consequence of this other theorem, whose proof is postponed.

Theorem 4.3. Let $\gamma > 1$ and ν be a vanishing γ -Carleson measure on \mathbb{D} . Assume that ν satisfies the following property:

$$(4.1) \quad \forall t > 0, \quad \exists \zeta \in \partial\mathbb{D} \quad \text{such that } \nu[S(\zeta, t)] > 0.$$

Then there exists a holomorphic function $u: \mathbb{D} \rightarrow \mathbb{C}$ such that

- (i) $u \in L^2(\nu)$;
- (ii) $\sup_{|\xi|=1, 0 < h \leq 1} \frac{1}{h^\gamma} \int_{S(\xi, h)} |u|^2 d\nu < \infty$;
- (iii) there exist $\delta > 0$ and two sequences (ζ_n) in $\partial\mathbb{D}$ and (t_n) in $(0, 1)$ with $t_n \xrightarrow[n \rightarrow \infty]{} 0^+$ such that

$$(4.2) \quad \frac{1}{t_n^\gamma} \int_{S(\zeta_n, t_n)} |u|^2 d\nu \geq \delta, \quad \text{for all } n \geq 1.$$

Proof of Theorem 4.2. It is plain that if $\|\varphi\|_\infty < 1$, then $M_w C_\varphi$ is compact for every weight $w \in \mathfrak{B}_\alpha^2$. In fact, if $\mu_{w, \varphi}$ is the measure defined in (2.3), then $\mu_{w, \varphi}[S(\xi, h)] = 0$ for $0 < h < 1 - \|\varphi\|_\infty$; hence Corollary 2.2 gives the result.

Conversely, assume that $\|\varphi\|_\infty = 1$.

Note that if $C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is not compact, it suffices to take $w = \mathbb{1}$; so we assume that C_φ is compact. Then $\nu := (A_\alpha)_\varphi = \varphi(dA_\alpha)$ is a vanishing $(\alpha + 2)$ -Carleson measure.

Since $\|\varphi\|_\infty = 1$, condition (4.1) is satisfied. Set $\gamma = \alpha + 2$ and u be the holomorphic function given by Theorem 4.3 and set $w = u \circ \varphi$. We have

$$\int_{\mathbb{D}} |w|^2 dA_\alpha = \int_{\mathbb{D}} |u \circ \varphi|^2 dA_\alpha = \int_{\mathbb{D}} |u|^2 d\nu < \infty;$$

so $w \in \mathfrak{B}_\alpha^2$.

Now, for every $\xi \in \partial\mathbb{D}$ and $h \in [0, 1)$, we have, with $\mu = \varphi(|w|^2 dA_\alpha)$:

$$\begin{aligned}\mu[S(\xi, h)] &= \int_{\varphi^{-1}[S(\xi, h)]} |w|^2 dA_\alpha = \int_{\mathbb{D}} (\mathbb{1}_{S(\xi, h)} \circ \varphi) |u \circ \varphi|^2 dA_\alpha \\ &= \int_{\mathbb{D}} \mathbb{1}_{S(\xi, h)} |u|^2 d\nu.\end{aligned}$$

Hence the properties (ii) and (iii) of Theorem 4.3 show that μ is a non-vanishing $(\alpha+2)$ -Carleson measure, and therefore that $M_w C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is bounded but not compact. \square

4.2 Proof of Theorem 4.3

To prove Theorem 4.3, we need several auxiliary results.

Lemma 4.4. *For every $\omega \in \partial\mathbb{D}$ and $r \in (0, 1)$, there exists a bounded analytic function $F \in H^\infty$ such that, for all $z \in \mathbb{D}$:*

- a) $\Re F(z) > 0$;
- b) $1/2 \leq |F(z)| \leq 2$;
- c) $|F(z)| < 1$ when $|z - \omega| > r$;
- d) $|F(z)| > 1$ when $|z - \omega| < r$.

Proof. By composing with a rotation, we can, and do, assume that $\omega = 1$.

Let $C = 1 - r$ and let A and B be the points of the intersection of the unit circle $\mathbb{T} = \partial\mathbb{D}$ with the circle of center 1 and radius r , with $\Im A > 0$ and $\Im B < 0$. Consider the Möbius transformation T sending A to 0, C to 1, and B to ∞ . The images by T of $\partial\mathbb{D}$ and $\partial D(1, r)$ are straight lines passing through 0. In fact the image of $\partial D(1, r)$ is the extended real line $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. Moreover $T[D(1, r)]$ is the open upper half-plane.

Define $g(z) = \sqrt{T(z)}$, where $\sqrt{\cdot}$ is the principal branch of the square root. Then, for $z \in \mathbb{D}$:

$$\begin{cases} \arg[g(z)] \in (0, \pi/2) & \text{if } z \in D(1, r), \\ \arg[g(z)] \in (-\pi/2, 0) & \text{if } z \in \mathbb{D} \setminus \overline{D(1, r)}. \end{cases}$$

Let now U be the Möbius transformation sending 0 to $i/2$, ∞ to $-i/2$, and 1 to 0. We have

- $|U[g(z)]| < 1/2$ for all $z \in \mathbb{D}$;
- $\Re U[g(z)] > 0$ for all $z \in \mathbb{D} \cap D(1, r) = S(1, r)$;
- $\Re U[g(z)] < 0$ for all $z \in \mathbb{D} \setminus \overline{D(1, r)}$.

Finally, the function F defined as $F(z) = \exp U[g(z)]$ suits. \square

Lemma 4.5. *Let $\gamma \geq 1$, ν be a γ -Carleson measure on \mathbb{D} and $F \in H^\infty$ such that $1/2 \leq |F(z)| \leq 2$ for all $z \in \mathbb{D}$. For given $\beta \in (0, 1]$, we define the function $\Phi: \mathbb{R}_+^* \rightarrow \mathbb{R}_+$ as:*

$$\Phi(\delta) = \sup_{|\xi|=1; 0 < h \leq \beta} \frac{1}{h^\gamma} \int_{S(\xi, h)} |F|^{2\delta} d\nu, \quad \text{for all } \delta > 0.$$

Then Φ is continuous.

Proof. First, we have $\Phi(\delta) < +\infty$ for all $\delta > 0$ because ν is a γ -Carleson measure; indeed, for all $\xi \in \partial\mathbb{D}$ and all $h \in (0, 1]$:

$$\frac{1}{h^\gamma} \int_{S(\xi, h)} |F|^{2\delta} d\nu \leq 4^\delta \frac{\nu[S(\xi, h)]}{h^\gamma} \leq C 4^\delta < +\infty.$$

Now, observe that, since $1/2 \leq |F(z)| \leq 2$, we have, for all $h \in (0, 1]$, all $\xi \in \partial\mathbb{D}$, and all $t \in \mathbb{R}$:

$$\frac{1}{4^{|t|}} \frac{1}{h^\gamma} \int_{S(\xi, h)} |F|^{2\delta} d\nu \leq \frac{1}{h^\gamma} \int_{S(\xi, h)} |F|^{2\delta+2t} d\nu \leq 4^{|t|} \frac{1}{h^\gamma} \int_{S(\xi, h)} |F|^{2\delta} d\nu.$$

Taking the supremum, we get

$$4^{-|t|} \Phi(\delta) \leq \Phi(\delta + t) \leq 4^{|t|} \Phi(\delta),$$

and that proves the continuity of Φ , since $\Phi(\delta) < +\infty$. \square

Proposition 4.6. *Let ν be a finite γ -Carleson measure on \mathbb{D} with property (4.1). Then, for every $\beta \in (0, 1]$ and every $\varepsilon \in (0, 1)$, there exists a function $v \in H^\infty$ satisfying:*

- (a) $|v(z)| < \varepsilon$ for all $z \in \mathbb{D}$ such that $|z| < 1 - \beta$;
- (b) $\frac{1}{h^\gamma} \int_{S(\xi, h)} |v|^2 d\nu \leq 1$ for all $h \in (0, 1]$ and all $\xi \in \partial\mathbb{D}$;
- (c) $\frac{1}{h^\gamma} \int_{S(\xi, h)} |v|^2 d\nu \leq \varepsilon^2$ for all $h \in (\beta, 1]$ and all $\xi \in \partial\mathbb{D}$;
- (d) there exists $t \in (0, \beta]$ and $\zeta \in \partial\mathbb{D}$ such that

$$\frac{1}{t^\gamma} \int_{S(\zeta, t)} |v|^2 d\nu \geq \left(\frac{3}{4}\right)^2.$$

Proof. Since ν is a γ -Carleson measure, there exists a positive constant C (and we can and do assume that $C \geq 1$) such that:

$$(4.3) \quad \nu[S(\xi, h)] \leq C h^\gamma, \quad \forall h \in (0, 1], \quad \forall \xi \in \partial\mathbb{D}.$$

Take $r = \beta(\varepsilon^2/2)^{1/\gamma}$.

By (4.1), there exists $\omega \in \partial\mathbb{D}$ such that

$$\nu[S(\omega, r)] > 0.$$

Let F be the function given by Lemma 4.4.

We define:

$$\Phi(\delta) = \sup_{0 < h \leq \beta, |\xi|=1} \frac{1}{h^\gamma} \int_{S(\xi, h)} |F|^{2\delta} d\nu.$$

Thanks to (4.3), we have, for all $\xi \in \partial\mathbb{D}$ and all $h \in (0, \beta]$:

$$\frac{1}{h^\gamma} \int_{S(\xi, h)} |F|^2 d\nu \leq 4 \frac{\nu[S(\xi, h)]}{h^\gamma} \leq 4C,$$

and we get $\Phi(1) \leq 4C$.

On the other hand, for all $\delta > 0$:

$$\Phi(\delta) \geq \frac{1}{r^\gamma} \int_{S(\omega, r)} |F|^{2\delta} d\nu.$$

Since $|F(z)| > 1$ for $z \in S(\omega, r)$ and $\nu[S(\omega, r)] > 0$, we get

$$\lim_{\delta \rightarrow +\infty} \int_{S(\omega, r)} |F|^{2\delta} d\nu = +\infty,$$

and consequently $\lim_{\delta \rightarrow +\infty} \Phi(\delta) = +\infty$. Since $\Phi(1) \leq 4C < (2C/\varepsilon)^2$ and, thanks to Lemma 4.5, Φ is continuous, there exists $\delta_0 > 1$ such that $\Phi(\delta_0) = (2C/\varepsilon)^2$.

Define

$$v = (\varepsilon/2C) F^{\delta_0}.$$

Observe that $r < \beta$; so $|z| < 1 - \beta$ implies $|z| < 1 - r$; hence $z \notin \overline{S(\omega, r)}$ and $|F(z)| < 1$. That means that $|v(z)| < \varepsilon/(2C) < \varepsilon$, and we have proved (a).

By definition of δ_0 , (b) is satisfied for all $h \in (0, \beta]$. It will be satisfied as well for $h \in (\beta, 1]$ once we have proved (c).

Let us prove (c). Take $\beta < h \leq 1$. Since $|F(z)| \leq 1$ for $z \in S(\xi, h) \setminus S(\omega, r)$, we have:

$$\begin{aligned} \int_{S(\xi, h)} |v|^2 d\nu &\leq \int_{S(\xi, h) \setminus S(\omega, r)} \left(\frac{\varepsilon}{2C}\right)^2 d\nu + \left(\frac{\varepsilon}{2C}\right)^2 \int_{S(\omega, r)} |F|^{2\delta_0} d\nu \\ &\leq \left(\frac{\varepsilon}{2C}\right)^2 \nu[S(\xi, h)] + \left(\frac{\varepsilon}{2C}\right)^2 r^\gamma \Phi(\delta_0) \\ &\leq \left(\frac{\varepsilon}{2C}\right)^2 C h^\gamma + r^\gamma = \left(\frac{\varepsilon}{2C}\right)^2 C h^\gamma + \beta^\gamma \frac{\varepsilon^2}{2} \\ &\leq \left(\frac{\varepsilon}{2C}\right)^2 C h^\gamma + h^\gamma \frac{\varepsilon^2}{2} \leq \left(\frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{2}\right) h^\gamma \leq \varepsilon^2 h^\gamma. \end{aligned}$$

Finally, by definition of δ_0 , there exist $t \in (0, \beta]$ and $\zeta \in \partial\mathbb{D}$ such that

$$\frac{1}{t^\gamma} \int_{S(\zeta, t)} |v|^2 d\nu = \left(\frac{\varepsilon}{2C}\right)^2 \frac{1}{t^\gamma} \int_{S(\zeta, t)} |F|^{2\delta_0} d\nu \geq \left(\frac{3}{4}\right)^2,$$

and (d) is proved. \square

Proof of Theorem 4.3. Consider a sequence $(\varepsilon_n)_{n \geq 1}$ of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < 1/4$.

Using Proposition 4.6, we are going to construct by induction four sequences $(v_n)_n$ in H^∞ , $(\beta_n)_n$, with $\beta_1 = 1$, and $(t_n)_n$ in $(0, 1]$, and $(\zeta_n)_n$ in $\partial\mathbb{D}$ such that, for all $n \geq 1$:

- (S1) $\beta_n \geq t_n > \beta_{n+1} \geq t_{n+1}$;
- (S2) $|v_n(z)| < \varepsilon_n$ for $|z| < 1 - \beta_n$;
- (S3) for all $\xi \in \partial\mathbb{D}$ and all $h \in (0, \beta_{n+1}] \cup [\beta_n, 1]$:

$$\frac{1}{h^\gamma} \int_{S(\xi, h)} |v_n|^2 d\nu \leq \varepsilon_n^2;$$

- (S4) $\frac{1}{h^\gamma} \int_{S(\xi, h)} |v_n|^2 d\nu \leq 1$ for all $h \in (\beta_{n+1}, \beta_n]$ and all $\xi \in \partial\mathbb{D}$;

- (S5) $\frac{1}{t_n^\gamma} \int_{S(\zeta_n, t_n)} |v_n|^2 d\nu \geq \left(\frac{3}{4}\right)^2$,

and

- (S6) $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} t_n = 0$.

Take $\beta_1 = 1$. With $\beta = \beta_1$ and $\varepsilon = \varepsilon_1$, let $v_1 = v$ be the function given by Proposition 4.6 and $\zeta_1 = \zeta$ and $t_1 = t \leq \beta_1$ the numbers given by part (d) of that proposition. By Proposition 4.6 (b) and (d) respectively, conditions (S4) and (S5) are satisfied for $n = 1$. Condition (S2) is void for $n = 1$. For condition (S3), note that since ν is a vanishing γ -Carleson measure, there exists $\beta_2 > 0$ such that

$$\frac{\nu[S(\xi, h)]}{h^\gamma} \leq \varepsilon_1^2 (1 + \|v_1\|_\infty^2)^{-1}$$

for all $h \in (0, \beta_2]$ and all $\xi \in \partial\mathbb{D}$. This implies, for these h 's and ξ 's:

$$\frac{1}{h^\gamma} \int_{S(\xi, h)} |v_1|^2 d\nu \leq \frac{\|v_1\|_\infty^2 \nu[S(\xi, h)]}{h^\gamma} \leq \varepsilon_1^2.$$

It follows, with (c) of Proposition 4.6, that (S3) is satisfied for $n = 1$.

We can of course ask that $\beta_2 \leq 1/2$.

Now, assume that v_1, \dots, v_{n+1} , $\beta_1, \dots, \beta_{n+1}$, t_1, \dots, t_{n+1} and $\zeta_1, \dots, \zeta_{n+1}$ satisfying (S1), (S2), (S3), (S4) and (S5) have been constructed.

As above, since ν is a vanishing γ -Carleson measure, there exists a positive number $\beta_{n+2} \leq \min(\beta_{n+1}, 1/(n+2))$ such that

$$\frac{\nu[S(\xi, h)]}{h^\gamma} \leq \varepsilon_n^2 (1 + \|v_{n+1}\|_\infty^2)^{-1}$$

for all $h \in (0, \beta_{n+2}]$ and all $\xi \in \partial\mathbb{D}$. Using Proposition 4.6 with $\beta = \beta_{n+2}$ and $\varepsilon = \varepsilon_{n+1}$, we get $v_{n+2} = v \in H^\infty$, $\zeta_{n+2} = \zeta \in \partial\mathbb{D}$ and $t_{n+2} = t \in (0, \beta_{n+2}]$ and the induction step follows.

We now set:

$$u(z) = \sum_{n=1}^{\infty} v_n(z), \quad z \in \mathbb{D}.$$

Thanks to (S2), this series converges uniformly on compact subsets of \mathbb{D} , so u is analytic in \mathbb{D} .

Take $\xi \in \partial\mathbb{D}$ and $h \in (0, 1]$. There exists a unique $n \geq 1$ such that $h \in (\beta_{n+1}, \beta_n]$. By the triangle inequality:

$$\begin{aligned} \left(\frac{1}{h^\gamma} \int_{S(\xi, h)} |u|^2 d\nu \right)^{1/2} &\leq \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \left(\frac{1}{h^\gamma} \int_{S(\xi, h)} |v_k|^2 d\nu \right)^{1/2} \\ &\quad + \left(\frac{1}{h^\gamma} \int_{S(\xi, h)} |v_n|^2 d\nu \right)^{1/2}. \end{aligned}$$

Now, since:

$$(4.4) \quad h \in (0, \beta_{k+1}] \cup (\beta_k, 1] \quad \text{for every } k \neq n,$$

we get, by (S3) and (S4):

$$\left(\frac{1}{h^\gamma} \int_{S(\xi, h)} |u|^2 d\nu \right)^{1/2} \leq \left(\sum_{k \neq n} \varepsilon_k \right) + 1 \leq \frac{1}{4} + 1 = \frac{5}{4}.$$

Consequently, u satisfies (ii) of Theorem 4.3. Then condition (ii) implies (i) because ν is a finite measure, u is bounded on $(1/2)\mathbb{D}$, and $\mathbb{D} \setminus (1/2)\mathbb{D}$ can be covered by a finite number of boxes $S(\xi, 1)$, with $\xi \in \partial\mathbb{D}$.

To obtain (iii), we use (4.4) again, with $h = t_n$, to get:

$$\begin{aligned} \left(\frac{1}{t_n^\gamma} \int_{S(\zeta_n, t_n)} |u|^2 d\nu \right)^{1/2} &\geq \left(\frac{1}{t_n^\gamma} \int_{S(\zeta_n, t_n)} |v_n|^2 d\nu \right)^{1/2} \\ &\quad - \sum_{k \neq n} \left(\frac{1}{t_n^\gamma} \int_{S(\zeta_n, t_n)} |v_k|^2 d\nu \right)^{1/2} \\ &\geq \frac{3}{4} - \sum_{k \neq n} \varepsilon_k \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \end{aligned}$$

and we have (iii). □

5 Hilbert-Schmidt regularization

We remarked in Section 3 that if $\varphi(z) = \frac{1+z}{2}$, then C_φ is compactifiable on \mathfrak{B}_α^2 by a weight in H^∞ . Actually, since $|\varphi(e^{it})| = \cos(t/2)$, we have $\int_{-\pi}^{\pi} \log \frac{1}{1-|\varphi(e^{it})|} dm(t) < \infty$, and [12, Theorem 4.1] tells that the composition

operator C_φ can be weighted to have a Hilbert-Schmidt operator on H^2 ; a fortiori, this weighted composition operator is Hilbert-Schmidt on \mathfrak{B}_α^2 (see [13, Theorem 3.12]). We can be more specific on an example, but unfortunately this example shows no difference between the Hardy and Bergman spaces.

Proposition 5.1. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be defined by $\varphi(z) = \frac{1+z}{2}$, and let $w(z) = (1-z)^\beta$ with $\beta > -1/2$, so that $w \in H^2$. Then the weighted composition operators $M_w C_\varphi: H^2 \rightarrow H^2$ and $M_w C_\varphi: \mathfrak{B}^2 \rightarrow \mathfrak{B}^2$ are Hilbert-Schmidt if and only if $\beta > 1/2$.*

Proof. The first item was proved in [9, Proposition 2.4]. For the second item, we have to determine those β such that

$$\int_{\mathbb{D}} \frac{|w(z)|^2}{(1-|\varphi(z)|^2)^2} dA(z) < \infty.$$

Since $|\varphi(z)|$ approaches 1 only when z approaches 1, we can as well consider

$$I := \int_{\Delta} \frac{|w(z)|^2}{(1-|\varphi(z)|^2)^2} dA(z),$$

where $\Delta = \mathbb{D} \cap D(1, 1)$. Passing in polar coordinates centered at 1, we write, for $z \in \Delta$: $z = 1 - r e^{i\theta}$ with $|\theta| < \pi/2$ and $r < 2 \cos \theta$. Then, $|\frac{1+z}{2}|^2 = 1 + \frac{r^2}{4} - r \cos \theta$ and

$$I = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \frac{r^{2\beta+1}}{r^2(\cos \theta - r/4)^2} dr \frac{d\theta}{\pi} = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{2 \cos \theta} \frac{r^{2\beta-1}}{(\cos \theta - r/4)^2} dr d\theta.$$

Making the change of variable $r = 2t \cos \theta$, $0 \leq t \leq 1$ in the inner integral and observing that $1 \geq 1 - t/2 \geq 1/2$, we see that

$$I \approx \int_0^{\pi/2} \int_0^1 \frac{t^{2\beta-1} (\cos \theta)^{2\beta}}{\cos^2 \theta} dt d\theta = \left(\int_0^1 t^{2\beta-1} dt \right) \left(\int_0^{\pi/2} (\sin \theta)^{2\beta-2} d\theta \right).$$

So, clearly, $I < \infty$ if and only if $\beta > 1/2$. \square

Fortunately, examples showing the difference between Hardy and Bergman spaces exist.

Theorem 5.2. *There exists a Blaschke product B which can be Hilbert-Schmidt regularized, and more, on \mathfrak{B}^2 , but not on H^2 .*

Proof. Any Blaschke product B is an inner function, i.e. $|B^*| = 1$ m -almost everywhere on the unit circle, implying, by [12, Theorem 3.1], that $M_w C_B$ is compact on H^2 for no weight $w \in H^2$, with $w \neq 0$.

On the other hand, as a consequence of [10, Theorem 3.1], we proved ([11, Theorem 4.4]; see also [14, Theorem 13]) that there exist Blaschke products B (which we called *slow Blaschke products*) such that C_B is compact on the Bergman-Orlicz space \mathfrak{B}^{Ψ_2} , and hence belong to every Schatten class S_p of \mathfrak{B}^2 . \square

Moreover, we can give the following quantitative precision to Theorem 5.2.

Theorem 5.3. *For any sequence (ε_n) of positive numbers with limit zero, there is a Blaschke product B such that*

$$a_n(C_B: \mathfrak{B}^2 \rightarrow \mathfrak{B}^2) \lesssim e^{-n\varepsilon_n}.$$

Proof. We can assume that ε_n decreases and that $n\varepsilon_n \uparrow \infty$ with $n\varepsilon_n \geq \sqrt{n}$.

For a given symbol φ , we set

$$\chi(h) = A(\{z; |\varphi(z)| \geq 1 - h\}).$$

We use [15, Theorem 5.1] which implies that

$$(5.1) \quad a_n(C_\varphi) \lesssim \inf_{0 < h < 1} \left[\sqrt{n} e^{-nh} + \sqrt{\frac{\chi(h)}{h^2}} \right].$$

Let $\delta: (0, 1) \rightarrow (0, 1)$ be a non-increasing and piecewise linear map, decreasing to 0 so slowly at the origin that

$$\delta(1 - |z|) \leq 4\varepsilon_n \quad \implies \quad 1 - |z| \leq \varepsilon_n^2 \exp(-2n\varepsilon_n).$$

By [10, Theorem 3.1] again, there exists a Blaschke product B such that $|B(z)| \leq \exp(-\delta(1 - |z|))$. Take $\varphi = B$ in (5.1) and observe that, for $h = 2\varepsilon_n \leq 1/2$, we have

$$|B(z)| \geq 1 - h \quad \implies \quad \exp(-\delta(1 - |z|)) \geq 1 - h \geq \exp(-2h).$$

Hence

$$\delta(1 - |z|) \leq 4\varepsilon_n \quad \text{and} \quad 1 - |z| \leq \varepsilon_n^2 \exp(-2n\varepsilon_n)$$

and

$$\chi(h) \leq 2\varepsilon_n^2 \exp(-2n\varepsilon_n).$$

Inserting this in (5.1), we get the result. \square

In order to find a necessary and sufficient condition for a symbol can be weighted in a Hilbert-Schmidt operator, we make some observations.

As recalled, a weighted composition operator which is Hilbert-Schmidt on H^2 is also Hilbert-Schmidt on \mathfrak{B}^2 . We know that C_φ is Hilbert-Schmidt on H^2 if and only if

$$\int_{\mathbb{T}} \frac{1}{1 - |\varphi|^2} dm < \infty.$$

Equivalently:

$$(5.2) \quad \sum_{n=0}^{\infty} \|\varphi^n\|_{H^2}^2 < \infty.$$

On the other hand, C_φ can be weighted to become a Hilbert-Schmidt operator on H^2 if and only if

$$\int_{\mathbb{T}} \log \frac{1}{1-|\varphi|} dm < \infty,$$

([12, Theorem 4.1]), which is equivalent to

$$(5.3) \quad \sum_{n=0}^{\infty} \frac{1}{n+1} \|C_\varphi(e_n)\|_{H^2}^2 < \infty.$$

Now, writing $e_n(z) = z^n$, and since $((n+1)^{(\alpha+1)/2} e_n)_n$ is an orthonormal basis of \mathfrak{B}_α^2 , C_φ is Hilbert-Schmidt on \mathfrak{B}_α^2 if and only if

$$\sum_{n=0}^{\infty} (n+1)^{\alpha+1} \|\varphi^n\|_{\mathfrak{B}_\alpha^2}^2 = \sum_{n=0}^{\infty} \|C_\varphi((n+1)^{(\alpha+1)/2} e_n)\|_{\mathfrak{B}_\alpha^2}^2 < \infty.$$

By comparison with (5.3), we might think that C_φ can be weighted to become a Hilbert-Schmidt operator on \mathfrak{B}_α^2 if and only if

$$(5.4) \quad \sum_{n=0}^{\infty} (n+1)^\alpha \|\varphi^n\|_{\mathfrak{B}_\alpha^2}^2 < \infty.$$

Since

$$(5.5) \quad \sum_{n=0}^{\infty} (n+1)^\alpha \|\varphi^n\|_{\mathfrak{B}_\alpha^2}^2 = \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+1}} dA(z),$$

this guesswork takes the following form: is it true that there exists a weight w such that $M_w C_\varphi: \mathfrak{B}_\alpha^2 \rightarrow \mathfrak{B}_\alpha^2$ is Hilbert-Schmidt if and only if

$$(5.6) \quad \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+1}} dA(z) < \infty \quad ?$$

We do not know if (5.6) implies the existence of a weight $w \not\equiv 0$ for which $M_w C_\varphi$ is Hilbert-Schmidt, but, in any case, it implies that C_φ is compactifiable.

Proposition 5.4. *If, for $\alpha > -1$, we have $\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+1}} dA(z) < \infty$, then*

$$\lim_{z \rightarrow \xi} \frac{1-|z|}{1-|\varphi(z)|} = 0 \quad \text{for almost all } \xi \in \mathbb{T}.$$

Recall that, by Theorem 3.3, this last condition means that C_φ is compactifiable on \mathfrak{B}_α^2 .

Proof. We set:

$$g(z) = \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^{\alpha+1}.$$

Let $r_n = 1 - 2^{-n}$ and:

$$\Gamma_n = \{z \in \mathbb{D}; r_n \leq |z| < r_{n+1}\}.$$

Now, $1/(1-|\varphi|^2)^{\alpha+1}$ is subharmonic (and even logarithmically-subharmonic), because we can write $1/(1-|\varphi|^2)^{\alpha+1} = \sum_{k=0}^{\infty} c_k(\alpha)|\varphi|^{2k}$ with $c_k(\alpha) \geq 0$. Hence, we have, since $A(\Gamma_n) \approx 1 - r_n^2$:

$$\begin{aligned} \int_{\mathbb{T}} g(r_n e^{i\theta}) d\theta &= \int_{\mathbb{T}} \frac{(1 - r_n^2)^{\alpha+1}}{(1 - |\varphi(r_n e^{i\theta})|^2)^{\alpha+1}} d\theta \\ &\lesssim \frac{1}{1 - r_n^2} \int_{\Gamma_n} \frac{(1 - r_n^2)^{\alpha+1}}{(1 - |\varphi(z)|^2)^{\alpha+1}} dA(z) \\ &\lesssim \int_{\Gamma_n} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\alpha+1}} dA(z) \end{aligned}$$

(we used that $1 - r_n^2 \leq 2(1 - r_n) = 2 \times 2^{-n} = 4(1 - r_{n+1}) \leq 4(1 - |z|) \leq 4(1 - |z|^2)$ for $z \in \Gamma_n$, so $(1 - r_n^2)^\alpha \lesssim (1 - |z|^2)^\alpha$ when $\alpha \geq 0$, and, when $-1 < \alpha < 0$, we used that $1 - |z|^2 \leq 1 - r_n^2$ for $z \in \Gamma_n$). The sets Γ_n being disjoint, we get that:

$$\begin{aligned} \int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} g(r_n e^{i\theta}) \right) d\theta &= \sum_{n=0}^{\infty} \int_{\mathbb{T}} g(r_n e^{i\theta}) d\theta \\ &\lesssim \sum_{n=0}^{\infty} \int_{\Gamma_n} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\alpha+1}} dA(z) \\ &= \int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\alpha+1}} dA(z) < \infty, \end{aligned}$$

meaning that the function $\sum_{n=0}^{\infty} g(r_n \cdot)$ is integrable on \mathbb{T} . It follows that $g(r_n \cdot) \xrightarrow[n \rightarrow \infty]{} 0$ almost everywhere. Since the existence of a radial limit implies that of an angular limit, we obtain that $\angle \lim_{z \rightarrow \xi} \frac{1 - |z|}{1 - |\varphi(z)|} = 0$ for almost all $\xi \in \mathbb{T}$. By the Julia-Caratheodory theorem, it follows that $\lim_{z \rightarrow \xi} \frac{1 - |z|}{1 - |\varphi(z)|} = 0$ for almost all $\xi \in \mathbb{T}$. \square

An a priori different condition than (5.6) appears in the following theorem.

Theorem 5.5. *Let $d\lambda_\alpha(r) = 2(\alpha + 1)(1 - r^2)^\alpha r dr$, be the marginal probability measure on $[0, 1)$ of dA_α , and*

$$(5.7) \quad G(\theta) = \int_0^1 \frac{d\lambda_\alpha(r)}{(1 - |\varphi(re^{i\theta})|^2)^{\alpha+2}}.$$

Then:

1) *If $\log G \in L^1(0, 2\pi)$, then there exists $w \in H^\infty$, $w \not\equiv 0$, such that $M_w C_\varphi$ is Hilbert-Schmidt on \mathfrak{B}_α^2 .*

2) *Conversely, if there exists such a weight w , then $\log G \in L^{1,\infty}(0, 2\pi)$.*

Note that $G \geq 1$, so $\log G \geq 0$.

Recall that $L^{1,\infty}(\mu)$ is the space of (classes of) measurable functions f such that $\sup_{a>0} a m(\{|f| > a\}) < \infty$, and that $L^1(\mu) \subseteq L^{1,\infty}(\mu)$, by Markov's inequality.

5.1 Proof of 1) of Theorem 5.5

For convenience, we set

$$(5.8) \quad U(z) = \frac{1}{(1 - |\varphi(z)|^2)^{\alpha+2}}.$$

We will use two lemmas. For that, we denote ρ the pseudo-hyperbolic metric on \mathbb{D} . Recall that

$$\rho(u, v) = \left| \frac{u - v}{1 - \bar{u}v} \right|, \quad u, v \in \mathbb{D}.$$

Lemma 5.6. *There is a positive constant $C = C(\alpha)$ such that, for $u, v \in \mathbb{D}$:*

$$(5.9) \quad \rho(u, v) \leq \frac{1}{2} \implies \frac{1}{C} \leq \frac{U(u)}{U(v)} \leq C.$$

Proof. Since

$$\left(\frac{1}{2} \times \frac{1 - |\varphi(v)|}{1 - |\varphi(u)|} \right)^{\alpha+2} \leq \frac{U(u)}{U(v)} \leq \left(2 \times \frac{1 - |\varphi(v)|}{1 - |\varphi(u)|} \right)^{\alpha+2},$$

it suffices to show that there is a positive constant such that

$$\frac{1}{C} \leq \frac{1 - |\varphi(v)|}{1 - |\varphi(u)|} \leq C$$

when $\rho(u, v) \leq 1/2$. Moreover, by the Schwarz-Pick inequality, we have:

$$\rho(|\varphi(u)|, |\varphi(v)|) \leq \rho(\varphi(u), \varphi(v)) \leq \rho(u, v),$$

it suffices to majorize $q := \frac{1-a}{1-b}$ when $\rho(a, b) \leq 1/2$ and $0 \leq a, b < 1$ (the minoration will come by exchanging a and b).

If $a \geq b$, then $q \leq 1$

If $a < b$, we remark that $\rho(a, b) \leq 1/2$ writes $T_a(b) := \frac{a-b}{1-ab} \geq -1/2$. Since T_a is decreasing on $[-1, 1]$, we get $b \leq T_a(-1/2)$, i.e. $b \leq \frac{1+2a}{2+a}$, and $1-b \geq \frac{1-a}{2+a}$. Therefore $q \leq 2+a \leq 3$. \square

Let, for $n \geq 0$:

$$r_n = \exp(-2^{-n})$$

and

$$\Gamma_n = \{z \in \mathbb{D}; r_n \leq |z| < r_{n+1}\}.$$

Lemma 5.7. For $r_n \leq u, v \leq r_{n+1}$, we have $\rho(u e^{i\theta}, v e^{i\theta}) \leq 1/2$, for every $\theta \in \mathbb{R}$.

Proof. It is a simple computation:

$$\rho(u e^{i\theta}, v e^{i\theta}) = \frac{|u - v|}{1 - uv} \leq \frac{r_{n+1} - r_n}{1 - r_{n+1}^2} = \frac{r_{n+1} - r_{n+1}^2}{1 - r_{n+1}^2} = \frac{r_{n+1}}{1 + r_{n+1}} \leq \frac{1}{2}. \quad \square$$

Now, we can finish the proof of 1) of Theorem 5.5.

We have to show that there exists a non-null function $w_0 \in H^\infty$ such that

$$(5.10) \quad \int_{\mathbb{D}} |w_0|^2 U dA_\alpha < \infty,$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$.

For every $w \in H^\infty$, we have:

$$\int_{\mathbb{D}} |w|^2 U dA_\alpha = \int_{D(0, e^{-1})} |w|^2 U dA_\alpha + \sum_{n=0}^{\infty} \int_{\Gamma_n} |w|^2 U dA_\alpha$$

For every $n \geq 0$:

$$\int_{\Gamma_n} |w|^2 U dA_\alpha = 2(\alpha + 1) \int_{r_n}^{r_{n+1}} \left(\frac{1}{2\pi} \int_0^{2\pi} |w(re^{i\theta})|^2 U(re^{i\theta}) d\theta \right) (1 - r^2)^\alpha r dr$$

As said in the proof of Proposition 5.4, U is logarithmically-subharmonic; hence the function $|w|^2 U$ is also logarithmically-subharmonic; in particular, it is subharmonic; so we have (see [3, Theorem 1.6, page 9]), for $r_n \leq r \leq r_{n+1}$:

$$\frac{1}{2\pi} \int_0^{2\pi} |w(re^{i\theta})|^2 U(re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |w(r_{n+1}e^{i\theta})|^2 U(r_{n+1}e^{i\theta}) d\theta.$$

By Lemma 5.6 and Lemma 5.7, we have $U(r_{n+1}e^{i\theta}) \leq C U(r_n e^{i\theta})$. But $r_n = r_{n+1}^2$, so $U(r_{n+1}e^{i\theta}) \leq C U(r_{n+1}^2 e^{i\theta})$, and hence

$$\frac{1}{2\pi} \int_0^{2\pi} |w(re^{i\theta})|^2 U(re^{i\theta}) d\theta \leq C \frac{1}{2\pi} \int_0^{2\pi} |w(r_{n+1}e^{i\theta})|^2 U(r_{n+1}^2 e^{i\theta}) d\theta.$$

By the subharmonicity of $|w|^2 U$ again, we obtain:

$$\frac{1}{2\pi} \int_0^{2\pi} |w(re^{i\theta})|^2 U(re^{i\theta}) d\theta \leq C \frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 U(r_{n+1}e^{i\theta}) d\theta.$$

Using Lemma 5.6 and Lemma 5.7 again, we have, for every $r_n \leq r < r_{n+1}$:

$$\frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 U(r_{n+1}e^{i\theta}) d\theta \leq C \frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 U(r e^{i\theta}) d\theta.$$

Therefore:

$$\int_{\Gamma_n} |w|^2 U dA_\alpha \leq C^2 \int_{r_n}^{r_{n+1}} \left(\frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 U(r e^{i\theta}) d\theta \right) d\lambda_\alpha(r).$$

Using the Fubini theorem, we finally obtain:

$$\begin{aligned} \int_{\mathbb{D}} |w|^2 U dA_\alpha &\leq \int_{D(0, e^{-1})} |w|^2 U dA_\alpha \\ &\quad + C^2 \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{e^{-1}}^1 U(r e^{i\theta}) d\lambda_\alpha(r) \right) |w(e^{i\theta})|^2 d\theta. \end{aligned}$$

Since

$$G(\theta) = \int_0^1 U(r e^{i\theta}) d\lambda_\alpha(r),$$

we have:

$$(5.11) \quad \int_{\mathbb{D}} |w|^2 U dA_\alpha \leq \int_{D(0, e^{-1})} |w|^2 U dA_\alpha + C^2 \frac{1}{2\pi} \int_0^{2\pi} G(\theta) |w(e^{i\theta})|^2 d\theta.$$

We now use Szegő's theorem (see [5, Theorem 3.1, Chapter IV, page 139], or [18, Section 8.3]):

$$\inf_{w \in H^\infty, w(0)=1} \frac{1}{2\pi} \int_0^{2\pi} |w(e^{i\theta})|^2 G(\theta) d\theta = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log G(\theta) d\theta \right).$$

Remarking that the hypothesis of the theorem writes:

$$\int_0^{2\pi} \log G(\theta) d\theta < \infty,$$

that shows that there exists $w_0 \in H^\infty$ with $w_0(0) = 1$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} |w_0(e^{i\theta})|^2 G(\theta) d\theta < \infty.$$

With (5.11), that shows that w_0 satisfies (5.10), and that ends the proof of 1) of Theorem 5.5.

Note that the proof shows that we can actually get a polynomial for w_0 .

5.2 Proof of 2) of Theorem 5.5

We may, and do, assume that $\|w\|_\infty = 1$.

By hypothesis, we have

$$\int_{\mathbb{D}} |w(z)|^2 \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\alpha+2}} dA(z) < \infty.$$

Setting, with U defined in (5.8):

$$(5.12) \quad \psi(\theta) = \int_{1/2}^1 |w(r e^{i\theta})|^2 U(r e^{i\theta}) d\lambda_\alpha(r),$$

we hence have $\psi \in L^1(0, 2\pi)$.

Let

$$\tilde{G}(\theta) = \int_{1/2}^1 U(r e^{i\theta}) d\lambda_\alpha(r)$$

and

$$J(\theta) = \inf\{|w(r e^{i\theta})|^2; 1/2 \leq r < 1\}.$$

We have $\psi(\theta) \geq \tilde{G}(\theta) J(\theta)$, so

$$\log \tilde{G} \leq \log \psi + \log(1/J) \leq \log^+ \psi + \log(1/J) \leq \psi + \log(1/J).$$

Since $U \geq 1$, we have $\tilde{G}(\theta) \geq C_\alpha$, with $C_\alpha = (3/4)^{\alpha+1} > 0$; hence $\log \tilde{G}(\theta) \geq \log C_\alpha > -\infty$. Therefore, to get $\log G \in L^{1,\infty}(0, 2\pi)$ and finish the proof of 2) of Theorem 5.5, it suffices to prove that $\log \tilde{G} \in L^{1,\infty}(0, 2\pi)$, and for that, to prove that $\log(1/J) \in L^{1,\infty}(0, 2\pi)$. This is the object of the following theorem.

Theorem 5.8. *Let $v \in H^\infty$ such that $\|v\|_\infty = 1$ and set*

$$(5.13) \quad I_v(\theta) = \inf\{|v(r e^{i\theta})|; 1/2 \leq r < 1\}.$$

Then $\log(1/I_v) \in L^{1,\infty}(0, 2\pi)$.

Proof. We can write $v(z) = B(z) v_0(z)$, where B is the Blaschke product whose zeros are those of v , and v_0 does not vanish. Since

$$I_v \geq I_B \times I_{v_0},$$

it suffices to prove that $\log(1/I_B) \in L^{1,\infty}(0, 2\pi)$ and $\log(1/I_{v_0}) \in L^{1,\infty}(0, 2\pi)$.

Case of a non vanishing function.

We can write $v_0 = \exp(-h)$, where $h: \mathbb{D} \rightarrow \{\Re z > 0\}$. We have $h = u + i\tilde{u}$, where $u = \Re h$ and \tilde{u} is the conjugate function of u . Since $u > 0$, $u = P[\mu]$ is the Poisson integral of a positive measure μ , and we have

$$u(r e^{i\theta}) \leq C M_\mu(\theta) \quad \forall r \in [0, 1),$$

where M_μ is the Hardy-Littlewood maximal function of μ . Then:

$$|v_0(r e^{i\theta})| \geq \exp(-C M_\mu(\theta));$$

so $I_{v_0}(\theta) \geq \exp(-C M_\mu(\theta))$, and $\log(1/I_{v_0}(\theta)) \leq C M_\mu(\theta)$. Since $M_\mu \in L^{1,\infty}(0, 2\pi)$, by Kolmogorov's theorem, we obtain that $\log(1/I_{v_0}) \in L^{1,\infty}(0, 2\pi)$.

Case of a Blaschke product.

This case will follow from the next result. We note $\arg z$ the principal argument of z : $-\pi < \arg z \leq \pi$.

Proposition 5.9. *Let B_0 be a Blaschke product whose zeros a_n have modulus greater or equal to some positive constant c , say $c = 3/4$. Then there exist $f \in L^1(-\pi, \pi)$ and $u = P[w]$, with $w \in L^1(-\pi, \pi)$, such that*

$$(5.14) \quad \log(1/|B_0(z)|) \leq f(\arg z) + u(z), \quad \text{for all } z \in \mathbb{D}.$$

For $a \in \mathbb{D}$, we denote $\Delta(a, 1/2)$ the pseudo-hyperbolic disk of center a and radius $1/2$.

We begin by two lemmas.

Lemma 5.10. *For $a \in \mathbb{D}$, we set*

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z},$$

as well as $I_a = I_{\varphi_a}$ and $G_a = \log(1/I_a)$. Then, for every $a \in \mathbb{D}$, we have $G_a \in L^1(-\pi, \pi)$.

First, we have $G_a \geq 0$. Then:

$$|\varphi_a(z)| = \frac{|a - z|}{|1 - \bar{a}z|} \geq \frac{|z - a|}{2};$$

so, it suffices to give a lower estimate of $|z - a|$.

We separate two cases.

- First case: $|a| \leq 1/4$. Then we have $|r e^{i\theta} - a| \geq 1/4$ when $1/2 \leq r < 1$; hence $G_a(\theta) \leq \log 8$ for all θ and $G_a \in L^1(-\pi, \pi)$.

- Second case: $|a| > 1/4$. We can assume that $1/4 < a < 1$. If $z = r e^{i\theta}$, then, for $|\theta| \leq \pi/2$:

$$|z - a| \geq \text{dist}(a, R_\theta) = a |\sin \theta|,$$

where R_θ is the ray passing through 0 and $e^{i\theta}$, so

$$G_a(\theta) \leq \left| \log \left(\frac{a}{2} |\sin \theta| \right) \right|$$

and $G_a \in L^1(-\pi, \pi)$. □

Lemma 5.11. *There is a positive constant C such that, for $3/4 \leq a < 1$ and $h = 1 - a$, we have:*

$$(5.15) \quad |\theta| \leq Ch \quad \text{when } z = r e^{i\theta} \in \Delta(a, 1/2).$$

Proof. The pseudo-hyperbolic disk $\Delta(a, 1/2)$ is equal to the Euclidean disk $D(\tilde{a}, R)$, with

$$\tilde{a} = \frac{3}{4 - |a|^2} a \quad \text{and} \quad R = 2 \frac{1 - |a|^2}{4 - |a|^2}$$

(see [5, page 3]). For $0 < a < 1$ and $h = 1 - a$, we have $R \leq 4h/3$ and $3a/4 \leq \tilde{a} \leq a$. Hence $\Delta(a, 1/2)$ is contained in the angular sector of vertex 0 and half-angle θ_a such that $\sin \theta_a = R/\tilde{a} \leq (4h/3)/(3a/4)$ (see Figure 1). For $3/4 \leq a < 1$, that gives $\sin \theta_a \leq (64/27)h$. It follows that there is $C > 0$ ($C = 64\pi/54$ works) such that $|\theta| \leq Ch$ when $z = r e^{i\theta} \in \Delta(a, 1/2)$. \square

Proof of Proposition 5.9. We restrict ourselves, for the time, to $3/4 \leq a < 1$.

Note that, since $3/4 \leq a < 1$ and $a = 1 - h$, we have $0 < h \leq 1/4$.

- Let $z = r e^{i\theta} \in \Delta(a, 1/2)$.

We write

$$|\varphi_a(z)| = \frac{|z - a|}{a \left| z - \frac{1}{a} \right|} \geq \frac{|z - a|}{a \left[|z - a| + \left(\frac{1}{a} - a \right) \right]} = \frac{1}{a + \frac{1 - a^2}{|z - a|}};$$

so, if $z \in \Delta(a, 1/2)$, and $z = r e^{i\theta}$, we have $|\theta| < \pi/2$; hence, when $\theta \neq 0$:

$$|\varphi_a(z)| \geq \frac{1}{a + \frac{1 - a^2}{a |\sin \theta|}}.$$

It follows from Lemma 5.11 that, for another constant C :

$$\frac{1}{I_a(\theta)} \leq a + \frac{1 - a^2}{a |\sin \theta|} \leq C \frac{h}{|\theta|},$$

so

$$G_a(\theta) \leq \log \left(C \frac{h}{|\theta|} \right)$$

and

$$\int_0^{Ch} G_a(\theta) d\theta \leq \left[\theta \log \left(C \frac{h}{\theta} \right) + \theta \right]_0^{Ch} = Ch.$$

Setting, for $|\theta| \leq \pi$ (recall that $a = 1 - h$):

$$(5.16) \quad f_a(\theta) = \log \left(C \frac{h}{|\theta|} \right) \mathbb{1}_{[-Ch, Ch]}(\theta),$$

we hence have:

$$(5.17) \quad \log \left(\frac{1}{|\varphi_a(z)|} \right) \leq f_a(\theta) \quad \text{for } z = r e^{i\theta} \in \Delta(a, 1/2)$$

(since then $|\theta| \leq Ch$). Moreover, we have

$$(5.18) \quad \|f_a\|_1 \leq 2Ch.$$

• Now, let $z \in \mathbb{D} \setminus \Delta(a, 1/2)$.

Let D_a be the (Euclidean) disk of diameter $[c, 1/c]$, where c is the point of the segment $\partial\Delta(a, 1/2) \cap [0, 1)$ such that $0 < c < a$, and

$$(5.19) \quad A_a = \partial\mathbb{D} \cap D_a.$$

We write simply $A_a = A$ thereafter.

We set

$$(5.20) \quad w_a = 2 \log 2 \mathbb{1}_A \quad \text{and} \quad u_a = P[w_a],$$

the Poisson integral of w_a .

We have, for some positive constant C :

$$(5.21) \quad \|w_a\|_1 = 2 \log 2 m(A) \leq Ch.$$

In fact, the diameter of D_a is $\frac{1}{c} - c$ and $\frac{1}{2} = |\varphi_a(c)| = \frac{a-c}{1-ac}$, so

$$c = \frac{2a-1}{2-a} = \frac{1-2h}{1+h} = 1 - 3h + o(h),$$

and the diameter of D_a is equal to $6h + o(h)$.

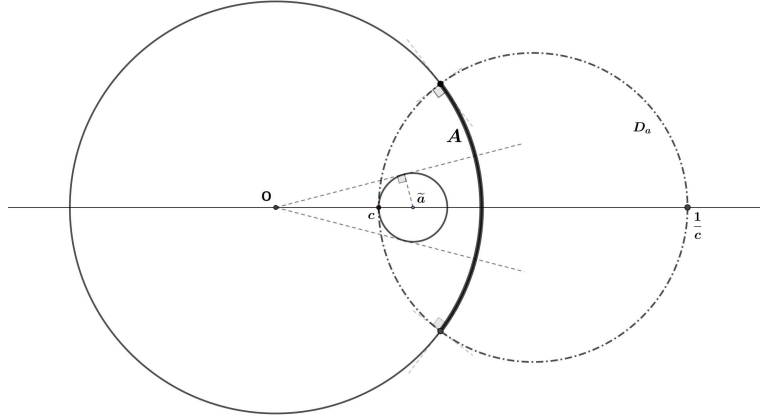


Figure 1: circles

Lemma 5.12. *Let $0 < \theta_0 < \pi/2$ and A be the arc of $\partial\mathbb{D}$ with end points $e^{-i\theta_0}$ and $e^{i\theta_0}$ and midpoint 1. Then, if D_A is the disk orthogonal to $\partial\mathbb{D}$ passing through $e^{-i\theta_0}$ and $e^{i\theta_0}$, we have*

$$P[\mathbb{1}_A] \geq 1/2 \quad \text{on} \quad \mathbb{D} \cap D_A.$$

Proof. Let $T: \overline{\mathbb{D}} \rightarrow \{z \in \mathbb{C}; \Im z \geq 0\} \cup \{\infty\}$ be the conformal map mapping A onto $\mathbb{R}_- \cup \{\infty\}$ and $\partial\mathbb{D} \setminus A$ onto \mathbb{R}_+^* . The unique bounded solution to the

Dirichlet problem with data 1 on $\mathbb{R}_- \cup \{\infty\}$ and 0 on \mathbb{R}_+^* is $U(\zeta) = \frac{1}{\pi} \arg \zeta$; hence $P[\mathbb{1}_A](z) = \frac{1}{\pi} \arg(Tz)$.

We have $U(\zeta) \geq 1/2$ if and only if ζ is in the closed left-hand side upper quadrant Q . But $T^{-1}(i\mathbb{R})$ is the arc orthogonal to $\partial\mathbb{D}$ and passing through $e^{-i\theta_0}$ and $e^{i\theta_0}$; hence $T^{-1}(i\mathbb{R}) = A$ and, since D_A is orthogonal to \mathbb{D} , we have $T(\overline{\mathbb{D} \cap D_A}) = Q \cup \{\infty\}$. Therefore $P[\mathbb{1}_A](z) \geq 1/2$ when $z \in \mathbb{D} \cap D_A$. \square

By this lemma, we have

$$u_a \geq \log 2 \quad \text{on } \mathbb{D} \cap D_a.$$

In particular, since $\Delta(a, 1/2) \subseteq \mathbb{D} \cap D_a$, we have $u_a \geq \log 2$ on $\partial\Delta(a, 1/2)$. Of course u_a is equal to $\mathbb{1}_A$ on $\partial\mathbb{D}$, so is positive on $\partial\mathbb{D}$.

On the other hand, the function $\log(1/|\varphi_a|)$ is harmonic in $\mathbb{D} \setminus \Delta(a, 1/2)$ and is equal to 0 on $\partial\mathbb{D}$ and to $\log 2$ on $\partial\Delta(a, 1/2)$. Therefore, since $\partial\mathbb{D} \cup \partial\Delta(a, 1/2)$ is the boundary of $\mathbb{D} \setminus \Delta(a, 1/2)$, we obtain that

$$(5.22) \quad u_a(z) \geq \log(1/|\varphi_a(z)|) \quad \text{for } z \in \mathbb{D} \setminus \Delta(a, 1/2).$$

- It follows from (5.17) and (5.22) that

$$(5.23) \quad \log(1/|\varphi_a(z)|) \leq f_a(\arg z) + u_a(z) \quad \text{for all } z \in \mathbb{D}.$$

- We are now able to finish the proof.

We write

$$B_0 = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \varphi_{a_n}$$

with $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$ and $|a_n| \geq 3/4$. We have, by (5.23):

$$\log(1/|B_0(z)|) \leq \sum_{n=1}^{\infty} f_{|a_n|}(\arg(\bar{a}_n z)) + \sum_{n=1}^{\infty} u_{|a_n|}(z e^{-i \arg a_n}),$$

that is

$$(5.24) \quad \log(1/|B_0(z)|) \leq f(\arg z) + u(z),$$

with

$$f(\theta) = \sum_{n=1}^{\infty} f_{|a_n|}(\arg(\bar{a}_n e^{i\theta}))$$

and $u = P[w]$, where

$$w = 2 \log 2 \sum_{n=1}^{\infty} \mathbb{1}_{e^{i \arg a_n} A_{|a_n|}}.$$

We have $f, w \in L^1(-\pi, \pi)$, by invariance of the Lebesgue measure and since, we have $\|f_{|a_n|}\|_1 \leq C(1 - |a_n|)$ and $\|w_{|a_n|}\|_1 \leq C(1 - |a_n|)$, and $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$. That finishes the proof of Proposition 5.9. \square

Now, it is easy to end the proof of Theorem 5.8 and hence that of Theorem 5.5.

End of the proof of Theorem 5.8. We only have to write

$$B = \left(\prod_{|a_n| < 3/4} \frac{|a_n|}{a_n} \varphi_{a_n} \right) \times B_0$$

where B_0 is the Blaschke product made with the zeros of B of modulus $\geq 3/4$ (as usual $|a_n|/a_n = 1$ if $a_n = 0$).

Then, with the notation of Lemma 5.10, if

$$G = \sum_{|a_n| < 3/4} G_{a_n},$$

we have, by Proposition 5.9, if $|z| \geq 1/2$:

$$\log(1/|B(z)|) \leq G(\arg z) + f(\arg z) + u(z),$$

with $f \in L^1(-\pi, \pi)$ and $u = P[w]$ with $w \in L^1(-\pi, \pi)$.

Since the maximal radial function of $u = P[w]$ is smaller than its Hardy-Littlewood maximal function M_w (actually equivalent: see [19, Theorem 11.20 and Exercise 19]), by a well-known theorem of Hardy and Littlewood, we get:

$$(5.25) \quad \sup_{1/2 \leq r < 1} \log \frac{1}{|B(re^{i\theta})|} \leq G(\theta) + f(\theta) + M_w(\theta).$$

Now, $G \in L^1(-\pi, \pi)$, by Lemma 5.10, and $M_w \in L^{1,\infty}(-\pi, \pi)$, by the Kolmogorov theorem; therefore $\log(1/|B|) \in L^{(1,\infty)}(-\pi, \pi)$, and that finishes the proof of Theorem 5.8. \square

Remark. The proofs of Theorem 5.5, 2) and Theorem 5.8 show that if the weighted Bergman space \mathfrak{B}_U^2 of analytic functions f such that $\int_{\mathbb{D}} |f|^2 U dA < \infty$, contains a function $v \in H^\infty$, with $v(0) = 1$, then $\log(1/I_U) \in L^{1,\infty}(0, 2\pi)$.

The result of Theorem 5.8 is essentially sharp, as said by the following result.

Theorem 5.13. *There exists $v \in H^\infty$, $v \not\equiv 0$, such that $\log(1/I_v) \notin L^1(0, 2\pi)$.*

Proof. We start with

$$\sigma(\theta) = \begin{cases} 1/[\theta(\log \theta)^2], & 0 < \theta \leq 1/2, \\ 0 & \text{elsewhere.} \end{cases}$$

We have $\sigma \in L^1(0, 2\pi)$, and we consider $u = P[\sigma]$. Then u is positive and, since the Poisson kernel is positive and decreasing on $[0, \pi]$, we have, for $0 < \theta \leq 1/2$:

$$\begin{aligned} u(\rho e^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} P_\rho(\theta - t) \sigma(t) dt \geq \frac{1}{2\pi} \int_0^\theta P_\rho(\theta - t) \sigma(t) dt \\ &\geq \frac{1}{2\pi} \int_0^\theta P_\rho(\theta) \sigma(t) dt = \frac{1}{2\pi} P_\rho(\theta) \frac{1}{\log(1/\theta)}. \end{aligned}$$

Taking $\rho = 1 - \theta$, we have, as θ goes to 0:

$$\frac{1 - \rho^2}{1 - 2\rho \cos \theta + \rho^2} = \frac{1 - \rho^2}{(1 - \rho)^2 + 2\rho(1 - \cos \theta)} \sim \frac{\theta(2 - \theta)}{\theta^2 + 2(1 - \theta)\theta^2/2} \sim \frac{1}{\theta}.$$

Hence:

$$u((1 - \theta) e^{i\theta}) \geq \frac{C}{\theta} \frac{1}{\log(1/\theta)}.$$

Therefore

$$\sup_{1/2 \leq \rho < 1} u(\rho e^{i\theta}) \geq \frac{C}{\theta \log(1/\theta)}.$$

Let now $g = u + i\tilde{u}$ and $v = \exp(-g)$. We have $|v| = e^{-u}$ and hence

$$I_v(\theta) \leq \exp\left(-\frac{C}{\theta \log(1/\theta)}\right)$$

and

$$\log(1/I_v) \geq \frac{C}{\theta \log(1/\theta)}.$$

Therefore $\log(1/I_v) \notin L^1(0, 2\pi)$. □

5.3 An example

The fact that, in Theorem 5.5, the function U has the particular form given in (5.8), in particular is logarithmically-subharmonic, is important. In fact, we have the following result.

Theorem 5.14. *There exist a continuous function $U: \mathbb{D} \rightarrow \mathbb{C}$, with $U \geq 1$ and an analytic function $w: \mathbb{D} \rightarrow \mathbb{C}$, $w \not\equiv 0$, such that:*

$$\int_{\mathbb{D}} |w|^2 U dA < \infty,$$

but

$$\int_0^1 U(r e^{i\theta}) dr = \infty, \quad \text{for almost all } \theta.$$

To prove this, we use the following weak form of a result of Kahane and Katznelson [8].

Theorem 5.15 (Kahane-Katznelson). *Given any positive increasing function $\omega: (0, 1) \rightarrow (0, \infty)$ such that $\omega(r) \xrightarrow[r \rightarrow 1]{} \infty$ and any pair of measurable functions $g, h: [0, 2\pi] \rightarrow \overline{\mathbb{R}} = [-\infty, +\infty]$, there exists an analytic function $F: \mathbb{D} \rightarrow \mathbb{C}$ such that*

- 1) $\max_{|z|=r} |F(z)| = o(\omega(r))$ as r goes to 1;
- 2) $\lim_{r \rightarrow 1} \Re F(r e^{i\theta}) = g(\theta)$ and $\lim_{r \rightarrow 1} \Im F(r e^{i\theta}) = h(\theta)$, for almost all $\theta \in [0, 2\pi]$.

Proof of Theorem 5.14. The Kahane-Katznelson theorem shows that there exists a function $w = \exp(-F)$ belonging to \mathfrak{B}^2 , and even in $\bigcap_{\beta > -1} \mathfrak{B}_\beta^2$, if we want, taking, for instance, $\omega(r) = \sqrt{\log(1/(1-r))}$, such that $\lim_{r \rightarrow 1} w(r e^{i\theta}) = 0$ for almost every $\theta \in [0, 2\pi]$. We may also assume that $w(0) = 1$.

By the Egorov theorem, we get, for all $n \geq 1$, numbers $\rho_n \in (1 - \frac{1}{n}, 1)$ and measurable sets $A_n \subseteq \mathbb{T}$ such that

- 1) $|w(r e^{i\theta})|^2 \leq 2^{-n}$ for $e^{i\theta} \in A_n$ and $\rho_n \leq r < 1$;
- 2) $m(\mathbb{T} \setminus A_n) < 2^{-n}$.

By the regularity of the measure, we can assume that the sets A_n are closed.

Let ρ'_n and ρ''_n such that $\rho_n < \rho'_n < \rho''_n < \frac{1+\rho_n}{2}$ and $\rho''_n - \rho'_n \geq (1 - \rho_n)/3$, and

$$E_n = \{r e^{i\theta}; e^{i\theta} \in A_n, \rho'_n \leq r \leq \rho''_n\}.$$

By the continuity of w , there is an open neighborhood G_n of E_n , with closure contained in \mathbb{D} , such that $\rho_n < |z| < (1 + \rho_n)/2$ and $|w(z)|^2 < 2^{-n+1}$ for $z \in G_n$. We have

$$\int_{G_n} \frac{2}{1 - \rho_n} |w(z)|^2 dA(z) \leq \frac{1}{\pi} \int_0^{2\pi} \left(\int_{\rho_n}^{\frac{1+\rho_n}{2}} \frac{2}{1 - \rho_n} 2^{-n+1} dr \right) d\theta \leq 2^{-n+2}.$$

The Urysohn lemma gives a continuous function $U_n: \mathbb{D} \rightarrow [0, \frac{2}{1-\rho_n}]$ such that:

- a) $U_n(z) = 0$ if $z \notin G_n$;
- b) $U_n(z) = 2/(1 - \rho_n)$ for $z \in E_n$;
- c) $\int_{\mathbb{D}} |w|^2 U_n dA \leq 2^{-n+2}$.

Then

$$U = 1 + \sum_{n=1}^{\infty} U_n$$

is continuous on \mathbb{D} , because the sum is locally finite, since $U_n = 0$ out of G_n . We have, by c), and since $w \in \mathfrak{B}^2$:

$$\int_{\mathbb{D}} |w|^2 U dA = \int_{\mathbb{D}} |w|^2 dA + \sum_{n=1}^{\infty} \int_{\mathbb{D}} |w|^2 U_n dA < \infty.$$

Moreover, since $\sum_{n=1}^{\infty} m(\mathbb{T} \setminus A_n) < \infty$, for almost all θ , there exists $N(\theta) \geq 1$ such that $e^{i\theta} \in A_n$ for all $n \geq N(\theta)$. Hence, for these θ :

$$\begin{aligned} \int_0^1 U(r e^{i\theta}) dr &\geq \sum_{n=N(\theta)}^{\infty} \int_{\rho'_n}^{\rho''_n} U_n(r e^{i\theta}) dr \geq \sum_{n=N(\theta)}^{\infty} \int_{\rho'_n}^{\rho''_n} \frac{2}{1 - \rho_n} dr \\ &= \sum_{n=N(\theta)}^{\infty} (\rho''_n - \rho'_n) \frac{2}{1 - \rho_n} \geq \sum_{n=N(\theta)}^{\infty} \frac{2}{3} = \infty. \end{aligned}$$

That finishes the proof of Theorem 5.14. \square

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