



HAL
open science

Boundedness of composition operators on general weighted Hardy spaces of analytic functions

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza, Pascal Lefèvre

► **To cite this version:**

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza, Pascal Lefèvre. Boundedness of composition operators on general weighted Hardy spaces of analytic functions. 2022. hal-03029931v2

HAL Id: hal-03029931

<https://univ-artois.hal.science/hal-03029931v2>

Preprint submitted on 14 Mar 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Boundedness of composition operators on general weighted Hardy spaces of analytic functions

*Pascal Lefèvre, Daniel Li,
Hervé Queffélec, Luis Rodríguez-Piazza*

March 14, 2022

Abstract. We characterize the (essentially) decreasing sequences of positive numbers $\beta = (\beta_n)$ for which all composition operators on $H^2(\beta)$ are bounded, where $H^2(\beta)$ is the space of analytic functions f in the unit disk such that $\sum_{n=0}^{\infty} |c_n|^2 \beta_n < \infty$ if $f(z) = \sum_{n=0}^{\infty} c_n z^n$. We also give conditions for the boundedness when β is not assumed essentially decreasing.

MSC 2010 primary: 47B33 ; secondary: 30H10

Key-words automorphism of the unit disk; composition operator; Δ_2 -condition; multipliers; weighted Hardy space

1 Introduction

Let $\beta = (\beta_n)_{n \geq 0}$ be a sequence of positive numbers such that

$$(1.1) \quad \liminf_{n \rightarrow \infty} \beta_n^{1/n} \geq 1.$$

The associated weighted Hardy space $H^2(\beta)$ is the set of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$(1.2) \quad \|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < \infty.$$

It is a Hilbert space of analytic functions on \mathbb{D} . When $\beta_n \equiv 1$, we recover the usual Hardy space H^2 .

Recall that a symbol is a (non constant) analytic self-map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, and the associated composition operator $C_\varphi: H^2(\beta) \rightarrow \mathcal{H}ol(\mathbb{D})$ is defined (formally) as:

$$(1.3) \quad C_\varphi(f) = f \circ \varphi.$$

An important question in the theory is to decide when C_φ is bounded on $H^2(\beta)$, i.e. when $C_\varphi: H^2(\beta) \rightarrow H^2(\beta)$. When $H^2(\beta)$ is the usual Hardy space H^2 (i.e. when $\beta_n \equiv 1$), it is well-known ([18, pp. 13–17]) that all symbols generate bounded composition operators. On the other hand, for the Dirichlet space, corresponding to $\beta_n = n + 1$, not all composition operators are bounded. Note that, by definition of the norm of $H^2(\beta)$, all rotations R_θ , $\theta \in \mathbb{R}$, induce bounded composition operators on $H^2(\beta)$ and send isometrically $H^2(\beta)$ into itself.

Our goal in this paper is characterizing the (non-increasing) sequences β for which all composition operators act boundedly on the space $H^2(\beta)$, i.e. send $H^2(\beta)$ into itself. We will prove the following result, where $T_a(z) = \frac{a+z}{1+\bar{a}z}$ for $a, z \in \mathbb{D}$.

Theorem 1.1. *Let β be an essentially decreasing sequence of positive numbers. The following assertions are equivalent:*

- 1) *all composition operators are bounded on $H^2(\beta)$;*
- 2) *all maps T_a , for $0 < a < 1$, induce bounded composition operators C_{T_a} on $H^2(\beta)$;*
- 3) *for some $a \in (0, 1)$, the map T_a induces a bounded composition operator C_{T_a} on $H^2(\beta)$;*
- 4) *β satisfies the Δ_2 -condition.*

For the notion of essentially decreasing sequence, see Definition 2.1; the Δ_2 -condition is defined in (2.4). Actually, we will obtain Theorem 1.1 by gluing Theorem 3.3 and Theorem 4.1.

Most of the existing works with weighted Hardy spaces concern the case

$$(1.4) \quad \beta_n = \int_0^1 t^n d\sigma(t)$$

where σ is a positive measure on $(0, 1)$. More specifically the following definition is often used. Let $G: (0, 1) \rightarrow \mathbb{R}_+$ be an integrable function and let H_G^2 be the space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that:

$$(1.5) \quad \|f\|_{H_G^2}^2 := \int_{\mathbb{D}} |f(z)|^2 G(1 - |z|^2) dA(z) < \infty.$$

Such weighted Bergman type spaces are used, for instance, in [11], [10] and in [14]. We have $H_G^2 = H^2(\beta)$ with:

$$(1.6) \quad \beta_n = 2 \int_0^1 r^{2n+1} G(1 - r^2) dr = \int_0^1 t^n G(1 - t) dt,$$

and the sequence $\beta = (\beta_n)_n$ is non-increasing.

In Shapiro's presentation, the main point is the case $\varphi(0) = 0$ and a subordination principle for subharmonic functions (Littlewood's subordination principle). The case of automorphisms is claimed simple, using an integral representation for the norm and some change of variable. When β is defined as in (1.4), one disposes of integral representations for the norm in $H^2(\beta)$, and, as in the Hardy space case, this integral representation rather easily gives the boundedness of C_{T_a} on H^2 , where

$$(1.7) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}$$

for $a \in \mathbb{D}$. But the above representation (1.4) is equivalent, by the Hausdorff moment theorem, to a high regularity of the sequence β , namely its *complete monotony*. When integral representations fail, we have to work with bare hands. If the symbol vanishes at the origin, Kacnel'son's theorem gives a positive answer when β is essentially decreasing (see [2] or [13, Theorem 3.12]). Actually that follows from an older theorem of Goluzin [8] (see [5, Theorem 6.3]), which itself uses a refinement by Rogosinski of Littlewood's principle ([5, Theorem 6.2]). So that the main issue remains the boundedness of C_{T_a} .

A polynomial minoration (see Definition 2.3 below) for β is necessary for any C_{T_a} to be bounded on $H^2(\beta)$ (Proposition 2.5) and we showed in [13, end of Section 3] that for $\beta_n = \exp(-\sqrt{n})$, C_{T_a} is never bounded on $H^2(\beta)$. But this polynomial minoration is not sufficient, as we will see in Theorem 1.1 below, which also evidences the basic role of the maps T_a in the question.

However, we construct a weight β which is not essentially decreasing and for which all composition operators with symbol vanishing at 0 are bounded (Theorem 3.9), though no map T_a with $0 < a < 1$ induces a bounded composition operator (Proposition 4.4).

For spaces of Bergman type $A_G^2 := H_G^2$, where $\tilde{G}(r) = G(1-r^2)$, defined as the spaces of analytic functions in \mathbb{D} such that $\int_{\mathbb{D}} |f(z)|^2 \tilde{G}(|z|) dA < \infty$, for a positive non-increasing continuous function \tilde{G} on $[0, 1)$, Kriete and MacCluer studied in [11] some analogous problems. They proved, in particular [11, Theorem 3] that, for:

$$\tilde{G}(r) = \exp\left(-B \frac{1}{(1-r)^\alpha}\right), \quad B > 0, \quad 0 < \alpha \leq 2,$$

and

$$\varphi(z) = z + t(1-z)^\beta, \quad 1 < \beta \leq 3, \quad 0 < t < 2^{1-\beta},$$

then C_φ is bounded on A_G^2 if and only if $\beta \geq \alpha + 1$.

Here

$$\beta_n = \int_0^1 t^n e^{-B/(1-\sqrt{t})^\alpha} dt;$$

and, since $\beta_n \approx \exp(-cn^{\alpha/(\alpha+1)})$, the sequence (β_n) does not satisfy the Δ_2 -condition, accordingly to our Theorem 3.3 below.

We end the paper with some miscellaneous remarks.

2 Definitions, notation, and preliminary results

The open unit disk of \mathbb{C} is denoted \mathbb{D} and we write \mathbb{T} its boundary $\partial\mathbb{D}$. We set $e_n(z) = z^n$, $n \geq 0$.

The weighted Hardy space $H^2(\beta)$ defined in the introduction is a Hilbert space with the canonical orthonormal basis

$$(2.1) \quad e_n^\beta(z) = \frac{1}{\sqrt{\beta_n}} z^n, \quad n \geq 0,$$

and the reproducing kernel K_w given for all $w \in \mathbb{D}$ by

$$(2.2) \quad K_w(z) = \sum_{n=0}^{\infty} e_n^\beta(z) \overline{e_n^\beta(w)} = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \bar{w}^n z^n.$$

Note that (1.1) is necessary for $H^2(\beta)$ to consist of analytic functions in \mathbb{D} . Indeed the fact that $\sum_{n \geq 1} \frac{1}{n\sqrt{\beta_n}} z^n$ belongs to $H^2(\beta)$ and is analytic in \mathbb{D} implies (1.1). Note also that H^2 is continuously embedded in $H^2(\beta)$ if and only if β is bounded above. In particular, this is the case when β is non-increasing. In this paper, we need a slightly more general notion.

Definition 2.1. *A sequence of positive numbers $\beta = (\beta_n)$ is said essentially decreasing if, for some constant $C \geq 1$, we have, for all $m \geq n \geq 0$:*

$$(2.3) \quad \beta_m \leq C \beta_n.$$

Note that saying that β is essentially decreasing means that the shift operator on $H^2(\beta)$ is power bounded.

If β is essentially decreasing, and if we set:

$$\tilde{\beta}_n = \sup_{m \geq n} \beta_m,$$

the sequence $\tilde{\beta} = (\tilde{\beta}_n)$ is non-increasing and we have $\beta_n \leq \tilde{\beta}_n \leq C \beta_n$. In particular, the space $H^2(\beta)$ is isomorphic to $H^2(\tilde{\beta})$ and H^2 is continuously embedded in $H^2(\beta)$.

Definition 2.2. *The sequence of positive numbers $\beta = (\beta_n)$ is said to satisfy the Δ_2 -condition if there is a positive constant $\delta < 1$ such that, for all integers $n \geq 0$:*

$$(2.4) \quad \beta_{2n} \geq \delta \beta_n.$$

This terminology is given by analogy with that used for Orlicz functions.

Definition 2.3. *The sequence of positive numbers $\beta = (\beta_n)$ is said to have a polynomial minoration if there are positive constants δ and α such that, for all integers $n \geq 1$:*

$$(2.5) \quad \beta_n \geq \delta n^{-\alpha}.$$

That means that $H^2(\beta)$ is continuously embedded in the weighted Bergman space \mathfrak{B}_α^2 of the analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that

$$\|f\|_{\mathfrak{B}_\alpha^2}^2 := (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty$$

since $\mathfrak{B}_\alpha^2 = H^2(\gamma)$ with $\gamma_n \approx n^{-\alpha}$.

The following simple proposition links those notions.

Proposition 2.4. *Let β be an essentially decreasing sequence of positive numbers. Then if β satisfies the Δ_2 -condition, it has a polynomial minoration.*

The converse does not hold.

Proof. Assume $\beta_m \leq C \beta_n$ for $m \geq n$ and $\beta_{2^k} \geq e^{-A} \beta_k$. Let now n be an integer ≥ 2 , and $k \geq 1$ the smallest integer such that $2^k \geq n$, so that $k \leq a \log n$ with a a positive constant. We get:

$$\beta_n \geq C^{-1} \beta_{2^k} \geq C^{-1} e^{-kA} \beta_1 \geq C^{-1} \beta_1 e^{-aA \log n} =: \rho n^{-\alpha},$$

with $\rho = C^{-1} \beta_1$ and $\alpha = aA$.

Let us now see that the converse does not hold. Let $\delta > 0$. We set $\beta_0 = \beta_1 = 1$ and for $n \geq 2$:

$$\beta_n = \frac{1}{(k!)^\delta} \quad \text{when } k! < n \leq (k+1)!$$

The sequence β is non-increasing.

For n and k as above, we have:

$$\beta_n = \frac{1}{(k!)^\delta} \geq \frac{1}{n^\delta};$$

hence β has arbitrary polynomial minoration. However we have, for $k \geq 2$:

$$\frac{\beta_{2(k!)}}{\beta_{k!}} = \frac{(k!)^{-\delta}}{[(k-1)!]^{-\delta}} = \frac{1}{k^\delta} \xrightarrow[k \rightarrow \infty]{} 0,$$

so β fails to satisfy the Δ_2 -condition. □

For $a \in \mathbb{D}$, we define:

$$(2.6) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}, \quad z \in \mathbb{D}.$$

Recall that T_a is an automorphism of \mathbb{D} and that $T_a(0) = a$ and $T_a(-a) = 0$.

Though we do not need this, we may remark that $(T_a)_{a \in (-1,1)}$ is a group and $(T_a)_{a \in (0,1)}$ is a semigroup. It suffices to see that $T_a \circ T_b = T_{a*b}$, with:

$$(2.7) \quad a * b = \frac{a+b}{1+ab}.$$

Proposition 2.5. *Let $a \in (0, 1)$ and assume that T_a induces a bounded composition operator on $H^2(\beta)$. Then β has a polynomial minoration.*

Proof. Since

$$\|K_x\|^2 = \sum_{n=0}^{\infty} \frac{x^{2n}}{\beta_n},$$

we have $\|K_x\| \leq \|K_y\|$ for $0 \leq x \leq y < 1$.

We define by induction a sequence $(u_n)_{n \geq 0}$ with:

$$u_0 = 0 \quad \text{and} \quad u_{n+1} = T_a(u_n).$$

Since $T_a(1) = 1$ (recall that $a \in (0, 1)$), we have:

$$1 - u_{n+1} = \int_{u_n}^1 T_a'(t) dt = \int_{u_n}^1 \frac{1 - a^2}{(1 + at)^2} dt;$$

hence

$$\frac{1 - a}{1 + a} (1 - u_n) \leq 1 - u_{n+1} \leq (1 - a^2)(1 - u_n).$$

Let $0 < x < 1$. We can find $N \geq 0$ such that $u_N \leq x < u_{N+1}$. Then:

$$1 - x \leq 1 - u_N \leq (1 - a^2)^N.$$

On the other hand, since $C_{T_a}^* K_z = K_{T_a(z)}$ for all $z \in \mathbb{D}$, we have:

$$\|K_x\| \leq \|K_{u_{N+1}}\| \leq \|C_{T_a}\| \|K_{u_N}\| \leq \|C_{T_a}\|^{N+1} \|K_{u_0}\| = \|C_{T_a}\|^{N+1}.$$

Let $s \geq 0$ such that $(1 - a^2)^{-s} = \|C_{T_a}\|$. We obtain:

$$(2.8) \quad \|K_x\| \leq \|C_{T_a}\| \frac{1}{(1 - x)^s}.$$

But

$$\|K_x\|^2 = \sum_{k=0}^{\infty} \frac{x^{2k}}{\beta_k};$$

so we get, for any $k \geq 2$:

$$\frac{x^{2k}}{\beta_k} \leq \|C_{T_a}\|^2 \frac{1}{(1 - x)^{2s}}.$$

Taking $x = 1 - \frac{1}{k}$, we obtain $\beta_k \geq C k^{-2s}$. □

Remarks. 1) For example, when $\beta_n = \exp[-c(\log(n+1))^2]$, with $c > 0$, no T_a induces a bounded composition operator on $H^2(\beta)$, though all symbols φ with $\varphi(0) = 0$ are bounded, since β is decreasing, as we will see in Proposition 3.2.

2) For the Dirichlet space \mathcal{D}^2 , we have $\beta_n = n+1$, but all the maps T_a induce bounded composition operators on \mathcal{D}^2 (see [13, Remark before Theorem 3.12]). In this case β has a polynomial minoration though it is not bounded above.

3) However, even for decreasing sequences, a polynomial minoration for β is not enough for some T_a to induce a bounded composition operator. Indeed, we saw in Proposition 2.4 an example of a decreasing sequence β with polynomial minoration, but not sharing the Δ_2 -condition, and we will see in Theorem 4.1 that the Δ_2 -condition is needed for having some T_a inducing a bounded composition operator.

4) In [7], Eva Gallardo-Gutiérrez and Jonathan Partington give estimates for the norm of C_{T_a} , with $a \in (0, 1)$, when C_{T_a} is bounded on $H^2(\beta)$. More precisely, they proved that if β is bounded above and C_{T_a} is bounded, then

$$\|C_{T_a}\| \geq \left(\frac{1+a}{1-a} \right)^\sigma,$$

where $\sigma = \inf\{s \geq 0; (1-z)^{-s} \notin H^2(\beta)\}$, and

$$\|C_{T_a}\| \leq \left(\frac{1+a}{1-a} \right)^\tau,$$

where $\tau = \frac{1}{2} \sup \Re W(A)$, with A the infinitesimal generator of the continuous semigroup (S_t) defined as $S_t = C_{T_{\tanh t}}$, namely $(Af)(z) = f'(z)(1-z^2)$, and $W(A)$ its numerical range.

For $\beta_n = 1/(n+1)^\nu$ with $0 \leq \nu \leq 1$, the two bounds coincide, so they get $\|C_{T_a}\| = \left(\frac{1+a}{1-a} \right)^{(\nu+1)/2}$.

5) We saw in the proof of Proposition 2.5 that if C_{T_a} is bounded on $H^2(\beta)$ for some $a \in (0, 1)$, then the reproducing kernels K_w have, by (2.8), a *slow growth*:

$$(2.9) \quad \|K_w\| \leq \frac{C}{(1-|w|)^s}$$

for positive constants C and s . Actually, we have the following equivalence.

Proposition 2.6. *The sequence β has a polynomial minoration if and only if the reproducing kernels K_w of $H^2(\beta)$ have a slow growth.*

Proof. The sufficiency is easy and seen at the end of the proof of Proposition 2.5. For the necessity, we only have to see that:

$$\|K_w\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{|w|^{2n}}{\beta_n} \leq \frac{1}{\beta_0} + \delta^{-1} \sum_{n=1}^{\infty} n^\alpha |w|^{2n} \leq \frac{C}{(1-|w|^2)^{\alpha+1}}. \quad \square$$

3 Boundedness of composition operators

We study in this section conditions ensuring that all composition operators on $H^2(\beta)$ are bounded.

3.1 Conditions on the weight

We begin with this simple observation.

Proposition 3.1. *If all composition operators, and even if all composition operators with symbol vanishing at 0, are bounded on $H^2(\beta)$, then the sequence β is bounded above.*

Proof. Let $f \in H^\infty$. Write $f = A\varphi + f(0)$ where A is a constant and φ a symbol vanishing at 0. We have $\varphi = C_\varphi(z) \in H^2(\beta)$, by hypothesis. So that $f \in H^2(\beta)$ and $H^\infty \subseteq H^2(\beta)$. It follows (by the closed graph theorem) that there exists a constant M such that $\|f\|_{H^2(\beta)} \leq M\|f\|_\infty$ for all $f \in H^\infty$. Testing that with $f(z) = z^n$, we get $\beta_n \leq M^2$. \square

For symbols vanishing at 0, we have the following characterization.

Proposition 3.2. *The following assertions are equivalent:*

1) *all symbols φ such that $\varphi(0) = 0$ induce bounded composition operators C_φ on $H^2(\beta)$ and*

$$(3.1) \quad \sup_{\varphi(0)=0} \|C_\varphi\| < \infty;$$

2) *β is an essentially decreasing sequence.*

Of course, by the uniform boundedness principle, (3.1) is equivalent to:

$$\sup_{\varphi(0)=0} \|f \circ \varphi\| < \infty \quad \text{for all } f \in H^2(\beta).$$

Proof. 2) \Rightarrow 1) We may assume that β is non-increasing. Then the Goluzin-Rogosinski theorem ([5, Theorem 6.3]) gives the result; in fact, writing $f(z) = \sum_{n=0}^\infty c_n z^n$ and $(C_\varphi f)(z) = \sum_{n=0}^\infty d_n z^n$, it says that:

$$\|C_\varphi f\|^2 = |d_0|^2 \beta_0 + \sum_{n=1}^\infty |d_n|^2 \beta_n \leq |c_0|^2 \beta_0 + \sum_{n=1}^\infty |c_n|^2 \beta_n = \|f\|^2,$$

leading to C_φ bounded and $\|C_\varphi\| \leq 1$. Alternatively, we can use a result of Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]). This result was also proved by C. Cowen [4, Corollary of Theorem 7].

1) \Rightarrow 2) Set $M = \sup_{\varphi(0)=0} \|C_\varphi\|$. Let $m > n$, and take:

$$\varphi(z) = \varphi_{m,n}(z) = z \left(\frac{1 + z^{m-n}}{2} \right)^{1/n}.$$

Then $\varphi(0) = 0$ and $[\varphi(z)]^n = \frac{z^n + z^m}{2}$; hence

$$\frac{1}{4}(\beta_n + \beta_m) = \|\varphi^n\|^2 = \|C_\varphi(e_n)\|^2 \leq \|C_\varphi\|^2 \|e_n\|^2 \leq M^2 \beta_n,$$

so β is essentially decreasing. \square

Boundedness of β_n does not suffice. For example, let (β_n) be a sequence such that $\beta_{2k+2}/\beta_{2k+1} \xrightarrow[k \rightarrow \infty]{} \infty$ (for instance $\beta_{2k} = 1$ and $\beta_{2k+1} = 1/(k+1)$); if $\varphi(z) = z^2$, then $\|C_\varphi(z^{2n+1})\|^2 = \|z^{2(2n+1)}\|^2 = \beta_{2(2n+1)}$; since $\|z^{2n+1}\|^2 = \beta_{2n+1}$, C_φ is not bounded on $H^2(\beta)$.

A more interesting example is the following. For $0 < r < 1$, let $\beta_n = \pi n r^{2n}$. This sequence is eventually decreasing, so it is essentially decreasing. The square $\|f\|_{H^2(\beta)}^2$ of the norm $\|f\|_{H^2(\beta)}$ is the area of the part of the Riemann surface on which $r\mathbb{D}$ is mapped by f . E. Reich [17], generalizing Goluzin's result [8] (see [5, Theorem 6.3]), proved that for all symbols φ such that $\varphi(0) = 0$, the composition operator C_φ is bounded on $H^2(\beta)$ and

$$\|C_\varphi\| \leq \sup_{n \geq 1} \sqrt{n} r^{n-1} \leq \frac{1}{\sqrt{2e}} \frac{1}{r \sqrt{\log(1/r)}}.$$

For $0 < r < 1/\sqrt{2}$, Goluzin's theorem asserts that $\|C_\varphi\| \leq 1$.

Note that this sequence β does not satisfy the Δ_2 -condition since $\beta_{2n}/\beta_n = 2r^{2n}$, Theorem 4.1 below states that no composition operator C_{T_a} is bounded.

However that the weight β is essentially decreasing is not necessary for the boundedness of all composition operators C_φ , with symbol φ vanishing at 0, as we will see later (Theorem 3.9).

3.2 Sufficient condition for the boundedness of composition operators – Part I

We now have one of the the main results of this section.

Theorem 3.3. *Let $H^2(\beta)$ be a weighted Hardy space with $\beta = (\beta_n)$ essentially decreasing and satisfying the Δ_2 -condition. Then all composition operators on $H^2(\beta)$ are bounded.*

For the proof, we need a lemma.

Lemma 3.4. *For $0 < a < 1$, we write:*

$$(3.2) \quad (T_a z)^n = \sum_{m=0}^{\infty} a_{m,n} z^m.$$

Then, there are constants $b > 0$, $0 < C_1 < 1$, $C_2 > 1$ such that

$$|a_{m,n}| \leq \begin{cases} e^{-bn} & \text{if } m \leq C_1 n, \\ e^{-bm} & \text{if } m \geq C_2 n. \end{cases}$$

Proof. First take $0 < r < 1$; let:

$$M(r) = \sup_{|z|=r} |T_a(z)| = \sup_{|z|=r} \left| \frac{z+a}{1+az} \right|.$$

We have $M(r) < 1$, so we can write $M(r) = r^\rho$, for some positive $\rho = \rho(a)$.
The Cauchy inequalities give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = r^{\rho n - m},$$

and we obtain the first inequality by taking $r = e^{-\alpha}$ and adjusting C_1 .

Next, we use that T_a is analytic on $D(0, 1/a)$. Fix $1 < r =: e^\beta < 1/a$, with $\beta > 0$. Let $M(r) = e^\alpha$ with $\alpha > 0$. The Cauchy inequalities again give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = e^{\alpha n - \beta m},$$

and we obtain the second inequality by adjusting C_2 . □

We will also need the following result of V. È Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]).

Theorem 3.5 (V. È Kacnel'son). *Let H be a separable complex Hilbert space and $(e_i)_{i \geq 0}$ a fixed orthonormal basis of H . Let $M: H \rightarrow H$ be a bounded linear operator. We assume that the matrix of M with respect to this basis is lower-triangular: $\langle M e_j | e_i \rangle = 0$ for $i < j$.*

Let $(\gamma_j)_{j \geq 0}$ be a non-decreasing sequence of positive real numbers and Γ the (possibly unbounded) diagonal operator such that $\Gamma(e_j) = \gamma_j e_j$, $j \geq 0$. Then the operator $\Gamma^{-1} M \Gamma: H \rightarrow H$ is bounded and moreover:

$$\|\Gamma^{-1} M \Gamma\| \leq \|M\|.$$

Proof of Theorem 3.3. We may, and do, assume that β is non-increasing.

Proposition 3.2 gives the result when $\varphi(0) = 0$.

It remains to show that all C_{T_a} , $a \in \mathbb{D}$, are bounded. Indeed, if $a = \varphi(0)$ and $\psi = T_{-a} \circ \varphi$, then $\psi(0) = 0$ and $\varphi = T_a \circ \psi$, so $C_\varphi = C_\psi \circ C_{T_a}$. Moreover, we have only to show that when $a \in [0, 1)$. Indeed, if $a \in \mathbb{D}$ and $a = |a| e^{i\theta}$, we have $T_a = R_\theta \circ T_{|a|} \circ R_{-\theta}$, so $C_{T_a} = C_{R_{-\theta}} \circ C_{T_{|a|}} \circ C_{R_\theta}$.

We consider the matrices

$$A = (a_{m,n})_{m,n \geq 0} \quad \text{and} \quad A_\beta = \left(\sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} \right)_{m,n \geq 0}.$$

Since $C_{T_a} e_n = T_a^n e_n$, the formula (3.2) shows that A is the matrix of C_{T_a} in H^2 with respect to the basis $(e_n)_{n \geq 0}$. On the other hand, A_β is the matrix of C_{T_a} in $H^2(\beta)$ with respect to the basis $(e_n^\beta)_{n \geq 0}$. We note that $A_\beta = B A B^{-1}$, where B is the diagonal matrix with values $\sqrt{\beta_0}, \sqrt{\beta_1}, \dots$ on the diagonal.

Since C_{T_a} is a bounded composition operators on H^2 , the matrix A defines a bounded operator on ℓ_2 . We have to show that A_β also, i.e. $\|A_\beta\| < \infty$.

For that purpose, we split A and A_β into several sub-matrices.

Let N be an integer such that $N \geq 2/C_1$, where C_1 is defined in Lemma 3.4 (actually, the proof of that lemma shows that we can take C_1 such that $1/C_1$ is

an integer, so we could take $N = 2/C_1$). Let $I_0 = [0, N[$ $J_0 = [N, +\infty[$ and for $k = 1, 2, \dots$:

$$I_k = [N^k, N^{k+1}[\quad \text{and} \quad J_k = [N^{k+1}, +\infty[.$$

We define the matrices D_β and R_β , whose entries are respectively:

$$d_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty} (I_k \times I_k) \\ 0 & \text{elsewhere;} \end{cases}$$

and

$$r_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty} (I_k \times I_{k+1}) \\ 0 & \text{elsewhere.} \end{cases}$$

We also define the matrix S_β with entries:

$$s_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty} (J_k \times I_k) \\ 0 & \text{elsewhere.} \end{cases}$$

Matrices D , R , and S are constructed in the same way from A and we set $U = A - (D + R + S)$.

Now, let H_k be the subspace of the sequences $(x_n)_{n \geq 0}$ in ℓ_2 such that $x_n = 0$ for $n \notin I_k$, i.e. $H_k = \text{span}\{e_n; n \in I_k\}$, and let P_k be (the matrix of) the orthogonal projection of ℓ_2 with range H_k . We have:

$$D = \sum_{k=0}^{\infty} P_k A P_k \quad \text{and} \quad R = \sum_{k=0}^{\infty} P_k A P_{k+1},$$

where $D_k = P_k A P_k$ is the matrix with entries $a_{m,n}$ when $(m,n) \in I_k \times I_k$ and 0 elsewhere, and $R_k = P_k A P_{k+1}$ the matrix with entries $a_{m,n}$ when $(m,n) \in I_k \times I_{k+1}$ and 0 elsewhere.

$$\left(\begin{array}{c|c|c|c} \boxed{D_0} & \boxed{R_0} & & \\ \hline & D_1 & \boxed{R_1} & \\ \hline S_0 & S_1 & D_2 & R_2 \\ \hline & & S_2 & \end{array} \right) U$$

Since the subspaces H_k are orthogonal, the matrices D and R induce bounded operators on ℓ_2 , and

$$(3.3) \quad \|D\| \leq \|A\|, \quad \|R\| \leq \|A\|.$$

Now, for $k \geq 1$, let B_k be the diagonal matrix whose entries are $b_{m,m} = \sqrt{\beta_m}$ if $m \in I_k$ and $b_{m,n} = 0$ otherwise.

Then $P_k D_\beta P_k = P_k B_k D B_k^{-1} P_k$, so

$$\|P_k D_\beta P_k\| \leq \|B_k\| \|B_k^{-1}\| \|D\| \leq \max_{j \in I_k} \sqrt{\beta_j} \max_{j \in I_k} \frac{1}{\sqrt{\beta_j}} \|A\|.$$

But the weight β satisfies the Δ_2 -condition: $\beta_{2l} \geq \delta_0 \beta_l$, and it follows that for every $l \geq 1$:

$$\beta_{N^{2l}} \geq \delta^2 \beta_l,$$

for some other constant δ , chosen small enough to have $\|P_0 D_\beta P_0\| \leq \delta^{-1} \|A\|$. Since β is non-increasing, we have $\beta_j \geq \delta^2 \beta_{N^k}$ for $N^k \leq j \leq N^{k+1}$. In particular $\max_{j \in I_k} \sqrt{\beta_j} \leq \delta^{-1} \min_{j \in I_k} \sqrt{\beta_j}$ and

$$\|P_k D_\beta P_k\| \leq \delta^{-1} \|A\|.$$

Hence, by orthogonality of the subspaces H_k :

$$(3.4) \quad \|D_\beta\| \leq \delta^{-1} \|D\|.$$

In the same way, we have $P_k R_\beta P_k = P_k B_k D B_{k+1}^{-1} P_k$, so:

$$\|P_k R_\beta P_k\| \leq \max_{j \in I_k} \sqrt{\beta_j} \max_{j \in I_{k+1}} \frac{1}{\sqrt{\beta_j}} \|A\| = \max_{(m,n) \in I_k \times I_{k+1}} \sqrt{\frac{\beta_m}{\beta_n}} \|A\|.$$

But, when $(m, n) \in I_k \times I_{k+1}$, we get

$$\beta_n \geq \beta_{N^{k+2}} \geq \delta^2 \beta_{N^k} \geq \delta^2 \beta_m.$$

Hence

$$(3.5) \quad \|R_\beta\| \leq \delta^{-1} \|R\|.$$

Next, consider $U = A - (D + R + S)$; we can compute its Hilbert-Schmidt norm using Lemma 3.4. Note that $u_{m,n} \neq 0$ only (if it happens) when $m \in I_k$ for some $k \geq 0$ and $n \geq \min I_{k+2} \geq N^{k+2} > Nm$, since $m \in I_k$, so only when $m \leq C_1 n$. We have, since then $u_{m,n} = a_{m,n}$:

$$\|U\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m \leq C_1 n} |a_{m,n}|^2 \leq \sum_{n=0}^{\infty} \sum_{m \leq C_1 n} e^{-2bn} \leq \sum_{n=0}^{\infty} C_1 n e^{-2bn} < \infty.$$

Consequently, with (3.3), we have $\|S\| \leq \|A\| + \|D\| + \|R\| + \|U\| < \infty$.

Now, since S is a lower-triangular matrix and β is non-increasing, we can use the result of V. È Kacnel'son (Theorem 3.5), with $\gamma_j = 1/\sqrt{\beta_j}$. We get that S_β defines a bounded operator and $\|S_\beta\| \leq \|S\|$.

Further, Proposition 2.4 says that β has a polynomial minoration:

$$\beta_n \geq c n^{-\sigma}$$

for positive constants c and σ . Then, if $U_\beta = A_\beta - (D_\beta + R_\beta + S_\beta)$, we have:

$$\|U_\beta\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m < \rho n/2} \frac{|a_{m,n}|^2}{\beta_n} \leq \sum_{n=0}^{\infty} \frac{\rho n}{2} e^{-\alpha \rho n} \frac{n^\sigma}{c} < \infty.$$

Putting this together with (3.4) and (3.5), we finally obtain that $A_\beta = S_\beta + D_\beta + R_\beta + U_\beta$ is the matrix of a bounded operator, and that ends the proof of Theorem 3.3. \square

Remark. We could have done here without the theorem of Kacnel'son, using Lemma 3.4 to show that the matrix $\left(a_{m,n} \sqrt{\frac{\beta_m}{\beta_n}}\right)$ is Hilbert-Schmidt as well “far below” the main diagonal. Indeed, we see from this lemma that

$$\sum_{m \geq C_2 n} |a_{m,n}|^2 \frac{\beta_m}{\beta_n} \lesssim \sum_{m \geq C_2 n} n^\sigma |a_{m,n}|^2 \lesssim \sum_{m \geq C_2 n} n^\sigma e^{-2bm} \lesssim \sum_{n \geq 0} n^\sigma e^{-2bC_2 n} < +\infty.$$

See also Theorem 6.2 of the final section. Kacnel'son's theorem will really be needed in the forthcoming Theorem 3.6.

3.3 Sufficient condition for the boundedness of composition operators – Part II

In this section, we give a sufficient condition for the boundedness of composition operators with symbol vanishing at 0, of a different nature than the one given in Proposition 3.2.

Theorem 3.6. *Let $\beta = (\beta_n)_{n=0}^\infty$ be a bounded sequence of positive numbers with a polynomial minoration. Assume that:*

$$(3.6) \quad \text{For every } \delta > 0, \text{ there exists a positive constant } C = C(\delta) \text{ such that} \\ \beta_m \leq C \beta_n \quad \text{whenever } m > (1 + \delta)n.$$

Then, for all symbols $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ vanishing at 0, the composition operator C_φ is bounded on $H^2(\beta)$.

To prove Theorem 3.6, we need several lemmas.

Lemma 3.7. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic self-map such that $\varphi(0) = 0$ and $|\varphi'(0)| < 1$. Then there exists $\rho > 0$ such that*

$$|\widehat{\varphi^n}(m)| \leq \exp\left(-\frac{1}{2}[(1 + \rho)n - m]\right).$$

Proof. It is the same as that of Lemma 3.4. Since $\varphi(0) = 0$, we can write $\varphi(z) = z \varphi_1(z)$. Since $|\varphi'(0)| < 1$, we have $\varphi_1: \mathbb{D} \rightarrow \mathbb{D}$. Let $M(r) = \sup_{|z|=r} |\varphi_1(z)|$. The Cauchy inequalities say that $|\widehat{\varphi_1^n}(m)| \leq [M(r)]^n / r^m$. We have $M(r) < 1$, so there exists a positive number $\rho = \rho(r)$ such that $M(r) = r^\rho$. We get:

$$|\widehat{\varphi^n}(m)| = |\widehat{\varphi_1^n}(m-n)| \leq \frac{r^{\rho n}}{r^{m-n}} = r^{(1+\rho)n-m},$$

and the result follows, by taking $r = e^{-1/2}$. \square

The next lemma is a variant of the result of V. È Kacnel'son quoted before.

Lemma 3.8. *Let $A: \ell_2 \rightarrow \ell_2$ be a bounded operator represented by the matrix $(a_{m,n})_{m,n}$, i.e. $a_{m,n} = \langle A e_n, e_m \rangle$, where $(e_n)_{n \geq 1}$ is the canonical basis of ℓ_2 .*

Let (d_n) be a sequence of positive numbers such that, for every m and n :

$$(3.7) \quad d_m < d_n \quad \implies \quad a_{m,n} = 0.$$

Then, D being the (possibly unbounded) diagonal operator with entries d_n , we have:

$$\|D^{-1}AD\| \leq \|A\|.$$

For the convenience of the reader, we reproduce the proof.

Proof. Let \mathbb{C}_0 be the right-half plane $\mathbb{C}_0 = \{z \in \mathbb{C}; \Re z > 0\}$. We set $H_N = \text{span}\{e_n; n \leq N\}$ and

$$A_N = P_N A J_N,$$

where P_N is the orthogonal projection from ℓ_2 onto H_N and J_N the canonical injection from H_N into ℓ_2 . We consider, for $z \in \overline{\mathbb{C}_0}$:

$$A_N(z) = D^{-z} A_N D^z: H_N \rightarrow H_N,$$

where $D^z(e_n) = d_n^z e_n$.

If $(a_{m,n}(z))_{m,n}$ is the matrix of $A_N(z)$ on the basis $\{e_n; n \leq N\}$ of H_N , we clearly have:

$$a_{m,n}(z) = a_{m,n}(d_n/d_m)^z.$$

In particular, we have, thanks to (3.7):

$$a_{m,n}(z) = 0 \quad \text{if } d_m < d_n,$$

and

$$|a_{m,n}(z)| \leq \sup_{k,l} |a_{k,l}| := M, \quad \text{for all } z \in \overline{\mathbb{C}_0}.$$

Since $\|A_N(z)\|^2 \leq \|A_N(z)\|_{HS}^2 = \sum_{m,n \leq N} |a_{m,n}(z)|^2 \leq (N+1)^2 M^2$, we get:

$$\|A_N(z)\| \leq (N+1) M \quad \text{for all } z \in \overline{\mathbb{C}_0}.$$

Let us consider the function $u: \overline{\mathbb{C}_0} \rightarrow \overline{\mathbb{C}_0}$ defined by:

$$(3.8) \quad u_N(z) = \|A_N(z)\|.$$

This function u_N is continuous on $\overline{\mathbb{C}_0}$, bounded above by $(N+1)M$, and subharmonic in \mathbb{C}_0 . Moreover, thanks to (3.7), the maximum principle gives:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) = \sup_{\partial\mathbb{C}_0} u_N(z).$$

Since $\|D^z\| = \|D^{-z}\| = 1$ for $z \in \partial\mathbb{C}_0$, we have $\|A_N(z)\| \leq \|A_N\|$ for $z \in \partial\mathbb{C}_0$, and we get:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) \leq \|A_N\| \leq \|A\|.$$

In particular $u_N(1) \leq \|A\|$, and, letting N going to infinity, we get $\|D^{-1}AD\| \leq \|A\|$. \square

Proof of Theorem 3.6. First, if $|\varphi'(0)| = 1$, we have $\varphi(z) = \alpha z$ for some α with $|\alpha| = 1$, and the result is trivial.

So, we assume that $|\varphi'(0)| < 1$. Then, by Lemma 3.7, there exists $\rho > 0$ such that, for all m, n :

$$|\widehat{\varphi^n}(m)| \leq \exp\left(-\frac{1}{2}[(1+\rho)n - m]\right).$$

Since $\varphi(0) = 0$, we also know that $\widehat{\varphi^n}(m) = 0$ if $m < n$.

Take $\delta = \rho/2$ and use property (3.6): there exists $C > 0$ such that:

$$\frac{\beta_m}{\beta_n} \leq C \quad \text{when } m \geq (1+\delta)n.$$

Define now a new sequence $\gamma = (\gamma_n)$ as:

$$\gamma_n = \max\left\{\beta_n, \sup_{m > (1+\delta)n} \beta_m\right\}.$$

We have:

- 1) $\beta_n \leq \gamma_n \leq C\beta_n$;
- 2) $\gamma_m \leq \gamma_n$ if $m \geq (1+\delta)n$.

Item 1) implies that $H^2(\gamma) = H^2(\beta)$, and we are reduced to prove that $C_\varphi: H^2(\gamma) \rightarrow H^2(\gamma)$ is bounded.

Let $A = (a_{m,n}) = (\widehat{\varphi^n}(m))$. We have to prove that

$$B = (\gamma_m^{1/2} \gamma_n^{-1/2} a_{m,n})_{m,n}$$

represents a bounded operator on ℓ_2 .

Define the matrix

$$A_1 = (a_{m,n} \mathbb{1}_{\{(m,n); m \leq (1+\delta)n\}})_{m,n}$$

and set $A_2 = A - A_1$. Define analogously B_1 and $B_2 = B - B_1$.

Then A_1 is a Hilbert-Schmidt operator, because (recall that $a_{m,n} = 0$ if $m < n$)

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{(1+\delta)n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \exp(-[(1+\rho)n - m]) \\ &\leq \sum_{n=1}^{\infty} (\delta n + 1) \exp(-\delta n) < \infty. \end{aligned}$$

Now, β is bounded above and has a polynomial minoration, so, for some positive constants C_1, C_2 , and α , we have:

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \frac{\gamma_m}{\gamma_n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \frac{C_1}{n^{-\alpha}} \exp(-\delta n) \\ &\leq \sum_{n=1}^{\infty} C_2 n^{\alpha+1} \exp(-\delta n) < \infty, \end{aligned}$$

meaning that B_1 is a Hilbert-Schmidt operator.

Since A is bounded, it follows that $A_2 = A - A_1$ is bounded. Remark that, writing $A_2 = (\alpha_{m,n})_{m,n}$, we have, with $d_n = 1/\sqrt{\gamma_n}$:

$$d_m < d_n \implies \gamma_m > \gamma_n \implies m < (1+\delta)n \implies \alpha_{m,n} = 0.$$

Hence we can apply Lemma 3.8 to the matrix A_2 , and it ensues that B_2 is bounded, and therefore that $B = B_1 + B_2$ is bounded as well, as wanted. \square

As a corollary of Theorem 3.6, we can provide the following example.

Theorem 3.9. *There exists a bounded sequence β , with a polynomial minoration, but which is not essentially decreasing, for which every composition operator with symbol vanishing at 0 is bounded on $H^2(\beta)$, with nevertheless $\sup_{\varphi(0)=0} \|C_\varphi\| = \infty$.*

It should be noted that for this weight, the composition operators are not all bounded, as we will see in Proposition 4.4.

Proof. Define $\beta_n = 1$ for $n \leq 3!$, and, for $k \geq 3$:

$$\left\{ \begin{array}{l} \beta_n = \frac{1}{k!} \quad \text{for } k! < n \leq (k+1)! - 2 \text{ and for } n = (k+1)! \\ \beta_n = \frac{1}{(k+1)!} \quad \text{for } n = (k+1)! - 1. \end{array} \right.$$

Note that, for $m > n$, we have $\beta_m > \beta_n$ only if $n = (k+1)! - 1$ and $m = (k+1)! = n + 1$, for some $k \geq 3$.

However, β is not essentially decreasing since, for every $k \geq 3$, we have $\beta_{n+1}/\beta_n = k + 1$ if $n = (k+1)! - 1$.

The sequence β has a polynomial minoration because $\beta_n \geq 1/(2n)$ for all $n \geq 1$. In fact, for $k \geq 3$, we have $\beta_n \geq (k+1)/n \geq 1/n$ if $k! < n \leq (k+1)! - 2$ or if $n = (k+1)!$; and for $n = (k+1)! - 1$, we have $n\beta_n = [(k+1)! - 1]/(k+1)! \geq 1/2$.

Now, it remains to check (3.6) in order to apply Theorem 3.6 and finish the proof of Theorem 3.9. Note first that we have $\beta_m/\beta_n \leq 1$ if $m \geq n + 2$. Next, for given $\delta > 0$, there exists an integer N such that $(1 + \delta)n \geq n + 2$ for every $n \geq N$, so $\beta_m/\beta_n \leq 1$ if $m \geq (1 + \delta)n$ and $n \geq N$. It suffices to take $C = \max_{1 \leq n \leq N} \beta_{n+1}/\beta_n$ to obtain (3.6). The last assertion follows from Proposition 3.2. \square

4 Necessity of the Δ_2 -condition

In this section, we will show that, for essentially decreasing sequences β , the Δ_2 -condition is necessary for having boundedness of composition operators on $H^2(\beta)$. We will indeed show slightly more.

Theorem 4.1. *Let β be such that, for some $a \in (0, 1)$, T_a induces a bounded composition operator on $H^2(\beta)$. Then β satisfies:*

$$(\exists 0 < \delta < 1/3) (\forall n \geq 1) (\exists E_n \subseteq [(1 - 2\delta)n, (1 - \delta)n]) \quad \text{with}$$

$$|E_n| \geq \delta n \quad \text{and} \quad \beta_n \geq \delta \frac{1}{|E_n|} \sum_{m \in E_n} \beta_m.$$

In particular, if β is essentially decreasing, then β satisfies the Δ_2 -condition.

In order to prove Theorem 4.1, we need several preliminary lemmas. The first one is standard, but we give it for convenience.

Lemma 4.2. *Let $a \in (0, 1)$ and let*

$$P_{-a}(x) = \frac{1 - a^2}{1 + 2a \cos x + a^2}$$

be the Poisson kernel at the point $-a$. Then, for all $x \in [-\pi, \pi]$:

$$(4.1) \quad T_a(e^{ix}) = \exp[i h_a(x)],$$

where

$$(4.2) \quad h_a(x) = \int_0^x P_{-a}(t) dt.$$

Proof. For $t \in [-\pi, \pi]$, write:

$$\psi(t) := \frac{e^{it} + a}{1 + a e^{it}} = \exp(i u(t)),$$

with u a real-valued, \mathcal{C}^1 function on $[-\pi, \pi]$ such that $u(0) = 0$. This is possible since $|\psi(e^{it})| = 1$ and $\psi(0) = 1$. Differentiating both sides with respect to t , we get:

$$i e^{it} \frac{1 - a^2}{(1 + a e^{it})^2} = i u'(t) \frac{e^{it} + a}{1 + a e^{it}}.$$

This implies

$$u'(t) = \frac{1 - a^2}{|1 + a e^{it}|^2} = P_{-a}(t),$$

and the result follows since $u(0) = 0 = h_a(0)$. \square

4.1 Main lemma and proof of Theorem 4.1

To prove Theorem 4.1, we need the following lemma. The proof of this lemma uses Lemma 4.2 and a van der Corput type estimate, inspired from [21, pp. 72–73]. We thank R. Zarouf [22] for interesting recent informations in this respect, related to his joint work with O. Szehr on the Schäffer problem (see [20], in which the authors are primarily concerned with upper bounds).

The first version of our paper was put on arXiv at the end of November 2020. Since then, the paper [1] was put on arXiv on July 2021, where sharp estimates of powers of Blaschke factors are given (see also K. Fouchet's thesis [6]), with different purposes (strongly annular analytic functions). However, in our case, our proof is much simpler.

Recall that we have set:

$$(4.3) \quad [T_a(z)]^n = \sum_{m=0}^{\infty} a_{m,n} z^m.$$

Lemma 4.3. *Let $a \in (0, 1)$. We set:*

$$(4.4) \quad \tau = \frac{1 + a}{1 - a} > 1$$

and write:

$$(4.5) \quad \tau^{-1} = 1 - 3\mu,$$

with $\mu = \mu_a \in (0, 1/3)$. For every fixed positive integer n , let:

$$(4.6) \quad J_n = [(1 - 2\mu)n, (1 - \mu)n].$$

Then, there exists $\delta = \delta_a > 0$ such that, for every n large enough, there exists a set of indices $E_n \subseteq J_n$ with cardinality $|E_n| \geq \delta n$ and such that:

$$(4.7) \quad m \in E_n \implies |a_{m,n}| \geq \delta n^{-1/2}.$$

Proof of Theorem 4.1. Set $M = \|C_{T_a}\|$. We have:

$$(4.8) \quad \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|T_a^n\|^2 = \|C_{T_a}(z^n)\|^2 \leq \|C_{T_a}\|^2 \|z^n\|^2 = M^2 \beta_n.$$

so, by Lemma 4.3, since $E_n \subseteq J_n = [(1 - 2\mu)n, (1 - \mu)n]$:

$$M^2 \beta_n \geq \sum_{m \in E_n} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} \sum_{m \in E_n} \beta_m \geq \delta^3 |E_n|^{-1} \sum_{m \in E_n} \beta_m.$$

This proves (changing δ) the first part of Theorem 4.1. Next, assume that β is essentially decreasing. We may, and do, assume that β is non-increasing. We set, for x not an integer, $\beta_x = \beta_k$ with k the least integer greater than x . The above implies, for all integers $n \geq 1$:

$$\beta_n \geq (\delta^3/M^2) |E_n|^{-1} |E_n| \beta_{(1-\mu)n} \geq c \beta_{(1-\mu)n}.$$

Let $r \geq 1$ such that $(1 - \mu)^r \leq 1/2$; we have:

$$\beta_n \geq c^r \beta_{(1-\mu)^r n} \geq c^r \beta_{n/2},$$

so β satisfies the Δ_2 -condition. □

A consequence of Theorem 4.1 is the following result.

Proposition 4.4. *For the weight β constructed in the proof of Theorem 3.9, no automorphism T_a with $0 < a < 1$ can be bounded.*

Proof. Consider the necessary condition for the boundedness of C_{T_a} in Theorem 4.1:

$$(4.9) \quad \beta_n \geq \delta \frac{1}{|E_n|} \sum_{m \in E_n} \beta_m.$$

For the weight β constructed in the proof of Theorem 3.9, we are going to see that this condition (4.9) is not satisfied for $n = (k + 1)! - 1 =: n_k$.

Indeed, for this n , the left-hand side of (4.9) is equal to $1/(k + 1)!$ and the right-hand side to $\delta/k!$ since $(1 - 2\delta)n_k > k!$ for k large, so that all β_m are equal to $1/k!$ for $m \in E_{n_k}$. This ends the proof. □

4.2 Proof of Lemma 4.3

To prove Lemma 4.3, we will use a variant of [21, Lemma 4.6 p. 72] on the stationary phase method. A careful reading of the proof in [21, p. 72] gives the version below, which allows the derivative F' of F to vanish at some point, as occurs in our situation. For sake of completeness, we will give a proof, however postponed.

Proposition 4.5 (Stationary phase). *Let F be real function on the interval $[A, B]$, with continuous derivatives up to the third order and $F'' > 0$ throughout $]A, B[$. Assume that there is a (unique) point c in $]A, B[$ such that $F'(c) = 0$, and that, for some positive numbers λ_2, λ_3 , and η , the following assertions hold:*

- 1) $[c - \eta, c + \eta] \subseteq [A, B]$;
- 2) $F''(x) \geq \lambda_2$ for all $x \in [c - \eta, c + \eta]$;
- 3) $|F'''(x)| \leq \lambda_3$ for all $x \in [A, B]$.

Then:

$$(4.10) \quad \int_A^B e^{iF(x)} dx = \sqrt{2\pi} \frac{e^{i(F(c)+\pi/4)}}{|F''(c)|^{1/2}} + O\left(\frac{1}{\eta\lambda_2} + \eta^4\lambda_3\right),$$

where the O involves an absolute constant.

Proof of Lemma 4.3. We turn to the problem of bounding $a_{m,n}$ from below, in the case $m \in J_n$, and only in that case. Since $\inf_{[0,\pi]} P_{-a} = \tau^{-1} < \sup_{[0,\pi]} P_{-a} = \tau$, there exists a unique point $x_m = x_{m,n} \in [0, \pi]$ such that

$$nP_{-a}(x_m) - m = n \frac{(1 - a^2)}{1 + 2a \cos x_m + a^2} - m = 0,$$

or else:

$$(4.11) \quad \cos x_m = \frac{n}{m} \frac{1 - a^2}{2a} - \frac{1 + a^2}{2a}.$$

The point is that if $m \in J_n$, x_m can approach neither 0 nor π , so that $\sin x_m \geq \delta_a > 0$; more precisely, the definition of J_n and (4.11) imply that $\pi/4 \leq x_m \leq \pi/2$.

With h_a the function of Lemma 4.2, the Fourier formulas give, since $a_{m,n}$ is real, or since $h_a(x) - mx$ is odd:

$$2\pi a_{m,n} = \int_{-\pi}^{\pi} \exp i[nh_a(x) - mx] dx = 2 \Re I_{m,n},$$

where

$$(4.12) \quad I_{m,n} = \int_0^{\pi} \exp i[nh_a(x) - mx] dx.$$

Write:

$$(4.13) \quad I_{m,n} = \int_0^{\pi} \exp[i F_m(x)] dx,$$

with:

$$(4.14) \quad F_m(x) = nh_a(x) - mx = n \int_0^x P_{-a}(t) dt - mx.$$

We have:

$$(4.15) \quad F'_m(x) = n P_{-a}(x) - m.$$

We will now proceed in two steps, first giving good lower bounds for $|I_{m,n}|$, then showing that the argument of $I_{m,n}$ is often far from $\pi/2 \pmod{\pi}$. Then, we will be done.

First step. We will prove that:

$$(4.16) \quad I_{m,n} = \sqrt{2\pi} n^{-1/2} \frac{e^{i(F_m(x_m)+\pi/4)}}{\sqrt{|h''_a(x_m)|}} + O(n^{-3/5}),$$

where the O only depends on a .

Note that $3/5 > 1/2$ and $F''_m = n h''_a$.

To get (4.16), we will show that Theorem 4.5 is applicable with:

$$[A, B] = [0, \pi], \quad c = x_m, \quad \lambda_2 = \kappa_0 n, \quad \lambda_3 = C_0 n, \quad \eta = (\lambda_2 \lambda_3)^{-1/5}.$$

The parameter η is chosen in order to make both error terms in Theorem 4.5 equal: $\frac{1}{\eta \lambda_2} = \eta^4 \lambda_3$; so:

$$\eta = \kappa n^{-2/5}$$

and

$$(4.17) \quad \frac{1}{\eta \lambda_2} + \eta^4 \lambda_3 = \tilde{\kappa} n^{-3/5} = O(n^{-3/5})$$

(with $\kappa = (\kappa_0 C_0)^{-1/5}$ and $\tilde{\kappa} = 2/\kappa_0 \kappa$).

The slight technical difficulty encountered here is that $F''_m(x)$ vanishes at 0 and π . But Theorem 4.5 covers this case. We have

$$F''_m(x) = n P'_{-a}(x) = 2a(1-a^2) \frac{\sin x}{(1+2a \cos x + a^2)^2} n,$$

and there are some positive κ_0 and σ such that

$$(4.18) \quad F''_m(x) \geq \kappa_0 n = \lambda_2 \quad \text{for } x \in [\sigma, \pi - \sigma].$$

Now (for n large enough), $[x_m - \eta, x_m + \eta] \subseteq [\sigma, \pi - \sigma]$. Hence the assumptions 1) and 2) of Proposition 4.5 are satisfied.

Finally, since $F_m(x) = n h_a(x) - m x$, and h_a is \mathcal{C}^∞ on \mathbb{R} , we have, for all $x \in [0, \pi]$:

$$|F'''_m(x)| \leq C_0 n = \lambda_3,$$

and assertion 3) of Proposition 4.5 holds.

With (4.17) this ends the proof of (4.16), once we remarked that $n h''_a(x_m) = F''_m(x_m)$.

Note that, since $|h_a''(x_m)| \leq M_a$, we get that $|I_{m,n}| \geq \delta n^{-1/2}$ when $m \in J_n$.

Second step. The mean-value theorem gives, for $m \in J_n$:

$$(4.19) \quad |\sin x_m| \geq \delta \quad \text{and} \quad x_{m+1} - x_m \approx \cos x_m - \cos x_{m+1}.$$

We also have, for $x \in \mathcal{J} = [1 - 2\mu, 1 - \mu]$, with another constant δ :

$$(4.20) \quad \delta \leq P'_{-a}(x) = 2a(1 - a^2) \frac{\sin x}{(1 + 2a \cos x + a^2)^2} \leq \delta^{-1}.$$

We now claim that

$$(4.21) \quad x_{m+1} - x_m \approx n^{-1} \quad \text{for } m \in J_n.$$

Indeed, since $m \in J_n$, we have, using (4.11):

$$\cos x_m - \cos x_{m+1} = \frac{1 - a^2}{2a} \frac{n}{m(m+1)} \approx \frac{n}{m^2} \approx n^{-1}.$$

In view of (4.19), this proves (4.21).

Now, according to (4.16), when $m \in J$, the main term in $I_{m,n}$ is

$$A_{m,n} := n^{-1/2} \frac{\sqrt{2\pi}}{\sqrt{|h_a''(x_m)|}} e^{i(F_m(x_m) + \pi/4)},$$

and its argument θ_m is $F_m(x_m) + \pi/4$. Going from m to $m+1$, the variation $F_{m+1}(x_{m+1}) - F_m(x_m)$ of this argument is

$$\begin{aligned} \theta_{m+1} - \theta_m &= n \int_{x_m}^{x_{m+1}} \left(P_{-a}(t) - \frac{m}{n} \right) dt - x_{m+1} \\ &= n \int_{x_m}^{x_{m+1}} \left(P_{-a}(t) - P_{-a}(x_m) \right) dt - x_{m+1}. \end{aligned}$$

But, due to (4.20), this implies:

$$\begin{aligned} \theta_{m+1} - \theta_m &= -x_{m+1} + O\left(n \int_{x_m}^{x_{m+1}} (t - x_m) dt\right) \\ &= -x_{m+1} + O(n(x_{m+1} - x_m)^2) = -x_{m+1} + O(n^{-1}). \end{aligned}$$

Since $-\sigma \leq -x_{m+1} \leq \pi - \sigma$, the variation of θ_m is thus regular. As a consequence, for a positive proportion E_n of the indices $m \in J_n$, the argument θ_m will belong to a subarc of \mathbb{T} which lies δ -apart from $\pm\pi/2$, implying $\cos \theta_m \geq \delta$, or else:

$$|E_n| \geq \delta n,$$

and, for all $m \in E_n$:

$$\Re A_{m,n} \geq \delta |A_{m,n}| \geq \delta n^{-1/2}.$$

It follows that, for $m \in E_n$:

$$\Re I_{m,n} \geq \delta n^{-1/2} - C n^{-3/5} \geq \tilde{\delta} n^{-1/2}.$$

Since $a_{m,n} = \pi^{-1} \Re I_{m,n}$, that ends the proof of Lemma 4.3. \square

We now pass to the proof of Proposition 4.5. The following lemma can be found in [15, Lemma 1, page 47].

Lemma 4.6. *Let $F: [u, v] \rightarrow \mathbb{R}$, with $u < v$, be a C^2 -function with $F'' > 0$, and F' not vanishing on $[u, v]$. Let*

$$J = \int_u^v e^{iF(x)} dx.$$

Then:

- a) if $F' > 0$ on $[u, v]$, then $|J| \leq \frac{2}{F'(u)}$;
- b) If $F' < 0$ on $[u, v]$, then $|J| \leq \frac{2}{|F'(v)|}$.

Proof of Proposition 4.5. Write now the integral I of Proposition 4.5 on $[A, B]$ as $I = I_1 + I_2 + I_3$ with:

$$I_1 = \int_A^{c-\eta} e^{iF(x)} dx, \quad I_2 = \int_{c-\eta}^{c+\eta} e^{iF(x)} dx, \quad I_3 = \int_{c+\eta}^B e^{iF(x)} dx.$$

Lemma 4.6 with $u = A$ and $v = c - \eta$ implies:

$$(4.22) \quad |I_1| \leq \frac{2}{|F'(c-\eta)|} \leq \frac{2}{\eta\lambda_2},$$

where, for the last inequality, we just have to write

$$|F'(c-\eta)| = F'(c) - F'(c-\eta) = \eta F''(\xi)$$

for some $\xi \in [c-\eta, c]$ so that $F''(\xi) \geq \lambda_2$.

Similarly, Lemma 4.6 with $u = c + \eta$ and $v = B$ implies

$$(4.23) \quad |I_3| \leq \frac{2}{F'(c+\eta)} \leq \frac{2}{\eta\lambda_2}.$$

We can now estimate I_2 . The Taylor formula shows that

$$F(x) = F(c) + \frac{(x-c)^2}{2} F''(c) + R,$$

with

$$|R| \leq \frac{|x-c|^3}{6} \lambda_3.$$

Hence

$$I_2 = e^{iF(c)} \int_0^\eta 2 \exp\left(\frac{i}{2} x^2 F''(c)\right) dx + S$$

with

$$|S| \leq \lambda_3 \int_0^\eta \frac{x^3}{3} dx = \frac{\eta^4}{12} \lambda_3.$$

Finally, set

$$K = \int_0^\eta 2 \exp\left(\frac{i}{2} x^2 F''(c)\right) dx.$$

We make the change of variable $x = \sqrt{\frac{2}{F''(c)}} \sqrt{t}$. Recall that $\int_0^\infty \frac{e^{it}}{\sqrt{t}} dt = \sqrt{\pi} e^{i\pi/4}$ is the classical Fresnel integral, and that an integration by parts gives, for $m > 0$:

$$\left| \int_m^\infty \frac{e^{it}}{\sqrt{t}} dt \right| \leq \frac{2}{\sqrt{m}}.$$

Therefore, with $m = \frac{\eta^2}{2} F''(c)$:

$$K = \sqrt{\frac{2}{F''(c)}} \int_0^m \frac{e^{it}}{\sqrt{t}} dt = \sqrt{\frac{2\pi}{F''(c)}} e^{i\pi/4} + R_m,$$

with

$$|R_m| \leq C \sqrt{\frac{1}{F''(c)}} \frac{1}{\sqrt{m}} \leq \frac{C}{\eta \lambda_2}.$$

All in all, we proved that

$$(4.24) \quad I_2 = \sqrt{\frac{2\pi}{F''(c)}} \exp[i(F(c) + \pi/4)] + O\left(\frac{1}{\eta \lambda_2} + \eta^4 \lambda_3\right).$$

and the same estimate holds for I , thanks to (4.22) and (4.23).

We have hence proved Proposition 4.5. □

5 Some results on multipliers

The set $\mathcal{M}(H^2(\beta))$ of multipliers of $H^2(\beta)$ is by definition the vector space of functions h analytic on \mathbb{D} and such that $hf \in H^2(\beta)$ for all $f \in H^2(\beta)$. When $h \in \mathcal{M}(H^2(\beta))$, the operator M_h of multiplication by h is bounded on $H^2(\beta)$ by the closed graph theorem. The space $\mathcal{M}(H^2(\beta))$ equipped with the operator norm is a Banach space. We note the obvious property:

$$(5.1) \quad \mathcal{M}(H^2(\beta)) \hookrightarrow H^\infty \quad \text{contractively.}$$

Indeed, if $h \in \mathcal{M}(H^2(\beta))$, we easily get for all $w \in \mathbb{D}$:

$$M_h^*(K_w) = \overline{h(w)} K_w;$$

so that taking norms and simplifying, we are left with $|h(w)| \leq \|M_h\|$, showing that $h \in H^\infty$ with $\|h\|_\infty \leq \|M_h\|$.

Proposition 5.1. *We have $\mathcal{M}(H^2(\beta)) = H^\infty$ isomorphically if and only if β is essentially decreasing.*

Proof. The sufficient condition is proved in [13, beginning of the proof of Proposition 3.16]. For the necessity, we have $\|M_h\| \approx \|h\|_\infty$ for every $h \in H^\infty$. Now, for $m > n$ (recall that $e_n(z) = z^n$):

$$e_m(z) = z^{m-n} z^n = (M_{e_{m-n}} e_n)(z);$$

so, since $\|M_{e_{m-n}}\| \leq C \|e_{m-n}\|_\infty = C$ for some positive constant C :

$$\beta_m = \|e_m\|^2 \leq C^2 \|e_n\|^2 = C^2 \beta_n. \quad \square$$

In [13, Section 3.6], we gave the following notion of *admissible* Hilbert space of analytic functions.

Definition 5.2. *A Hilbert space H of analytic functions on \mathbb{D} , containing the constants, and with reproducing kernels K_a , $a \in \mathbb{D}$, is said admissible if:*

- (i) H^2 is continuously embedded in H ;
- (ii) $\mathcal{M}(H) = H^\infty$;
- (iii) the automorphisms of \mathbb{D} induce bounded composition operators on H ;
- (iv) $\frac{\|K_a\|_H}{\|K_b\|_H} \leq h\left(\frac{1-|b|}{1-|a|}\right)$ for $a, b \in \mathbb{D}$, where $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-decreasing function.

We proved in that paper that every weighted Hilbert space $H^2(\beta)$ with β non-increasing is admissible, under the additional hypothesis that the automorphisms of \mathbb{D} induce bounded composition operators. In view of Theorem 3.3, we get the following result.

Proposition 5.3. *Let β be essentially decreasing that satisfies the Δ_2 -condition. Then $H^2(\beta)$ is admissible.*

Let us give a different proof.

Proof. Because β is essentially decreasing, item (i) holds, as well as item (ii), by Proposition 5.1. Item (iii) is Theorem 3.3. It remains to show (iv). We may assume that β is non-increasing.

Let $0 < s < r < 1$.

Without loss of generality, we may assume that $r, s \geq 1/2$. It is enough to prove:

$$(5.2) \quad \|K_r\|^2 \leq C \|K_{r^2}\|^2$$

for some constant $C > 1$. Indeed, iteration of (5.2) gives:

$$\|K_r\|^2 \leq C^k \|K_{r^{2^k}}\|^2$$

and if k is the smallest integer such that $r^{2^k} \leq s$, we have $2^{k-1} \log r > \log s$ and $2^k \leq D \frac{1-s}{1-r}$ where D is a numerical constant. Writing $C = 2^\alpha$ with $\alpha > 1$, we obtain:

$$\left(\frac{\|K_r\|}{\|K_s\|} \right)^2 \leq C^k = (2^k)^\alpha \leq D^\alpha \left(\frac{1-s}{1-r} \right)^\alpha.$$

To prove (5.2), we pick some $M > 1$ such that $\beta_{2n} \geq M^{-1}\beta_n$ for all $n \geq 1$ and write $t = r^2$. We have:

$$\|K_r\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_{2n}} + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\beta_{2n-1}},$$

implying, since $\beta_{2n-1} \geq \beta_{2n} \geq M^{-1}\beta_n$ and $t^{2n-1} \leq 4t^{2n}$:

$$\|K_r\|^2 \leq \frac{1}{\beta_0} + M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} + 4M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} \leq 5M \|K_t\|^2. \quad \square$$

The notion of admissible Hilbert space H is useful for the set of conditional multipliers:

$$\mathcal{M}(H, \varphi) = \{w \in H; w(f \circ \varphi) \in H \text{ for all } f \in H\}.$$

As a corollary of [13, Theorem 3.18] we get:

Corollary 5.4. *Let β be essentially decreasing and satisfying the Δ_2 -condition. Then:*

- 1) $\mathcal{M}(H^2, \varphi) \subseteq \mathcal{M}(H^2(\beta), \varphi)$;
- 2) $\mathcal{M}(H^2(\beta), \varphi) = H^2(\beta)$ if and only if $\|\varphi\|_\infty < 1$;
- 3) $\mathcal{M}(H^2(\beta), \varphi) = H^\infty$ if and only if φ is a finite Blaschke product.

6 Miscellaneous remarks

Some of the results of this paper can slightly be improved.

6.1 Conditions on the weight β

First, we say that a sequence (β_n) of positive numbers is *slowly oscillating* if there is a function $\rho: (0, \infty) \rightarrow (0, \infty)$ that is bounded on each compact subset of $(0, \infty)$ for which:

$$\frac{\beta_m}{\beta_n} \leq \rho\left(\frac{m}{n}\right).$$

This clearly amounts to say that, for some positive constants $C_1 < C_2$, we have:

$$C_1 \leq \frac{\beta_m}{\beta_n} \leq C_2 \quad \text{when } n/2 \leq m \leq 2n.$$

Every essentially decreasing sequence with the Δ_2 -condition is slowly oscillating.

Proposition 6.1. *The following holds:*

- 1) *every slowly oscillating sequence has a polynomial minoration;*
- 2) *there are bounded sequences which are slowly oscillating, but not essentially decreasing.*

Proof. 1) is clear, because if $2^j \leq n < 2^{j+1}$, then

$$\beta_n \geq C^{-1} \beta_{2^j} \geq C^{-j-1} \beta_1 \geq C^{-1} \beta_1 n^{-\alpha},$$

with $\alpha = \log C / \log 2$.

2) We define β_n as follows. Let (a_k) be an increasing sequence of positive square integers such that $\lim_{k \rightarrow \infty} a_{k+1}/a_k = \infty$, for example $a_k = 4^{k^2}$, and let $b_k = \sqrt{a_k a_{k+1}}$; with our choice, this is an integer and we clearly have $a_k < b_k < a_{k+1}$. We set:

$$\beta_n = \begin{cases} a_k/n & \text{for } a_k \leq n < b_k \\ (a_k/b_k^2)n = (1/a_{k+1})n & \text{for } b_k \leq n < a_{k+1}. \end{cases}$$

This sequence (β_n) is slowly oscillating by construction. Indeed, it suffices to check that for $a_k \leq n/2 < b_k \leq n < a_{k+1}$, the quotient β_m/β_n remains lower and upper bounded when $n/2 \leq m \leq n$. But for $n/2 \leq m < b_k$, we have

$$\frac{\beta_m}{\beta_n} = \frac{a_k/m}{n/a_{k+1}} = \frac{a_k a_{k+1}}{mn} = \frac{b_k^2}{mn},$$

which is $\leq 2b_k^2/n^2 \leq 2$ and $\geq b_k^2/n^2 \geq (n/2)^2/n^2 = 1/4$; and for $b_k \leq m$, we have

$$\frac{\beta_m}{\beta_n} = \frac{m/a_{k+1}}{n/a_{k+1}} = \frac{m}{n} \in [1/2, 1].$$

However, though (β_n) is bounded, since $\beta_n \leq 1$ for $a_k \leq n < b_k$ and, for $b_k \leq n < a_{k+1}$,

$$\beta_n \leq \beta_{a_{k+1}-1} = \frac{1}{a_{k+1}} (a_{k+1} - 1) \leq 1,$$

it is not essentially decreasing, since

$$\frac{\beta_{a_{k+1}-1}}{\beta_{b_k}} = \frac{1}{\sqrt{a_k a_{k+1}}} (a_{k+1} - 1) \sim \sqrt{\frac{a_{k+1}}{a_k}} \xrightarrow{k \rightarrow \infty} \infty. \quad \square$$

By a slight modification (change the value of the constants in the definition of β_n), we could obtain a sequence which is slowly oscillating, tends to zero, yet again not essentially decreasing.

Now, Theorem 3.3 admits the following variant.

Theorem 6.2. *Let (β_n) be a sequence of positive numbers which is bounded above and slowly oscillating. Then all symbols that extend analytically in a neighborhood of \mathbb{D} induce a bounded composition operator on $H^2(\beta)$.*

It is the case, for example, for finite Blaschke products. The proof follows that of Theorem 3.3, with the help of the following lemma.

Lemma 6.3. *Let (β_n) be a sequence of positive numbers which is bounded above and slowly oscillating. Let $A = (a_{m,n})_{m,n}$ be the matrix of a bounded operator on ℓ_2 . Assume that, for constants $C_1 < 1$, $C_2 > 1$, and c, b , we have:*

- 1) $|a_{m,n}| \leq c e^{-bn}$ for $m \leq C_1 n$;
- 2) $|a_{m,n}| \leq c e^{-bm}$ for $m \geq C_2 n$.

Then the matrix $\tilde{A} = \left(a_{m,n} \sqrt{\frac{\beta_m}{\beta_n}} \right)$ also defines a bounded operator on ℓ_2 .

Sketch of proof. The matrix \tilde{A} is Hilbert-Schmidt far from the diagonal since, with $\lambda \geq \beta_n \geq \delta n^{-\alpha}$, we have:

$$\sum_{m < C_1 n} |a_{m,n}|^2 \beta_m / \beta_n \leq \sum_{m < C_1 n} \lambda \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n \geq 1} n^{\alpha+1} e^{-bn} < \infty,$$

and

$$\sum_{m > C_2 n} |a_{m,n}|^2 \beta_m / \beta_n \leq \sum_{m > C_2 n} \lambda \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n \geq 1} n^\alpha \left(\sum_{m > C_2 n} e^{-bm} \right).$$

Since β_m / β_n remains bounded from above and below around the diagonal, the matrix \tilde{A} behaves like A near the diagonal. \square

Remark. The proof shows that, instead of 1) and 2), it is enough to have:

$$\sum_{m < C_1 n} n^{\alpha+1} |a_{m,n}|^2 < \infty \quad \text{and} \quad \sum_{m > C_2 n} m^\alpha |a_{m,n}|^2 < \infty.$$

Moreover the proof also shows that when β is slowly oscillating, if we set $E = \{(m, n); C_1 n \leq m \leq C_2 n\}$, then the matrix $(\sqrt{\beta_m / \beta_n} \mathbb{1}_E(m, n))$ is a Schur multiplier over *all* the bounded matrices, while Kacnel'son's theorem (Theorem 3.5) says that, if $\gamma = (\gamma_n)$ is non-increasing, the matrix (γ_m / γ_n) is a Schur multiplier of all bounded *lower-triangular* matrices.

Proof of Theorem 6.2. We first prove that the assumptions of Lemma 6.3 are satisfied with $a_{m,n} = \widehat{\varphi^n}(m)$. This works nearly as in Lemma 3.4. For every symbol φ , let $M(r) = \sup_{|z|=r} |\varphi(z)|$ and write $M(1/e) = e^{-\delta}$, with $\delta > 0$. Let A be the matrix of C_φ , with respect to the canonical basis of H^2 . The Cauchy inequalities give, if $m \leq (\delta/2)n$:

$$|a_{m,n}| \leq [M(1/e)]^n e^m \leq e^{m-\delta n} \leq e^{-\delta/2n}$$

When φ is analytic in a neighborhood of $\overline{D(0, R)}$ with $R > 1$, the Cauchy inequalities give, writing $M(R) = e^\rho$ and $R = e^\delta$, for $m \geq C_2 n$, and a suitable constant C_2 (take $C_2 = (2\rho)/\delta$ for instance):

$$|a_{m,n}| \leq \frac{[M(R)]^n}{R^m} \leq e^{n\rho - \delta m} \leq e^{-(\delta/2)m}.$$

We now conclude with Lemma 6.3. \square

6.2 Singular inner functions

Let $a > 0$ and let I_a be the singular inner function defined by

$$(6.1) \quad I_a(z) = \exp\left(-a \frac{1+z}{1-z}\right) = \sum_{m=0}^{\infty} c_m(a) z^m.$$

If n is a positive integer, we have $I_n(z) = [I_1(z)]^n$ and we write

$$I_n(z) = \sum_{m=0}^{\infty} a_{m,n} z^m, \quad \text{with } a_{m,n} = c_m(n).$$

We rely on the following lemma, familiar to experts in orthogonal polynomials and special functions, but maybe not so much as regards the uniformity, essential for our present purposes (see [16] or [12]).

Lemma 6.4. *It holds*

$$(6.2) \quad c_m(n) = c n^{1/4} m^{-3/4} \cos(2\sqrt{2nm} + \pi/4) + R_m(n) =: M_m(n) + R_m(n),$$

where $c = \pi^{-1/2} 2^{1/4}$ and where $|R_m(n)| \leq K \sqrt{n} m^{-5/4}$, with K some numerical constant.

We use the following [16, p. 253] and [19, p. 198], where $L_m^{(\alpha)}$ denotes the generalized Laguerre polynomial of degree m with parameter α .

Theorem 6.5. *With the notation of (6.1), we have*

$$c_m(a) = e^{-a} L_m^{(-1)}(2a).$$

Moreover, we have, uniformly for $0 < \varepsilon \leq x \leq M < \infty$:

$$L_m^{(\alpha)}(x) = \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} m^{\alpha/2-1/4} \cos(2\sqrt{mx} - \alpha\pi/2 - \pi/4) + R_m(x),$$

where $|R_m(x)| \leq K_\alpha m^{\alpha/2-3/4}$, and $K_\alpha > 0$ only depending on α .

Now, using Theorem 6.5 with $\alpha = -1$ and $a = n$, we get Lemma 6.4. Actually, to get Lemma 6.4, we need some uniformity with respect to a in Theorem 6.5, which is not given by the above statement of that theorem; but provided we change $m^{-5/4}$ into $\sqrt{n} m^{-5/4}$ in R_m , a careful examination of Fejér's proof of Theorem 6.5 shows that this uniformity holds. Alternatively, write $c_m(a)$ as a Fourier coefficient

$$(6.3) \quad c_m(n) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp[i(n \cot x + 2mx)] dx$$

and use the previous van der Corput estimates.

Once this lemma is at our disposal, we can prove again the following theorem.

Theorem 6.6. *Assume that β is a non-increasing sequence and that the composition operator C_{I_1} maps $H^2(\beta)$ to itself. Then β satisfies the Δ_2 -condition.*

Proof. Our assumption implies, with $M = \|C_{I_1}\|^2$:

$$(6.4) \quad \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m \leq M \beta_n.$$

We use Lemma 6.4 with $C^{-2}n \leq m < n$, where $C > 1$ satisfies $C\sqrt{2} < \pi/2$, which is possible since $2\sqrt{2} < \pi$. The term $R_m(n)$ is dominated by $KC^{-1/2}m^{-3/4}$. Let $\theta_m = 2\sqrt{2}\sqrt{nm} + \pi/4$ be the argument of the number appearing in (6.2). We have

$$\theta_{m+1} - \theta_m = \frac{2\sqrt{2}\sqrt{n}}{\sqrt{m} + \sqrt{m+1}};$$

hence

$$\sqrt{2} \leq \theta_{m+1} - \theta_m \leq C\sqrt{2} < \frac{\pi}{2}.$$

The argument θ_m then varies regularly, and there is a positive constant δ and a subset E of integers in the interval $[C^{-2}n, n]$ such that $|E| \geq \delta n$ and

$$M_m(n) \geq 2\delta n^{1/4}m^{-3/4} \geq 2\delta n^{-1/2}$$

for all $m \in E$. Therefore, for n large enough, we have, for all $m \in E$:

$$|a_{m,n}| \geq 2\delta n^{-1/2} - K\sqrt{n}m^{-5/4} \geq 2\delta n^{-1/2} - KC^{5/2}n^{-3/4} \geq \delta n^{-1/2}.$$

With this information, (6.4) gives:

$$M\beta_n \geq \sum_{m \in E} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} |E| \beta_{\lfloor C^{-2}n \rfloor} \geq \delta^3 \beta_{\lfloor C^{-2}n \rfloor},$$

where $\lfloor \cdot \rfloor$ stands for the integer part, and this proves the theorem. \square

Acknowledgements. We warmly thank R. Zarouf for useful discussions and informations.

L. Rodríguez-Piazza is partially supported by the project PGC2018-094215-B-I00 (Spanish Ministerio de Ciencia, Innovación y Universidades, and FEDER funds). Parts of this paper was made when he visited the Université d'Artois in Lens and the Université de Lille in January 2020. It is his pleasure to thank all his colleagues in these universities for their warm welcome.

The third-named author was partly supported by the Labex CEMPI (ANR-LABX-0007-01).

This work is also partially supported by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front).

References

- [1] A. A. Borichev, K. Fouchet, R. Zarouf, On the Fourier coefficients of powers of a Blaschke factor and strongly annular functions, *preprint*, arXiv:2107.00405 (2021).
- [2] I. Chalendar, J. R. Partington, Norm estimates for weighted composition operators on spaces of holomorphic functions, *Complex Anal. Oper. Theory* 8, no. 5 (2014), 1087–1095.
- [3] I. Chalendar, J. R. Partington, Compactness and norm estimates for weighted composition operators on spaces of holomorphic functions, *Harmonic analysis, function theory, operator theory, and their applications*, 81–89, *Theta Ser. Adv. Math.* 19, Theta, Bucharest (2017).
- [4] C. C. Cowen, An application of Hadamard multiplication to operators on weighted Hardy spaces, *Linear Algebra Appl.* 133 (1990), 21–32.
- [5] P. L. Duren, *Univalent functions*, *Grundlehren der Mathematischen Wissenschaften* 259, Springer-Verlag, New York (1983).
- [6] K. Fouchet, Puissances de facteurs et de produits de Blaschke : coefficients de Fourier et applications, thèse de doctorat, Université Aix-Marseille (8 décembre 2021).
- [7] E. A. Gallardo-Gutiérrez, J. R. Partington, Norms of composition operators on weighted Hardy spaces, *Israel J. Math.* 196, no. 1 (2013), 273–283.
- [8] G. M. Goluzin, On majorants of subordinate analytic functions I, *Mat. Sbornik*, N.S. 29 (1951), 209–224.
- [9] V. È Kacnel’son, A remark on canonical factorization in certain spaces of analytic functions (Russian), in: *Investigations on linear operators and the theory of functions III*, edited by N. K. Nikol’skiĭ, *Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)* 30 (1972), 163–164. Translation: *J. Soviet Math.* 4 (1975), no. 2 (1976), 444–445.
- [10] K. Kellay, P. Lefèvre, Compact composition operators on weighted Hilbert spaces of analytic functions, *J. Math. Anal. Appl.* 386 (2012), 718–727.
- [11] T. L. Kriete, B. D. MacCluer, A rigidity theorem for composition operators on certain Bergman spaces, *Michigan Math. J.* 42 (1995), 379–386.
- [12] N. N. Lebedev, *Special functions and their applications*, Revised edition, Dover Publications Inc., New-York (1972).
- [13] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Comparison of singular numbers of composition operators on different Hilbert spaces of analytic functions, *J. Funct. Anal.* 280, no. 3 (2021), article 108834 – <https://doi.org/10.1016/j.jfa.2020.108834>.

- [14] D. Li, H. Queffélec, L. Rodríguez-Piazza, A spectral radius type formula for approximation numbers of composition operators, *J. Funct. Anal.* 267, no. 12 (2014), 4753–4774.
- [15] H. L. Montgomery, Ten lectures on the interface between analytic number theory and harmonic analysis, *CBMS Regional Conference Series in Mathematics* 84, American Mathematical Society, Providence RI (1994).
- [16] D. Newman, H. Shapiro, The Taylor coefficients of inner functions, *Michigan Math. J.* 9 (1962), 249–255.
- [17] E. Reich, An inequality for subordinate analytic functions, *Pacific J. Math.* 4, no. 2 (1954), 259–274.
- [18] J. H. Shapiro, Composition operators and classical function theory, *Universitext, Tracts in Mathematics*, Springer-Verlag, New York (1993).
- [19] G. Szegő, Orthogonal polynomials, *American Mathematical Society Colloquium Publications*, v. 23, American Mathematical Society, New York (1939).
- [20] O. Szehr, R. Zarouf, A constructive approach to Schaeffer’s conjecture, *J. Math. Pures Appl.* (9) 146 (2021), 1–30.
- [21] E. C. Titchmarsh, The theory of the Riemann Zeta-function, Second edition revised by D. R. Heath-Brown, *Oxford Science Publications* (1986).
- [22] R. Zarouf, *Private communication*.

Pascal Lefèvre

Univ. Artois, Laboratoire de Mathématiques de Lens (LML) UR 2462, & Fédération Mathématique des Hauts-de-France FR 2037 CNRS, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE
 pascal.lefevre@univ-artois.fr

Daniel Li

Univ. Artois, Laboratoire de Mathématiques de Lens (LML) UR 2462, & Fédération Mathématique des Hauts-de-France FR 2037 CNRS, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE
 daniel.li@univ-artois.fr

Hervé Queffélec

Univ. Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération Mathématique des Hauts-de-France FR 2037 CNRS, F-59 655 VILLENEUVE D’ASCQ Cedex, FRANCE
 Herve.Queffelec@univ-lille.fr

Luis Rodríguez-Piazza

Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Calle Tarfia s/n 41 012 SEVILLA, SPAIN
 piazza@us.es