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# Boundedness of composition operators on general weighted Hardy spaces of analytic functions

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**Abstract.** We characterize the (essentially) decreasing sequences of positive numbers  $\beta = (\beta_n)$  for which all composition operators on  $H^2(\beta)$  are bounded, where  $H^2(\beta)$  is the space of analytic functions  $f$  in the unit disk such that  $\sum_{n=0}^{\infty} |c_n|^2 \beta_n < \infty$  if  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ . We also give conditions for the boundedness when  $\beta$  is not assumed essentially decreasing.

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**Key-words** automorphism of the unit disk; composition operator;  $\Delta_2$ -condition; multipliers; weighted Hardy space

## 1 Introduction

Let  $\beta = (\beta_n)_{n \geq 0}$  be a sequence of positive numbers such that

$$(1.1) \quad \liminf_{n \rightarrow \infty} \beta_n^{1/n} \geq 1.$$

The associated weighted Hardy space  $H^2(\beta)$  is the set of analytic functions  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  such that

$$(1.2) \quad \|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < \infty.$$

It is a Hilbert space of analytic functions on  $\mathbb{D}$ . When  $\beta_n \equiv 1$ , we recover the usual Hardy space  $H^2$ .

Recall that a symbol is a (non constant) analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ , and the associated composition operator  $C_\varphi: H^2(\beta) \rightarrow \mathcal{H}ol(\mathbb{D})$  is defined (formally) as:

$$(1.3) \quad C_\varphi(f) = f \circ \varphi.$$

An important question in the theory is to decide when  $C_\varphi$  is bounded on  $H^2(\beta)$ , i.e. when  $C_\varphi: H^2(\beta) \rightarrow H^2(\beta)$ . When  $H^2(\beta)$  is the usual Hardy space  $H^2$  (i.e. when  $\beta_n \equiv 1$ ), it is well-known ([18, pp. 13–17]) that all symbols generate bounded composition operators. On the other hand, for the Dirichlet space, corresponding to  $\beta_n = n + 1$ , not all composition operators are bounded. Note that, by definition of the norm of  $H^2(\beta)$ , all rotations  $R_\theta$ ,  $\theta \in \mathbb{R}$ , induce bounded composition operators on  $H^2(\beta)$  and send isometrically  $H^2(\beta)$  into itself.

Our goal in this paper is characterizing the (non-increasing) sequences  $\beta$  for which all composition operators act boundedly on the space  $H^2(\beta)$ , i.e. send  $H^2(\beta)$  into itself. We will prove the following result, where  $T_a(z) = \frac{a+z}{1+\bar{a}z}$  for  $a, z \in \mathbb{D}$ .

**Theorem 1.1.** *Let  $\beta$  be an essentially decreasing sequence of positive numbers. The following assertions are equivalent:*

- 1) *all composition operators are bounded on  $H^2(\beta)$ ;*
- 2) *all maps  $T_a$ , for  $0 < a < 1$ , induce bounded composition operators  $C_{T_a}$  on  $H^2(\beta)$ ;*
- 3) *for some  $a \in (0, 1)$ , the map  $T_a$  induces a bounded composition operator  $C_{T_a}$  on  $H^2(\beta)$ ;*
- 4)  *$\beta$  satisfies the  $\Delta_2$ -condition.*

For the notion of essentially decreasing sequence, see Definition 2.1; the  $\Delta_2$ -condition is defined in (2.4). Actually, we will obtain Theorem 1.1 by gluing Theorem 3.3 and Theorem 4.1.

Most of the existing works with weighted Hardy spaces concern the case

$$(1.4) \quad \beta_n = \int_0^1 t^n d\sigma(t)$$

where  $\sigma$  is a positive measure on  $(0, 1)$ . More specifically the following definition is often used. Let  $G: (0, 1) \rightarrow \mathbb{R}_+$  be an integrable function and let  $H_G^2$  be the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that:

$$(1.5) \quad \|f\|_{H_G^2}^2 := \int_{\mathbb{D}} |f(z)|^2 G(1 - |z|^2) dA(z) < \infty.$$

Such weighted Bergman type spaces are used, for instance, in [11], [10] and in [14]. We have  $H_G^2 = H^2(\beta)$  with:

$$(1.6) \quad \beta_n = 2 \int_0^1 r^{2n+1} G(1 - r^2) dr = \int_0^1 t^n G(1 - t) dt,$$

and the sequence  $\beta = (\beta_n)_n$  is non-increasing.

In Shapiro's presentation, the main point is the case  $\varphi(0) = 0$  and a subordination principle for subharmonic functions (Littlewood's subordination principle). The case of automorphisms is claimed simple, using an integral representation for the norm and some change of variable. When  $\beta$  is defined as in (1.4), one disposes of integral representations for the norm in  $H^2(\beta)$ , and, as in the Hardy space case, this integral representation rather easily gives the boundedness of  $C_{T_a}$  on  $H^2$ , where

$$(1.7) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}$$

for  $a \in \mathbb{D}$ . But the above representation (1.4) is equivalent, by the Hausdorff moment theorem, to a high regularity of the sequence  $\beta$ , namely its *complete monotony*. When integral representations fail, we have to work with bare hands. If the symbol vanishes at the origin, Kacnel'son's theorem gives a positive answer when  $\beta$  is essentially decreasing (see [2] or [13, Theorem 3.12]). Actually that follows from an older theorem of Goluzin [8] (see [5, Theorem 6.3]), which itself uses a refinement by Rogosinski of Littlewood's principle ([5, Theorem 6.2]). So that the main issue remains the boundedness of  $C_{T_a}$ .

A polynomial minoration (see Definition 2.3 below) for  $\beta$  is necessary for any  $C_{T_a}$  to be bounded on  $H^2(\beta)$  (Proposition 2.5) and we showed in [13, end of Section 3] that for  $\beta_n = \exp(-\sqrt{n})$ ,  $C_{T_a}$  is never bounded on  $H^2(\beta)$ . But this polynomial minoration is not sufficient, as we will see in Theorem 1.1 below, which also evidences the basic role of the maps  $T_a$  in the question.

However, we construct a weight  $\beta$  which is not essentially decreasing and for which all composition operators with symbol vanishing at 0 are bounded (Theorem 3.9), though no map  $T_a$  with  $0 < a < 1$  induces a bounded composition operator (Proposition 4.4).

For spaces of Bergman type  $A_G^2 := H_G^2$ , where  $\tilde{G}(r) = G(1-r^2)$ , defined as the spaces of analytic functions in  $\mathbb{D}$  such that  $\int_{\mathbb{D}} |f(z)|^2 \tilde{G}(|z|) dA < \infty$ , for a positive non-increasing continuous function  $\tilde{G}$  on  $[0, 1)$ , Kriete and MacCluer studied in [11] some analogous problems. They proved, in particular [11, Theorem 3] that, for:

$$\tilde{G}(r) = \exp\left(-B \frac{1}{(1-r)^\alpha}\right), \quad B > 0, \quad 0 < \alpha \leq 2,$$

and

$$\varphi(z) = z + t(1-z)^\beta, \quad 1 < \beta \leq 3, \quad 0 < t < 2^{1-\beta},$$

then  $C_\varphi$  is bounded on  $A_G^2$  if and only if  $\beta \geq \alpha + 1$ .

Here

$$\beta_n = \int_0^1 t^n e^{-B/(1-\sqrt{t})^\alpha} dt;$$

and, since  $\beta_n \approx \exp(-cn^{\alpha/(\alpha+1)})$ , the sequence  $(\beta_n)$  does not satisfy the  $\Delta_2$ -condition, accordingly to our Theorem 3.3 below.

We end the paper with some miscellaneous remarks.

## 2 Definitions, notation, and preliminary results

The open unit disk of  $\mathbb{C}$  is denoted  $\mathbb{D}$  and we write  $\mathbb{T}$  its boundary  $\partial\mathbb{D}$ . We set  $e_n(z) = z^n$ ,  $n \geq 0$ .

The weighted Hardy space  $H^2(\beta)$  defined in the introduction is a Hilbert space with the canonical orthonormal basis

$$(2.1) \quad e_n^\beta(z) = \frac{1}{\sqrt{\beta_n}} z^n, \quad n \geq 0,$$

and the reproducing kernel  $K_w$  given for all  $w \in \mathbb{D}$  by

$$(2.2) \quad K_w(z) = \sum_{n=0}^{\infty} e_n^\beta(z) \overline{e_n^\beta(w)} = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \bar{w}^n z^n.$$

Note that (1.1) is necessary for  $H^2(\beta)$  to consist of analytic functions in  $\mathbb{D}$ . Indeed the fact that  $\sum_{n \geq 1} \frac{1}{n\sqrt{\beta_n}} z^n$  belongs to  $H^2(\beta)$  and is analytic in  $\mathbb{D}$  implies (1.1). Note also that  $H^2$  is continuously embedded in  $H^2(\beta)$  if and only if  $\beta$  is bounded above. In particular, this is the case when  $\beta$  is non-increasing. In this paper, we need a slightly more general notion.

**Definition 2.1.** *A sequence of positive numbers  $\beta = (\beta_n)$  is said essentially decreasing if, for some constant  $C \geq 1$ , we have, for all  $m \geq n \geq 0$ :*

$$(2.3) \quad \beta_m \leq C \beta_n.$$

Note that saying that  $\beta$  is essentially decreasing means that the shift operator on  $H^2(\beta)$  is power bounded.

If  $\beta$  is essentially decreasing, and if we set:

$$\tilde{\beta}_n = \sup_{m \geq n} \beta_m,$$

the sequence  $\tilde{\beta} = (\tilde{\beta}_n)$  is non-increasing and we have  $\beta_n \leq \tilde{\beta}_n \leq C \beta_n$ . In particular, the space  $H^2(\beta)$  is isomorphic to  $H^2(\tilde{\beta})$  and  $H^2$  is continuously embedded in  $H^2(\beta)$ .

**Definition 2.2.** *The sequence of positive numbers  $\beta = (\beta_n)$  is said to satisfy the  $\Delta_2$ -condition if there is a positive constant  $\delta < 1$  such that, for all integers  $n \geq 0$ :*

$$(2.4) \quad \beta_{2n} \geq \delta \beta_n.$$

This terminology is given by analogy with that used for Orlicz functions.

**Definition 2.3.** *The sequence of positive numbers  $\beta = (\beta_n)$  is said to have a polynomial minoration if there are positive constants  $\delta$  and  $\alpha$  such that, for all integers  $n \geq 1$ :*

$$(2.5) \quad \beta_n \geq \delta n^{-\alpha}.$$

That means that  $H^2(\beta)$  is continuously embedded in the weighted Bergman space  $\mathfrak{B}_\alpha^2$  of the analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that

$$\|f\|_{\mathfrak{B}_\alpha^2}^2 := (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty$$

since  $\mathfrak{B}_\alpha^2 = H^2(\gamma)$  with  $\gamma_n \approx n^{-\alpha}$ .

The following simple proposition links those notions.

**Proposition 2.4.** *Let  $\beta$  be an essentially decreasing sequence of positive numbers. Then if  $\beta$  satisfies the  $\Delta_2$ -condition, it has a polynomial minoration.*

*The converse does not hold.*

*Proof.* Assume  $\beta_m \leq C \beta_n$  for  $m \geq n$  and  $\beta_{2^k} \geq e^{-A} \beta_k$ . Let now  $n$  be an integer  $\geq 2$ , and  $k \geq 1$  the smallest integer such that  $2^k \geq n$ , so that  $k \leq a \log n$  with  $a$  a positive constant. We get:

$$\beta_n \geq C^{-1} \beta_{2^k} \geq C^{-1} e^{-kA} \beta_1 \geq C^{-1} \beta_1 e^{-aA \log n} =: \rho n^{-\alpha},$$

with  $\rho = C^{-1} \beta_1$  and  $\alpha = aA$ .

Let us now see that the converse does not hold. Let  $\delta > 0$ . We set  $\beta_0 = \beta_1 = 1$  and for  $n \geq 2$ :

$$\beta_n = \frac{1}{(k!)^\delta} \quad \text{when } k! < n \leq (k+1)!$$

The sequence  $\beta$  is non-increasing.

For  $n$  and  $k$  as above, we have:

$$\beta_n = \frac{1}{(k!)^\delta} \geq \frac{1}{n^\delta};$$

hence  $\beta$  has arbitrary polynomial minoration. However we have, for  $k \geq 2$ :

$$\frac{\beta_{2(k!)}}{\beta_{k!}} = \frac{(k!)^{-\delta}}{[(k-1)!]^{-\delta}} = \frac{1}{k^\delta} \xrightarrow[k \rightarrow \infty]{} 0,$$

so  $\beta$  fails to satisfy the  $\Delta_2$ -condition. □

For  $a \in \mathbb{D}$ , we define:

$$(2.6) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}, \quad z \in \mathbb{D}.$$

Recall that  $T_a$  is an automorphism of  $\mathbb{D}$  and that  $T_a(0) = a$  and  $T_a(-a) = 0$ .

Though we do not need this, we may remark that  $(T_a)_{a \in (-1,1)}$  is a group and  $(T_a)_{a \in (0,1)}$  is a semigroup. It suffices to see that  $T_a \circ T_b = T_{a*b}$ , with:

$$(2.7) \quad a * b = \frac{a+b}{1+ab}.$$

**Proposition 2.5.** *Let  $a \in (0, 1)$  and assume that  $T_a$  induces a bounded composition operator on  $H^2(\beta)$ . Then  $\beta$  has a polynomial minoration.*

*Proof.* Since

$$\|K_x\|^2 = \sum_{n=0}^{\infty} \frac{x^{2n}}{\beta_n},$$

we have  $\|K_x\| \leq \|K_y\|$  for  $0 \leq x \leq y < 1$ .

We define by induction a sequence  $(u_n)_{n \geq 0}$  with:

$$u_0 = 0 \quad \text{and} \quad u_{n+1} = T_a(u_n).$$

Since  $T_a(1) = 1$  (recall that  $a \in (0, 1)$ ), we have:

$$1 - u_{n+1} = \int_{u_n}^1 T_a'(t) dt = \int_{u_n}^1 \frac{1 - a^2}{(1 + at)^2} dt;$$

hence

$$\frac{1 - a}{1 + a} (1 - u_n) \leq 1 - u_{n+1} \leq (1 - a^2)(1 - u_n).$$

Let  $0 < x < 1$ . We can find  $N \geq 0$  such that  $u_N \leq x < u_{N+1}$ . Then:

$$1 - x \leq 1 - u_N \leq (1 - a^2)^N.$$

On the other hand, since  $C_{T_a}^* K_z = K_{T_a(z)}$  for all  $z \in \mathbb{D}$ , we have:

$$\|K_x\| \leq \|K_{u_{N+1}}\| \leq \|C_{T_a}\| \|K_{u_N}\| \leq \|C_{T_a}\|^{N+1} \|K_{u_0}\| = \|C_{T_a}\|^{N+1}.$$

Let  $s \geq 0$  such that  $(1 - a^2)^{-s} = \|C_{T_a}\|$ . We obtain:

$$(2.8) \quad \|K_x\| \leq \|C_{T_a}\| \frac{1}{(1 - x)^s}.$$

But

$$\|K_x\|^2 = \sum_{k=0}^{\infty} \frac{x^{2k}}{\beta_k};$$

so we get, for any  $k \geq 2$ :

$$\frac{x^{2k}}{\beta_k} \leq \|C_{T_a}\|^2 \frac{1}{(1 - x)^{2s}}.$$

Taking  $x = 1 - \frac{1}{k}$ , we obtain  $\beta_k \geq C k^{-2s}$ . □

**Remarks.** 1) For example, when  $\beta_n = \exp[-c(\log(n+1))^2]$ , with  $c > 0$ , no  $T_a$  induces a bounded composition operator on  $H^2(\beta)$ , though all symbols  $\varphi$  with  $\varphi(0) = 0$  are bounded, since  $\beta$  is decreasing, as we will see in Proposition 3.2.

2) For the Dirichlet space  $\mathcal{D}^2$ , we have  $\beta_n = n+1$ , but all the maps  $T_a$  induce bounded composition operators on  $\mathcal{D}^2$  (see [13, Remark before Theorem 3.12]). In this case  $\beta$  has a polynomial minoration though it is not bounded above.

3) However, even for decreasing sequences, a polynomial minoration for  $\beta$  is not enough for some  $T_a$  to induce a bounded composition operator. Indeed, we saw in Proposition 2.4 an example of a decreasing sequence  $\beta$  with polynomial minoration, but not sharing the  $\Delta_2$ -condition, and we will see in Theorem 4.1 that the  $\Delta_2$ -condition is needed for having some  $T_a$  inducing a bounded composition operator.

4) In [7], Eva Gallardo-Gutiérrez and Jonathan Partington give estimates for the norm of  $C_{T_a}$ , with  $a \in (0, 1)$ , when  $C_{T_a}$  is bounded on  $H^2(\beta)$ . More precisely, they proved that if  $\beta$  is bounded above and  $C_{T_a}$  is bounded, then

$$\|C_{T_a}\| \geq \left(\frac{1+a}{1-a}\right)^\sigma,$$

where  $\sigma = \inf\{s \geq 0; (1-z)^{-s} \notin H^2(\beta)\}$ , and

$$\|C_{T_a}\| \leq \left(\frac{1+a}{1-a}\right)^\tau,$$

where  $\tau = \frac{1}{2} \sup \Re W(A)$ , with  $A$  the infinitesimal generator of the continuous semigroup  $(S_t)$  defined as  $S_t = C_{T_{\tanh t}}$ , namely  $(Af)(z) = f'(z)(1-z^2)$ , and  $W(A)$  its numerical range.

For  $\beta_n = 1/(n+1)^\nu$  with  $0 \leq \nu \leq 1$ , the two bounds coincide, so they get  $\|C_{T_a}\| = \left(\frac{1+a}{1-a}\right)^{(\nu+1)/2}$ .

5) We saw in the proof of Proposition 2.5 that if  $C_{T_a}$  is bounded on  $H^2(\beta)$  for some  $a \in (0, 1)$ , then the reproducing kernels  $K_w$  have, by (2.8), a *slow growth*:

$$(2.9) \quad \|K_w\| \leq \frac{C}{(1-|w|)^s}$$

for positive constants  $C$  and  $s$ . Actually, we have the following equivalence.

**Proposition 2.6.** *The sequence  $\beta$  has a polynomial minoration if and only if the reproducing kernels  $K_w$  of  $H^2(\beta)$  have a slow growth.*

*Proof.* The sufficiency is easy and seen at the end of the proof of Proposition 2.5. For the necessity, we only have to see that:

$$\|K_w\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{|w|^{2n}}{\beta_n} \leq \frac{1}{\beta_0} + \delta^{-1} \sum_{n=1}^{\infty} n^\alpha |w|^{2n} \leq \frac{C}{(1-|w|^2)^{\alpha+1}}. \quad \square$$

### 3 Boundedness of composition operators

We study in this section conditions ensuring that all composition operators on  $H^2(\beta)$  are bounded.



### 3.1 Conditions on the weight

We begin with this simple observation.

**Proposition 3.1.** *If all composition operators, and even if all composition operators with symbol vanishing at 0, are bounded on  $H^2(\beta)$ , then the sequence  $\beta$  is bounded above.*

*Proof.* Let  $f \in H^\infty$ . Write  $f = A\varphi + f(0)$  where  $A$  is a constant and  $\varphi$  a symbol vanishing at 0. We have  $\varphi = C_\varphi(z) \in H^2(\beta)$ , by hypothesis. So that  $f \in H^2(\beta)$  and  $H^\infty \subseteq H^2(\beta)$ . It follows (by the closed graph theorem) that there exists a constant  $M$  such that  $\|f\|_{H^2(\beta)} \leq M\|f\|_\infty$  for all  $f \in H^\infty$ . Testing that with  $f(z) = z^n$ , we get  $\beta_n \leq M^2$ .  $\square$

For symbols vanishing at 0, we have the following characterization.

**Proposition 3.2.** *The following assertions are equivalent:*

1) *all symbols  $\varphi$  such that  $\varphi(0) = 0$  induce bounded composition operators  $C_\varphi$  on  $H^2(\beta)$  and*

$$(3.1) \quad \sup_{\varphi(0)=0} \|C_\varphi\| < \infty;$$

2)  *$\beta$  is an essentially decreasing sequence.*

Of course, by the uniform boundedness principle, (3.1) is equivalent to:

$$\sup_{\varphi(0)=0} \|f \circ \varphi\| < \infty \quad \text{for all } f \in H^2(\beta).$$

*Proof.* 2)  $\Rightarrow$  1) We may assume that  $\beta$  is non-increasing. Then the Goluzin-Rogosinski theorem ([5, Theorem 6.3]) gives the result; in fact, writing  $f(z) = \sum_{n=0}^\infty c_n z^n$  and  $(C_\varphi f)(z) = \sum_{n=0}^\infty d_n z^n$ , it says that:

$$\|C_\varphi f\|^2 = |d_0|^2 \beta_0 + \sum_{n=1}^\infty |d_n|^2 \beta_n \leq |c_0|^2 \beta_0 + \sum_{n=1}^\infty |c_n|^2 \beta_n = \|f\|^2,$$

leading to  $C_\varphi$  bounded and  $\|C_\varphi\| \leq 1$ . Alternatively, we can use a result of Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]). This result was also proved by C. Cowen [4, Corollary of Theorem 7].

1)  $\Rightarrow$  2) Set  $M = \sup_{\varphi(0)=0} \|C_\varphi\|$ . Let  $m > n$ , and take:

$$\varphi(z) = \varphi_{m,n}(z) = z \left( \frac{1 + z^{m-n}}{2} \right)^{1/n}.$$

Then  $\varphi(0) = 0$  and  $[\varphi(z)]^n = \frac{z^n + z^m}{2}$ ; hence

$$\frac{1}{4}(\beta_n + \beta_m) = \|\varphi^n\|^2 = \|C_\varphi(e_n)\|^2 \leq \|C_\varphi\|^2 \|e_n\|^2 \leq M^2 \beta_n,$$

so  $\beta$  is essentially decreasing.  $\square$

Boundedness of  $\beta_n$  does not suffice. For example, let  $(\beta_n)$  be a sequence such that  $\beta_{2k+2}/\beta_{2k+1} \xrightarrow[k \rightarrow \infty]{} \infty$  (for instance  $\beta_{2k} = 1$  and  $\beta_{2k+1} = 1/(k+1)$ ); if  $\varphi(z) = z^2$ , then  $\|C_\varphi(z^{2n+1})\|^2 = \|z^{2(2n+1)}\|^2 = \beta_{2(2n+1)}$ ; since  $\|z^{2n+1}\|^2 = \beta_{2n+1}$ ,  $C_\varphi$  is not bounded on  $H^2(\beta)$ .

A more interesting example is the following. For  $0 < r < 1$ , let  $\beta_n = \pi n r^{2n}$ . This sequence is eventually decreasing, so it is essentially decreasing. The square  $\|f\|_{H^2(\beta)}^2$  of the norm  $\|f\|_{H^2(\beta)}$  is the area of the part of the Riemann surface on which  $r\mathbb{D}$  is mapped by  $f$ . E. Reich [17], generalizing Goluzin's result [8] (see [5, Theorem 6.3]), proved that for all symbols  $\varphi$  such that  $\varphi(0) = 0$ , the composition operator  $C_\varphi$  is bounded on  $H^2(\beta)$  and

$$\|C_\varphi\| \leq \sup_{n \geq 1} \sqrt{n} r^{n-1} \leq \frac{1}{\sqrt{2e}} \frac{1}{r \sqrt{\log(1/r)}}.$$

For  $0 < r < 1/\sqrt{2}$ , Goluzin's theorem asserts that  $\|C_\varphi\| \leq 1$ .

Note that this sequence  $\beta$  does not satisfy the  $\Delta_2$ -condition since  $\beta_{2n}/\beta_n = 2r^{2n}$ , Theorem 4.1 below states that no composition operator  $C_{T_a}$  is bounded.

However that the weight  $\beta$  is essentially decreasing is not necessary for the boundedness of all composition operators  $C_\varphi$ , with symbol  $\varphi$  vanishing at 0, as we will see later (Theorem 3.9).

### 3.2 Sufficient condition for the boundedness of composition operators – Part I

We now have one of the the main results of this section.

**Theorem 3.3.** *Let  $H^2(\beta)$  be a weighted Hardy space with  $\beta = (\beta_n)$  essentially decreasing and satisfying the  $\Delta_2$ -condition. Then all composition operators on  $H^2(\beta)$  are bounded.*

For the proof, we need a lemma.

**Lemma 3.4.** *For  $0 < a < 1$ , we write:*

$$(3.2) \quad (T_a z)^n = \sum_{m=0}^{\infty} a_{m,n} z^m.$$

*Then, there are constants  $b > 0$ ,  $0 < C_1 < 1$ ,  $C_2 > 1$  such that*

$$|a_{m,n}| \leq \begin{cases} e^{-bn} & \text{if } m \leq C_1 n, \\ e^{-bm} & \text{if } m \geq C_2 n. \end{cases}$$

*Proof.* First take  $0 < r < 1$ ; let:

$$M(r) = \sup_{|z|=r} |T_a(z)| = \sup_{|z|=r} \left| \frac{z+a}{1+az} \right|.$$

We have  $M(r) < 1$ , so we can write  $M(r) = r^\rho$ , for some positive  $\rho = \rho(a)$ .

The Cauchy inequalities give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = r^{\rho n - m},$$

and we obtain the first inequality by taking  $r = e^{-\alpha}$  and adjusting  $C_1$ .

Next, we use that  $T_a$  is analytic on  $D(0, 1/a)$ . Fix  $1 < r =: e^\beta < 1/a$ , with  $\beta > 0$ . Let  $M(r) = e^\alpha$  with  $\alpha > 0$ . The Cauchy inequalities again give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = e^{\alpha n - \beta m},$$

and we obtain the second inequality by adjusting  $C_2$ . □

We will also need the following result of V. È Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]).

**Theorem 3.5** (V. È Kacnel'son). *Let  $H$  be a separable complex Hilbert space and  $(e_i)_{i \geq 0}$  a fixed orthonormal basis of  $H$ . Let  $M: H \rightarrow H$  be a bounded linear operator. We assume that the matrix of  $M$  with respect to this basis is lower-triangular:  $\langle M e_j | e_i \rangle = 0$  for  $i < j$ .*

*Let  $(\gamma_j)_{j \geq 0}$  be a non-decreasing sequence of positive real numbers and  $\Gamma$  the (possibly unbounded) diagonal operator such that  $\Gamma(e_j) = \gamma_j e_j$ ,  $j \geq 0$ . Then the operator  $\Gamma^{-1} M \Gamma: H \rightarrow H$  is bounded and moreover:*

$$\|\Gamma^{-1} M \Gamma\| \leq \|M\|.$$

*Proof of Theorem 3.3.* We may, and do, assume that  $\beta$  is non-increasing.

Proposition 3.2 gives the result when  $\varphi(0) = 0$ .

It remains to show that all  $C_{T_a}$ ,  $a \in \mathbb{D}$ , are bounded. Indeed, if  $a = \varphi(0)$  and  $\psi = T_{-a} \circ \varphi$ , then  $\psi(0) = 0$  and  $\varphi = T_a \circ \psi$ , so  $C_\varphi = C_\psi \circ C_{T_a}$ . Moreover, we have only to show that when  $a \in [0, 1)$ . Indeed, if  $a \in \mathbb{D}$  and  $a = |a| e^{i\theta}$ , we have  $T_a = R_\theta \circ T_{|a|} \circ R_{-\theta}$ , so  $C_{T_a} = C_{R_{-\theta}} \circ C_{T_{|a|}} \circ C_{R_\theta}$ .

We consider the matrices

$$A = (a_{m,n})_{m,n \geq 0} \quad \text{and} \quad A_\beta = \left( \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} \right)_{m,n \geq 0}.$$

Since  $C_{T_a} e_n = T_a^n e_n$ , the formula (3.2) shows that  $A$  is the matrix of  $C_{T_a}$  in  $H^2$  with respect to the basis  $(e_n)_{n \geq 0}$ . On the other hand,  $A_\beta$  is the matrix of  $C_{T_a}$  in  $H^2(\beta)$  with respect to the basis  $(e_n^\beta)_{n \geq 0}$ . We note that  $A_\beta = B A B^{-1}$ , where  $B$  is the diagonal matrix with values  $\sqrt{\beta_0}, \sqrt{\beta_1}, \dots$  on the diagonal.

Since  $C_{T_a}$  is a bounded composition operators on  $H^2$ , the matrix  $A$  defines a bounded operator on  $\ell_2$ . We have to show that  $A_\beta$  also, i.e.  $\|A_\beta\| < \infty$ .

For that purpose, we split  $A$  and  $A_\beta$  into several sub-matrices.

Let  $N$  be an integer such that  $N \geq 2/C_1$ , where  $C_1$  is defined in Lemma 3.4 (actually, the proof of that lemma shows that we can take  $C_1$  such that  $1/C_1$  is

an integer, so we could take  $N = 2/C_1$ ). Let  $I_0 = [0, N[$   $J_0 = [N, +\infty[$  and for  $k = 1, 2, \dots$ :

$$I_k = [N^k, N^{k+1}[ \quad \text{and} \quad J_k = [N^{k+1}, +\infty[.$$

We define the matrices  $D_\beta$  and  $R_\beta$ , whose entries are respectively:

$$d_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty} (I_k \times I_k) \\ 0 & \text{elsewhere;} \end{cases}$$

and

$$r_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty} (I_k \times I_{k+1}) \\ 0 & \text{elsewhere.} \end{cases}$$

We also define the matrix  $S_\beta$  with entries:

$$s_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty} (J_k \times I_k) \\ 0 & \text{elsewhere.} \end{cases}$$

Matrices  $D$ ,  $R$ , and  $S$  are constructed in the same way from  $A$  and we set  $U = A - (D + R + S)$ .

Now, let  $H_k$  be the subspace of the sequences  $(x_n)_{n \geq 0}$  in  $\ell_2$  such that  $x_n = 0$  for  $n \notin I_k$ , i.e.  $H_k = \text{span}\{e_n; n \in I_k\}$ , and let  $P_k$  be (the matrix of) the orthogonal projection of  $\ell_2$  with range  $H_k$ . We have:

$$D = \sum_{k=0}^{\infty} P_k A P_k \quad \text{and} \quad R = \sum_{k=0}^{\infty} P_k A P_{k+1},$$

where  $D_k = P_k A P_k$  is the matrix with entries  $a_{m,n}$  when  $(m,n) \in I_k \times I_k$  and 0 elsewhere, and  $R_k = P_k A P_{k+1}$  the matrix with entries  $a_{m,n}$  when  $(m,n) \in I_k \times I_{k+1}$  and 0 elsewhere.

$$\left( \begin{array}{c|c|c|c} \boxed{D_0} & \boxed{R_0} & & \\ \hline & D_1 & \boxed{R_1} & \\ \hline S_0 & S_1 & D_2 & R_2 \\ \hline & & S_2 & \end{array} \right) U$$

Since the subspaces  $H_k$  are orthogonal, the matrices  $D$  and  $R$  induce bounded operators on  $\ell_2$ , and

$$(3.3) \quad \|D\| \leq \|A\|, \quad \|R\| \leq \|A\|.$$

Now, for  $k \geq 1$ , let  $B_k$  be the diagonal matrix whose entries are  $b_{m,m} = \sqrt{\beta_m}$  if  $m \in I_k$  and  $b_{m,n} = 0$  otherwise.

Then  $P_k D_\beta P_k = P_k B_k D B_k^{-1} P_k$ , so

$$\|P_k D_\beta P_k\| \leq \|B_k\| \|B_k^{-1}\| \|D\| \leq \max_{j \in I_k} \sqrt{\beta_j} \max_{j \in I_k} \frac{1}{\sqrt{\beta_j}} \|A\|.$$

But the weight  $\beta$  satisfies the  $\Delta_2$ -condition:  $\beta_{2l} \geq \delta_0 \beta_l$ , and it follows that for every  $l \geq 1$ :

$$\beta_{N^{2l}} \geq \delta^2 \beta_l,$$

for some other constant  $\delta$ , chosen small enough to have  $\|P_0 D_\beta P_0\| \leq \delta^{-1} \|A\|$ . Since  $\beta$  is non-increasing, we have  $\beta_j \geq \delta^2 \beta_{N^k}$  for  $N^k \leq j \leq N^{k+1}$ . In particular  $\max_{j \in I_k} \sqrt{\beta_j} \leq \delta^{-1} \min_{j \in I_k} \sqrt{\beta_j}$  and

$$\|P_k D_\beta P_k\| \leq \delta^{-1} \|A\|.$$

Hence, by orthogonality of the subspaces  $H_k$ :

$$(3.4) \quad \|D_\beta\| \leq \delta^{-1} \|D\|.$$

In the same way, we have  $P_k R_\beta P_k = P_k B_k D B_{k+1}^{-1} P_k$ , so:

$$\|P_k R_\beta P_k\| \leq \max_{j \in I_k} \sqrt{\beta_j} \max_{j \in I_{k+1}} \frac{1}{\sqrt{\beta_j}} \|A\| = \max_{(m,n) \in I_k \times I_{k+1}} \sqrt{\frac{\beta_m}{\beta_n}} \|A\|.$$

But, when  $(m, n) \in I_k \times I_{k+1}$ , we get

$$\beta_n \geq \beta_{N^{k+2}} \geq \delta^2 \beta_{N^k} \geq \delta^2 \beta_m.$$

Hence

$$(3.5) \quad \|R_\beta\| \leq \delta^{-1} \|R\|.$$

Next, consider  $U = A - (D + R + S)$ ; we can compute its Hilbert-Schmidt norm using Lemma 3.4. Note that  $u_{m,n} \neq 0$  only (if it happens) when  $m \in I_k$  for some  $k \geq 0$  and  $n \geq \min I_{k+2} \geq N^{k+2} > Nm$ , since  $m \in I_k$ , so only when  $m \leq C_1 n$ . We have, since then  $u_{m,n} = a_{m,n}$ :

$$\|U\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m \leq C_1 n} |a_{m,n}|^2 \leq \sum_{n=0}^{\infty} \sum_{m \leq C_1 n} e^{-2bn} \leq \sum_{n=0}^{\infty} C_1 n e^{-2bn} < \infty.$$

Consequently, with (3.3), we have  $\|S\| \leq \|A\| + \|D\| + \|R\| + \|U\| < \infty$ .

Now, since  $S$  is a lower-triangular matrix and  $\beta$  is non-increasing, we can use the result of V. È Kacnel'son (Theorem 3.5), with  $\gamma_j = 1/\sqrt{\beta_j}$ . We get that  $S_\beta$  defines a bounded operator and  $\|S_\beta\| \leq \|S\|$ .

Further, Proposition 2.4 says that  $\beta$  has a polynomial minoration:

$$\beta_n \geq c n^{-\sigma}$$

for positive constants  $c$  and  $\sigma$ . Then, if  $U_\beta = A_\beta - (D_\beta + R_\beta + S_\beta)$ , we have:

$$\|U_\beta\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m < \rho n/2} \frac{|a_{m,n}|^2}{\beta_n} \leq \sum_{n=0}^{\infty} \frac{\rho n}{2} e^{-\alpha \rho n} \frac{n^\sigma}{c} < \infty.$$

Putting this together with (3.4) and (3.5), we finally obtain that  $A_\beta = S_\beta + D_\beta + R_\beta + U_\beta$  is the matrix of a bounded operator, and that ends the proof of Theorem 3.3.  $\square$

**Remark.** We could have done here without the theorem of Kacnel'son, using Lemma 3.4 to show that the matrix  $\left(a_{m,n} \sqrt{\frac{\beta_m}{\beta_n}}\right)$  is Hilbert-Schmidt as well “far below” the main diagonal. Indeed, we see from this lemma that

$$\sum_{m \geq C_2 n} |a_{m,n}|^2 \frac{\beta_m}{\beta_n} \lesssim \sum_{m \geq C_2 n} n^\sigma |a_{m,n}|^2 \lesssim \sum_{m \geq C_2 n} n^\sigma e^{-2bm} \lesssim \sum_{n \geq 0} n^\sigma e^{-2bC_2 n} < +\infty.$$

See also Theorem 6.2 of the final section. Kacnel'son's theorem will really be needed in the forthcoming Theorem 3.6.

### 3.3 Sufficient condition for the boundedness of composition operators – Part II

In this section, we give a sufficient condition for the boundedness of composition operators with symbol vanishing at 0, of a different nature than the one given in Proposition 3.2.

**Theorem 3.6.** *Let  $\beta = (\beta_n)_{n=0}^\infty$  be a bounded sequence of positive numbers with a polynomial minoration. Assume that:*

$$(3.6) \quad \text{For every } \delta > 0, \text{ there exists a positive constant } C = C(\delta) \text{ such that} \\ \beta_m \leq C \beta_n \quad \text{whenever } m > (1 + \delta)n.$$

*Then, for all symbols  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  vanishing at 0, the composition operator  $C_\varphi$  is bounded on  $H^2(\beta)$ .*

To prove Theorem 3.6, we need several lemmas.

**Lemma 3.7.** *Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  be an analytic self-map such that  $\varphi(0) = 0$  and  $|\varphi'(0)| < 1$ . Then there exists  $\rho > 0$  such that*

$$|\widehat{\varphi^n}(m)| \leq \exp\left(-\frac{1}{2}[(1 + \rho)n - m]\right).$$

*Proof.* It is the same as that of Lemma 3.4. Since  $\varphi(0) = 0$ , we can write  $\varphi(z) = z \varphi_1(z)$ . Since  $|\varphi'(0)| < 1$ , we have  $\varphi_1: \mathbb{D} \rightarrow \mathbb{D}$ . Let  $M(r) = \sup_{|z|=r} |\varphi_1(z)|$ . The Cauchy inequalities say that  $|\widehat{\varphi_1^n}(m)| \leq [M(r)]^n / r^m$ . We have  $M(r) < 1$ , so there exists a positive number  $\rho = \rho(r)$  such that  $M(r) = r^\rho$ . We get:

$$|\widehat{\varphi^n}(m)| = |\widehat{\varphi_1^n}(m-n)| \leq \frac{r^{\rho n}}{r^{m-n}} = r^{(1+\rho)n-m},$$

and the result follows, by taking  $r = e^{-1/2}$ .  $\square$

The next lemma is a variant of the result of V. È Kacnel'son quoted before.

**Lemma 3.8.** *Let  $A: \ell_2 \rightarrow \ell_2$  be a bounded operator represented by the matrix  $(a_{m,n})_{m,n}$ , i.e.  $a_{m,n} = \langle A e_n, e_m \rangle$ , where  $(e_n)_{n \geq 1}$  is the canonical basis of  $\ell_2$ .*

*Let  $(d_n)$  be a sequence of positive numbers such that, for every  $m$  and  $n$ :*

$$(3.7) \quad d_m < d_n \quad \implies \quad a_{m,n} = 0.$$

*Then,  $D$  being the (possibly unbounded) diagonal operator with entries  $d_n$ , we have:*

$$\|D^{-1}AD\| \leq \|A\|.$$

For the convenience of the reader, we reproduce the proof.

*Proof.* Let  $\mathbb{C}_0$  be the right-half plane  $\mathbb{C}_0 = \{z \in \mathbb{C}; \Re z > 0\}$ . We set  $H_N = \text{span}\{e_n; n \leq N\}$  and

$$A_N = P_N A J_N,$$

where  $P_N$  is the orthogonal projection from  $\ell_2$  onto  $H_N$  and  $J_N$  the canonical injection from  $H_N$  into  $\ell_2$ . We consider, for  $z \in \overline{\mathbb{C}_0}$ :

$$A_N(z) = D^{-z} A_N D^z: H_N \rightarrow H_N,$$

where  $D^z(e_n) = d_n^z e_n$ .

If  $(a_{m,n}(z))_{m,n}$  is the matrix of  $A_N(z)$  on the basis  $\{e_n; n \leq N\}$  of  $H_N$ , we clearly have:

$$a_{m,n}(z) = a_{m,n}(d_n/d_m)^z.$$

In particular, we have, thanks to (3.7):

$$a_{m,n}(z) = 0 \quad \text{if } d_m < d_n,$$

and

$$|a_{m,n}(z)| \leq \sup_{k,l} |a_{k,l}| := M, \quad \text{for all } z \in \overline{\mathbb{C}_0}.$$

Since  $\|A_N(z)\|^2 \leq \|A_N(z)\|_{HS}^2 = \sum_{m,n \leq N} |a_{m,n}(z)|^2 \leq (N+1)^2 M^2$ , we get:

$$\|A_N(z)\| \leq (N+1) M \quad \text{for all } z \in \overline{\mathbb{C}_0}.$$

Let us consider the function  $u: \overline{\mathbb{C}_0} \rightarrow \overline{\mathbb{C}_0}$  defined by:

$$(3.8) \quad u_N(z) = \|A_N(z)\|.$$

This function  $u_N$  is continuous on  $\overline{\mathbb{C}_0}$ , bounded above by  $(N+1)M$ , and subharmonic in  $\mathbb{C}_0$ . Moreover, thanks to (3.7), the maximum principle gives:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) = \sup_{\partial\mathbb{C}_0} u_N(z).$$

Since  $\|D^z\| = \|D^{-z}\| = 1$  for  $z \in \partial\mathbb{C}_0$ , we have  $\|A_N(z)\| \leq \|A_N\|$  for  $z \in \partial\mathbb{C}_0$ , and we get:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) \leq \|A_N\| \leq \|A\|.$$

In particular  $u_N(1) \leq \|A\|$ , and, letting  $N$  going to infinity, we get  $\|D^{-1}AD\| \leq \|A\|$ .  $\square$

*Proof of Theorem 3.6.* First, if  $|\varphi'(0)| = 1$ , we have  $\varphi(z) = \alpha z$  for some  $\alpha$  with  $|\alpha| = 1$ , and the result is trivial.

So, we assume that  $|\varphi'(0)| < 1$ . Then, by Lemma 3.7, there exists  $\rho > 0$  such that, for all  $m, n$ :

$$|\widehat{\varphi^n}(m)| \leq \exp\left(-\frac{1}{2}[(1+\rho)n - m]\right).$$

Since  $\varphi(0) = 0$ , we also know that  $\widehat{\varphi^n}(m) = 0$  if  $m < n$ .

Take  $\delta = \rho/2$  and use property (3.6): there exists  $C > 0$  such that:

$$\frac{\beta_m}{\beta_n} \leq C \quad \text{when } m \geq (1+\delta)n.$$

Define now a new sequence  $\gamma = (\gamma_n)$  as:

$$\gamma_n = \max\left\{\beta_n, \sup_{m > (1+\delta)n} \beta_m\right\}.$$

We have:

- 1)  $\beta_n \leq \gamma_n \leq C\beta_n$ ;
- 2)  $\gamma_m \leq \gamma_n$  if  $m \geq (1+\delta)n$ .

Item 1) implies that  $H^2(\gamma) = H^2(\beta)$ , and we are reduced to prove that  $C_\varphi: H^2(\gamma) \rightarrow H^2(\gamma)$  is bounded.

Let  $A = (a_{m,n}) = (\widehat{\varphi^n}(m))$ . We have to prove that

$$B = (\gamma_m^{1/2} \gamma_n^{-1/2} a_{m,n})_{m,n}$$

represents a bounded operator on  $\ell_2$ .



Define the matrix

$$A_1 = (a_{m,n} \mathbb{1}_{\{(m,n); m \leq (1+\delta)n\}})_{m,n}$$

and set  $A_2 = A - A_1$ . Define analogously  $B_1$  and  $B_2 = B - B_1$ .

Then  $A_1$  is a Hilbert-Schmidt operator, because (recall that  $a_{m,n} = 0$  if  $m < n$ )

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{(1+\delta)n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \exp(-[(1+\rho)n - m]) \\ &\leq \sum_{n=1}^{\infty} (\delta n + 1) \exp(-\delta n) < \infty. \end{aligned}$$

Now,  $\beta$  is bounded above and has a polynomial minoration, so, for some positive constants  $C_1, C_2$ , and  $\alpha$ , we have:

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \frac{\gamma_m}{\gamma_n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \frac{C_1}{n^{-\alpha}} \exp(-\delta n) \\ &\leq \sum_{n=1}^{\infty} C_2 n^{\alpha+1} \exp(-\delta n) < \infty, \end{aligned}$$

meaning that  $B_1$  is a Hilbert-Schmidt operator.

Since  $A$  is bounded, it follows that  $A_2 = A - A_1$  is bounded. Remark that, writing  $A_2 = (\alpha_{m,n})_{m,n}$ , we have, with  $d_n = 1/\sqrt{\gamma_n}$ :

$$d_m < d_n \implies \gamma_m > \gamma_n \implies m < (1+\delta)n \implies \alpha_{m,n} = 0.$$

Hence we can apply Lemma 3.8 to the matrix  $A_2$ , and it ensues that  $B_2$  is bounded, and therefore that  $B = B_1 + B_2$  is bounded as well, as wanted.  $\square$

As a corollary of Theorem 3.6, we can provide the following example.

**Theorem 3.9.** *There exists a bounded sequence  $\beta$ , with a polynomial minoration, but which is not essentially decreasing, for which every composition operator with symbol vanishing at 0 is bounded on  $H^2(\beta)$ , with nevertheless  $\sup_{\varphi(0)=0} \|C_\varphi\| = \infty$ .*

It should be noted that for this weight, the composition operators are not all bounded, as we will see in Proposition 4.4.

*Proof.* Define  $\beta_n = 1$  for  $n \leq 3!$ , and, for  $k \geq 3$ :

$$\left\{ \begin{array}{l} \beta_n = \frac{1}{k!} \quad \text{for } k! < n \leq (k+1)! - 2 \text{ and for } n = (k+1)! \\ \beta_n = \frac{1}{(k+1)!} \quad \text{for } n = (k+1)! - 1. \end{array} \right.$$

Note that, for  $m > n$ , we have  $\beta_m > \beta_n$  only if  $n = (k+1)! - 1$  and  $m = (k+1)! = n + 1$ , for some  $k \geq 3$ .

However,  $\beta$  is not essentially decreasing since, for every  $k \geq 3$ , we have  $\beta_{n+1}/\beta_n = k + 1$  if  $n = (k+1)! - 1$ .

The sequence  $\beta$  has a polynomial minoration because  $\beta_n \geq 1/(2n)$  for all  $n \geq 1$ . In fact, for  $k \geq 3$ , we have  $\beta_n \geq (k+1)/n \geq 1/n$  if  $k! < n \leq (k+1)! - 2$  or if  $n = (k+1)!$ ; and for  $n = (k+1)! - 1$ , we have  $n\beta_n = [(k+1)! - 1]/(k+1)! \geq 1/2$ .

Now, it remains to check (3.6) in order to apply Theorem 3.6 and finish the proof of Theorem 3.9. Note first that we have  $\beta_m/\beta_n \leq 1$  if  $m \geq n + 2$ . Next, for given  $\delta > 0$ , there exists an integer  $N$  such that  $(1 + \delta)n \geq n + 2$  for every  $n \geq N$ , so  $\beta_m/\beta_n \leq 1$  if  $m \geq (1 + \delta)n$  and  $n \geq N$ . It suffices to take  $C = \max_{1 \leq n \leq N} \beta_{n+1}/\beta_n$  to obtain (3.6). The last assertion follows from Proposition 3.2.  $\square$

## 4 Necessity of the $\Delta_2$ -condition

In this section, we will show that, for essentially decreasing sequences  $\beta$ , the  $\Delta_2$ -condition is necessary for having boundedness of composition operators on  $H^2(\beta)$ . We will indeed show slightly more.

**Theorem 4.1.** *Let  $\beta$  be such that, for some  $a \in (0, 1)$ ,  $T_a$  induces a bounded composition operator on  $H^2(\beta)$ . Then  $\beta$  satisfies:*

$$(\exists 0 < \delta < 1/3) (\forall n \geq 1) (\exists E_n \subseteq [(1 - 2\delta)n, (1 - \delta)n]) \quad \text{with}$$

$$|E_n| \geq \delta n \quad \text{and} \quad \beta_n \geq \delta \frac{1}{|E_n|} \sum_{m \in E_n} \beta_m.$$

*In particular, if  $\beta$  is essentially decreasing, then  $\beta$  satisfies the  $\Delta_2$ -condition.*

In order to prove Theorem 4.1, we need several preliminary lemmas. The first one is standard, but we give it for convenience.

**Lemma 4.2.** *Let  $a \in (0, 1)$  and let*

$$P_{-a}(x) = \frac{1 - a^2}{1 + 2a \cos x + a^2}$$

*be the Poisson kernel at the point  $-a$ . Then, for all  $x \in [-\pi, \pi]$ :*

$$(4.1) \quad T_a(e^{ix}) = \exp[i h_a(x)],$$

*where*

$$(4.2) \quad h_a(x) = \int_0^x P_{-a}(t) dt.$$

*Proof.* For  $t \in [-\pi, \pi]$ , write:

$$\psi(t) := \frac{e^{it} + a}{1 + a e^{it}} = \exp(i u(t)),$$

with  $u$  a real-valued,  $\mathcal{C}^1$  function on  $[-\pi, \pi]$  such that  $u(0) = 0$ . This is possible since  $|\psi(e^{it})| = 1$  and  $\psi(0) = 1$ . Differentiating both sides with respect to  $t$ , we get:

$$i e^{it} \frac{1 - a^2}{(1 + a e^{it})^2} = i u'(t) \frac{e^{it} + a}{1 + a e^{it}}.$$

This implies

$$u'(t) = \frac{1 - a^2}{|1 + a e^{it}|^2} = P_{-a}(t),$$

and the result follows since  $u(0) = 0 = h_a(0)$ .  $\square$

## 4.1 Main lemma and proof of Theorem 4.1

To prove Theorem 4.1, we need the following lemma. The proof of this lemma uses Lemma 4.2 and a van der Corput type estimate, inspired from [21, pp. 72–73]. We thank R. Zarouf [22] for interesting recent informations in this respect, related to his joint work with O. Szehr on the Schäffer problem (see [20], in which the authors are primarily concerned with upper bounds).

The first version of our paper was put on arXiv at the end of November 2020. Since then, the paper [1] was put on arXiv on July 2021, where sharp estimates of powers of Blaschke factors are given (see also K. Fouchet's thesis [6]), with different purposes (strongly annular analytic functions). However, in our case, our proof is much simpler.

Recall that we have set:

$$(4.3) \quad [T_a(z)]^n = \sum_{m=0}^{\infty} a_{m,n} z^m.$$

**Lemma 4.3.** *Let  $a \in (0, 1)$ . We set:*

$$(4.4) \quad \tau = \frac{1 + a}{1 - a} > 1$$

and write:

$$(4.5) \quad \tau^{-1} = 1 - 3\mu,$$

with  $\mu = \mu_a \in (0, 1/3)$ . For every fixed positive integer  $n$ , let:

$$(4.6) \quad J_n = [(1 - 2\mu)n, (1 - \mu)n].$$

Then, there exists  $\delta = \delta_a > 0$  such that, for every  $n$  large enough, there exists a set of indices  $E_n \subseteq J_n$  with cardinality  $|E_n| \geq \delta n$  and such that:

$$(4.7) \quad m \in E_n \implies |a_{m,n}| \geq \delta n^{-1/2}.$$

*Proof of Theorem 4.1.* Set  $M = \|C_{T_a}\|$ . We have:

$$(4.8) \quad \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|T_a^n\|^2 = \|C_{T_a}(z^n)\|^2 \leq \|C_{T_a}\|^2 \|z^n\|^2 = M^2 \beta_n.$$

so, by Lemma 4.3, since  $E_n \subseteq J_n = [(1 - 2\mu)n, (1 - \mu)n]$ :

$$M^2 \beta_n \geq \sum_{m \in E_n} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} \sum_{m \in E_n} \beta_m \geq \delta^3 |E_n|^{-1} \sum_{m \in E_n} \beta_m.$$

This proves (changing  $\delta$ ) the first part of Theorem 4.1. Next, assume that  $\beta$  is essentially decreasing. We may, and do, assume that  $\beta$  is non-increasing. We set, for  $x$  not an integer,  $\beta_x = \beta_k$  with  $k$  the least integer greater than  $x$ . The above implies, for all integers  $n \geq 1$ :

$$\beta_n \geq (\delta^3/M^2) |E_n|^{-1} |E_n| \beta_{(1-\mu)n} \geq c \beta_{(1-\mu)n}.$$

Let  $r \geq 1$  such that  $(1 - \mu)^r \leq 1/2$ ; we have:

$$\beta_n \geq c^r \beta_{(1-\mu)^r n} \geq c^r \beta_{n/2},$$

so  $\beta$  satisfies the  $\Delta_2$ -condition.  $\square$

A consequence of Theorem 4.1 is the following result.

**Proposition 4.4.** *For the weight  $\beta$  constructed in the proof of Theorem 3.9, no automorphism  $T_a$  with  $0 < a < 1$  can be bounded.*

*Proof.* Consider the necessary condition for the boundedness of  $C_{T_a}$  in Theorem 4.1:

$$(4.9) \quad \beta_n \geq \delta \frac{1}{|E_n|} \sum_{m \in E_n} \beta_m.$$

For the weight  $\beta$  constructed in the proof of Theorem 3.9, we are going to see that this condition (4.9) is not satisfied for  $n = (k + 1)! - 1 =: n_k$ .

Indeed, for this  $n$ , the left-hand side of (4.9) is equal to  $1/(k + 1)!$  and the right-hand side to  $\delta/k!$  since  $(1 - 2\delta)n_k > k!$  for  $k$  large, so that all  $\beta_m$  are equal to  $1/k!$  for  $m \in E_{n_k}$ . This ends the proof.  $\square$

## 4.2 Proof of Lemma 4.3

To prove Lemma 4.3, we will use a variant of [21, Lemma 4.6 p. 72] on the stationary phase method. A careful reading of the proof in [21, p. 72] gives the version below, which allows the derivative  $F'$  of  $F$  to vanish at some point, as occurs in our situation. For sake of completeness, we will give a proof, however postponed.

**Proposition 4.5** (Stationary phase). *Let  $F$  be real function on the interval  $[A, B]$ , with continuous derivatives up to the third order and  $F'' > 0$  throughout  $]A, B[$ . Assume that there is a (unique) point  $c$  in  $]A, B[$  such that  $F'(c) = 0$ , and that, for some positive numbers  $\lambda_2, \lambda_3$ , and  $\eta$ , the following assertions hold:*

- 1)  $[c - \eta, c + \eta] \subseteq [A, B]$ ;
- 2)  $F''(x) \geq \lambda_2$  for all  $x \in [c - \eta, c + \eta]$ ;
- 3)  $|F'''(x)| \leq \lambda_3$  for all  $x \in [A, B]$ .

Then:

$$(4.10) \quad \int_A^B e^{iF(x)} dx = \sqrt{2\pi} \frac{e^{i(F(c)+\pi/4)}}{|F''(c)|^{1/2}} + O\left(\frac{1}{\eta\lambda_2} + \eta^4\lambda_3\right),$$

where the  $O$  involves an absolute constant.

*Proof of Lemma 4.3.* We turn to the problem of bounding  $a_{m,n}$  from below, in the case  $m \in J_n$ , and only in that case. Since  $\inf_{[0,\pi]} P_{-a} = \tau^{-1} < \sup_{[0,\pi]} P_{-a} = \tau$ , there exists a unique point  $x_m = x_{m,n} \in [0, \pi]$  such that

$$nP_{-a}(x_m) - m = n \frac{(1 - a^2)}{1 + 2a \cos x_m + a^2} - m = 0,$$

or else:

$$(4.11) \quad \cos x_m = \frac{n}{m} \frac{1 - a^2}{2a} - \frac{1 + a^2}{2a}.$$

The point is that if  $m \in J_n$ ,  $x_m$  can approach neither 0 nor  $\pi$ , so that  $\sin x_m \geq \delta_a > 0$ ; more precisely, the definition of  $J_n$  and (4.11) imply that  $\pi/4 \leq x_m \leq \pi/2$ .

With  $h_a$  the function of Lemma 4.2, the Fourier formulas give, since  $a_{m,n}$  is real, or since  $h_a(x) - mx$  is odd:

$$2\pi a_{m,n} = \int_{-\pi}^{\pi} \exp i[nh_a(x) - mx] dx = 2 \Re I_{m,n},$$

where

$$(4.12) \quad I_{m,n} = \int_0^{\pi} \exp i[nh_a(x) - mx] dx.$$

Write:

$$(4.13) \quad I_{m,n} = \int_0^{\pi} \exp[i F_m(x)] dx,$$

with:

$$(4.14) \quad F_m(x) = nh_a(x) - mx = n \int_0^x P_{-a}(t) dt - mx.$$

We have:

$$(4.15) \quad F'_m(x) = n P_{-a}(x) - m.$$

We will now proceed in two steps, first giving good lower bounds for  $|I_{m,n}|$ , then showing that the argument of  $I_{m,n}$  is often far from  $\pi/2 \pmod{\pi}$ . Then, we will be done.

**First step.** We will prove that:

$$(4.16) \quad I_{m,n} = \sqrt{2\pi} n^{-1/2} \frac{e^{i(F_m(x_m) + \pi/4)}}{\sqrt{|h''_a(x_m)|}} + O(n^{-3/5}),$$

where the  $O$  only depends on  $a$ .

Note that  $3/5 > 1/2$  and  $F''_m = n h''_a$ .

To get (4.16), we will show that Theorem 4.5 is applicable with:

$$[A, B] = [0, \pi], \quad c = x_m, \quad \lambda_2 = \kappa_0 n, \quad \lambda_3 = C_0 n, \quad \eta = (\lambda_2 \lambda_3)^{-1/5}.$$

The parameter  $\eta$  is chosen in order to make both error terms in Theorem 4.5 equal:  $\frac{1}{\eta \lambda_2} = \eta^4 \lambda_3$ ; so:

$$\eta = \kappa n^{-2/5}$$

and

$$(4.17) \quad \frac{1}{\eta \lambda_2} + \eta^4 \lambda_3 = \tilde{\kappa} n^{-3/5} = O(n^{-3/5})$$

(with  $\kappa = (\kappa_0 C_0)^{-1/5}$  and  $\tilde{\kappa} = 2/\kappa_0 \kappa$ ).

The slight technical difficulty encountered here is that  $F''_m(x)$  vanishes at 0 and  $\pi$ . But Theorem 4.5 covers this case. We have

$$F''_m(x) = n P'_{-a}(x) = 2a(1-a^2) \frac{\sin x}{(1+2a \cos x + a^2)^2} n,$$

and there are some positive  $\kappa_0$  and  $\sigma$  such that

$$(4.18) \quad F''_m(x) \geq \kappa_0 n = \lambda_2 \quad \text{for } x \in [\sigma, \pi - \sigma].$$

Now (for  $n$  large enough),  $[x_m - \eta, x_m + \eta] \subseteq [\sigma, \pi - \sigma]$ . Hence the assumptions 1) and 2) of Proposition 4.5 are satisfied.

Finally, since  $F_m(x) = n h_a(x) - m x$ , and  $h_a$  is  $\mathcal{C}^\infty$  on  $\mathbb{R}$ , we have, for all  $x \in [0, \pi]$ :

$$|F'''_m(x)| \leq C_0 n = \lambda_3,$$

and assertion 3) of Proposition 4.5 holds.

With (4.17) this ends the proof of (4.16), once we remarked that  $n h''_a(x_m) = F''_m(x_m)$ .

Note that, since  $|h_a''(x_m)| \leq M_a$ , we get that  $|I_{m,n}| \geq \delta n^{-1/2}$  when  $m \in J_n$ .

**Second step.** The mean-value theorem gives, for  $m \in J_n$ :

$$(4.19) \quad |\sin x_m| \geq \delta \quad \text{and} \quad x_{m+1} - x_m \approx \cos x_m - \cos x_{m+1}.$$

We also have, for  $x \in \mathcal{J} = [1 - 2\mu, 1 - \mu]$ , with another constant  $\delta$ :

$$(4.20) \quad \delta \leq P'_{-a}(x) = 2a(1 - a^2) \frac{\sin x}{(1 + 2a \cos x + a^2)^2} \leq \delta^{-1}.$$

We now claim that

$$(4.21) \quad x_{m+1} - x_m \approx n^{-1} \quad \text{for } m \in J_n.$$

Indeed, since  $m \in J_n$ , we have, using (4.11):

$$\cos x_m - \cos x_{m+1} = \frac{1 - a^2}{2a} \frac{n}{m(m+1)} \approx \frac{n}{m^2} \approx n^{-1}.$$

In view of (4.19), this proves (4.21).

Now, according to (4.16), when  $m \in J$ , the main term in  $I_{m,n}$  is

$$A_{m,n} := n^{-1/2} \frac{\sqrt{2\pi}}{\sqrt{|h_a''(x_m)|}} e^{i(F_m(x_m) + \pi/4)},$$

and its argument  $\theta_m$  is  $F_m(x_m) + \pi/4$ . Going from  $m$  to  $m+1$ , the variation  $F_{m+1}(x_{m+1}) - F_m(x_m)$  of this argument is

$$\begin{aligned} \theta_{m+1} - \theta_m &= n \int_{x_m}^{x_{m+1}} \left( P_{-a}(t) - \frac{m}{n} \right) dt - x_{m+1} \\ &= n \int_{x_m}^{x_{m+1}} \left( P_{-a}(t) - P_{-a}(x_m) \right) dt - x_{m+1}. \end{aligned}$$

But, due to (4.20), this implies:

$$\begin{aligned} \theta_{m+1} - \theta_m &= -x_{m+1} + O\left(n \int_{x_m}^{x_{m+1}} (t - x_m) dt\right) \\ &= -x_{m+1} + O(n(x_{m+1} - x_m)^2) = -x_{m+1} + O(n^{-1}). \end{aligned}$$

Since  $-\sigma \leq -x_{m+1} \leq \pi - \sigma$ , the variation of  $\theta_m$  is thus regular. As a consequence, for a positive proportion  $E_n$  of the indices  $m \in J_n$ , the argument  $\theta_m$  will belong to a subarc of  $\mathbb{T}$  which lies  $\delta$ -apart from  $\pm\pi/2$ , implying  $\cos \theta_m \geq \delta$ , or else:

$$|E_n| \geq \delta n,$$

and, for all  $m \in E_n$ :

$$\Re A_{m,n} \geq \delta |A_{m,n}| \geq \delta n^{-1/2}.$$

It follows that, for  $m \in E_n$ :

$$\Re I_{m,n} \geq \delta n^{-1/2} - C n^{-3/5} \geq \tilde{\delta} n^{-1/2}.$$

Since  $a_{m,n} = \pi^{-1} \Re I_{m,n}$ , that ends the proof of Lemma 4.3.  $\square$

We now pass to the proof of Proposition 4.5. The following lemma can be found in [15, Lemma 1, page 47].

**Lemma 4.6.** *Let  $F: [u, v] \rightarrow \mathbb{R}$ , with  $u < v$ , be a  $C^2$ -function with  $F'' > 0$ , and  $F'$  not vanishing on  $[u, v]$ . Let*

$$J = \int_u^v e^{iF(x)} dx.$$

Then:

- a) if  $F' > 0$  on  $[u, v]$ , then  $|J| \leq \frac{2}{F'(u)}$ ;
- b) If  $F' < 0$  on  $[u, v]$ , then  $|J| \leq \frac{2}{|F'(v)|}$ .

*Proof of Proposition 4.5.* Write now the integral  $I$  of Proposition 4.5 on  $[A, B]$  as  $I = I_1 + I_2 + I_3$  with:

$$I_1 = \int_A^{c-\eta} e^{iF(x)} dx, \quad I_2 = \int_{c-\eta}^{c+\eta} e^{iF(x)} dx, \quad I_3 = \int_{c+\eta}^B e^{iF(x)} dx.$$

Lemma 4.6 with  $u = A$  and  $v = c - \eta$  implies:

$$(4.22) \quad |I_1| \leq \frac{2}{|F'(c-\eta)|} \leq \frac{2}{\eta\lambda_2},$$

where, for the last inequality, we just have to write

$$|F'(c-\eta)| = F'(c) - F'(c-\eta) = \eta F''(\xi)$$

for some  $\xi \in [c-\eta, c]$  so that  $F''(\xi) \geq \lambda_2$ .

Similarly, Lemma 4.6 with  $u = c + \eta$  and  $v = B$  implies

$$(4.23) \quad |I_3| \leq \frac{2}{F'(c+\eta)} \leq \frac{2}{\eta\lambda_2}.$$

We can now estimate  $I_2$ . The Taylor formula shows that

$$F(x) = F(c) + \frac{(x-c)^2}{2} F''(c) + R,$$

with

$$|R| \leq \frac{|x-c|^3}{6} \lambda_3.$$

Hence

$$I_2 = e^{iF(c)} \int_0^\eta 2 \exp\left(\frac{i}{2} x^2 F''(c)\right) dx + S$$

with

$$|S| \leq \lambda_3 \int_0^\eta \frac{x^3}{3} dx = \frac{\eta^4}{12} \lambda_3.$$



Finally, set

$$K = \int_0^\eta 2 \exp\left(\frac{i}{2} x^2 F''(c)\right) dx.$$

We make the change of variable  $x = \sqrt{\frac{2}{F''(c)}} \sqrt{t}$ . Recall that  $\int_0^\infty \frac{e^{it}}{\sqrt{t}} dt = \sqrt{\pi} e^{i\pi/4}$  is the classical Fresnel integral, and that an integration by parts gives, for  $m > 0$ :

$$\left| \int_m^\infty \frac{e^{it}}{\sqrt{t}} dt \right| \leq \frac{2}{\sqrt{m}}.$$

Therefore, with  $m = \frac{\eta^2}{2} F''(c)$ :

$$K = \sqrt{\frac{2}{F''(c)}} \int_0^m \frac{e^{it}}{\sqrt{t}} dt = \sqrt{\frac{2\pi}{F''(c)}} e^{i\pi/4} + R_m,$$

with

$$|R_m| \leq C \sqrt{\frac{1}{F''(c)}} \frac{1}{\sqrt{m}} \leq \frac{C}{\eta \lambda_2}.$$

All in all, we proved that

$$(4.24) \quad I_2 = \sqrt{\frac{2\pi}{F''(c)}} \exp[i(F(c) + \pi/4)] + O\left(\frac{1}{\eta \lambda_2} + \eta^4 \lambda_3\right).$$

and the same estimate holds for  $I$ , thanks to (4.22) and (4.23).

We have hence proved Proposition 4.5. □

## 5 Some results on multipliers

The set  $\mathcal{M}(H^2(\beta))$  of multipliers of  $H^2(\beta)$  is by definition the vector space of functions  $h$  analytic on  $\mathbb{D}$  and such that  $hf \in H^2(\beta)$  for all  $f \in H^2(\beta)$ . When  $h \in \mathcal{M}(H^2(\beta))$ , the operator  $M_h$  of multiplication by  $h$  is bounded on  $H^2(\beta)$  by the closed graph theorem. The space  $\mathcal{M}(H^2(\beta))$  equipped with the operator norm is a Banach space. We note the obvious property:

$$(5.1) \quad \mathcal{M}(H^2(\beta)) \hookrightarrow H^\infty \quad \text{contractively.}$$

Indeed, if  $h \in \mathcal{M}(H^2(\beta))$ , we easily get for all  $w \in \mathbb{D}$ :

$$M_h^*(K_w) = \overline{h(w)} K_w;$$

so that taking norms and simplifying, we are left with  $|h(w)| \leq \|M_h\|$ , showing that  $h \in H^\infty$  with  $\|h\|_\infty \leq \|M_h\|$ .

**Proposition 5.1.** *We have  $\mathcal{M}(H^2(\beta)) = H^\infty$  isomorphically if and only if  $\beta$  is essentially decreasing.*

*Proof.* The sufficient condition is proved in [13, beginning of the proof of Proposition 3.16]. For the necessity, we have  $\|M_h\| \approx \|h\|_\infty$  for every  $h \in H^\infty$ . Now, for  $m > n$  (recall that  $e_n(z) = z^n$ ):

$$e_m(z) = z^{m-n} z^n = (M_{e_{m-n}} e_n)(z);$$

so, since  $\|M_{e_{m-n}}\| \leq C \|e_{m-n}\|_\infty = C$  for some positive constant  $C$ :

$$\beta_m = \|e_m\|^2 \leq C^2 \|e_n\|^2 = C^2 \beta_n. \quad \square$$

In [13, Section 3.6], we gave the following notion of *admissible* Hilbert space of analytic functions.

**Definition 5.2.** *A Hilbert space  $H$  of analytic functions on  $\mathbb{D}$ , containing the constants, and with reproducing kernels  $K_a$ ,  $a \in \mathbb{D}$ , is said admissible if:*

- (i)  $H^2$  is continuously embedded in  $H$ ;
- (ii)  $\mathcal{M}(H) = H^\infty$ ;
- (iii) the automorphisms of  $\mathbb{D}$  induce bounded composition operators on  $H$ ;
- (iv)  $\frac{\|K_a\|_H}{\|K_b\|_H} \leq h\left(\frac{1-|b|}{1-|a|}\right)$  for  $a, b \in \mathbb{D}$ , where  $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a non-decreasing function.

We proved in that paper that every weighted Hilbert space  $H^2(\beta)$  with  $\beta$  non-increasing is admissible, under the additional hypothesis that the automorphisms of  $\mathbb{D}$  induce bounded composition operators. In view of Theorem 3.3, we get the following result.

**Proposition 5.3.** *Let  $\beta$  be essentially decreasing that satisfies the  $\Delta_2$ -condition. Then  $H^2(\beta)$  is admissible.*

Let us give a different proof.

*Proof.* Because  $\beta$  is essentially decreasing, item (i) holds, as well as item (ii), by Proposition 5.1. Item (iii) is Theorem 3.3. It remains to show (iv). We may assume that  $\beta$  is non-increasing.

Let  $0 < s < r < 1$ .

Without loss of generality, we may assume that  $r, s \geq 1/2$ . It is enough to prove:

$$(5.2) \quad \|K_r\|^2 \leq C \|K_{r^2}\|^2$$

for some constant  $C > 1$ . Indeed, iteration of (5.2) gives:

$$\|K_r\|^2 \leq C^k \|K_{r^{2^k}}\|^2$$

and if  $k$  is the smallest integer such that  $r^{2^k} \leq s$ , we have  $2^{k-1} \log r > \log s$  and  $2^k \leq D \frac{1-s}{1-r}$  where  $D$  is a numerical constant. Writing  $C = 2^\alpha$  with  $\alpha > 1$ , we obtain:

$$\left( \frac{\|K_r\|}{\|K_s\|} \right)^2 \leq C^k = (2^k)^\alpha \leq D^\alpha \left( \frac{1-s}{1-r} \right)^\alpha.$$

To prove (5.2), we pick some  $M > 1$  such that  $\beta_{2n} \geq M^{-1}\beta_n$  for all  $n \geq 1$  and write  $t = r^2$ . We have:

$$\|K_r\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_{2n}} + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\beta_{2n-1}},$$

implying, since  $\beta_{2n-1} \geq \beta_{2n} \geq M^{-1}\beta_n$  and  $t^{2n-1} \leq 4t^{2n}$ :

$$\|K_r\|^2 \leq \frac{1}{\beta_0} + M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} + 4M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} \leq 5M \|K_t\|^2. \quad \square$$

The notion of admissible Hilbert space  $H$  is useful for the set of conditional multipliers:

$$\mathcal{M}(H, \varphi) = \{w \in H; w(f \circ \varphi) \in H \text{ for all } f \in H\}.$$

As a corollary of [13, Theorem 3.18] we get:

**Corollary 5.4.** *Let  $\beta$  be essentially decreasing and satisfying the  $\Delta_2$ -condition. Then:*

- 1)  $\mathcal{M}(H^2, \varphi) \subseteq \mathcal{M}(H^2(\beta), \varphi)$ ;
- 2)  $\mathcal{M}(H^2(\beta), \varphi) = H^2(\beta)$  if and only if  $\|\varphi\|_\infty < 1$ ;
- 3)  $\mathcal{M}(H^2(\beta), \varphi) = H^\infty$  if and only if  $\varphi$  is a finite Blaschke product.

## 6 Miscellaneous remarks

Some of the results of this paper can slightly be improved.

### 6.1 Conditions on the weight $\beta$

First, we say that a sequence  $(\beta_n)$  of positive numbers is *slowly oscillating* if there is a function  $\rho: (0, \infty) \rightarrow (0, \infty)$  that is bounded on each compact subset of  $(0, \infty)$  for which:

$$\frac{\beta_m}{\beta_n} \leq \rho\left(\frac{m}{n}\right).$$

This clearly amounts to say that, for some positive constants  $C_1 < C_2$ , we have:

$$C_1 \leq \frac{\beta_m}{\beta_n} \leq C_2 \quad \text{when } n/2 \leq m \leq 2n.$$

Every essentially decreasing sequence with the  $\Delta_2$ -condition is slowly oscillating.

**Proposition 6.1.** *The following holds:*

- 1) *every slowly oscillating sequence has a polynomial minoration;*
- 2) *there are bounded sequences which are slowly oscillating, but not essentially decreasing.*

*Proof.* 1) is clear, because if  $2^j \leq n < 2^{j+1}$ , then

$$\beta_n \geq C^{-1} \beta_{2^j} \geq C^{-j-1} \beta_1 \geq C^{-1} \beta_1 n^{-\alpha},$$

with  $\alpha = \log C / \log 2$ .

2) We define  $\beta_n$  as follows. Let  $(a_k)$  be an increasing sequence of positive square integers such that  $\lim_{k \rightarrow \infty} a_{k+1}/a_k = \infty$ , for example  $a_k = 4^{k^2}$ , and let  $b_k = \sqrt{a_k a_{k+1}}$ ; with our choice, this is an integer and we clearly have  $a_k < b_k < a_{k+1}$ . We set:

$$\beta_n = \begin{cases} a_k/n & \text{for } a_k \leq n < b_k \\ (a_k/b_k^2)n = (1/a_{k+1})n & \text{for } b_k \leq n < a_{k+1}. \end{cases}$$

This sequence  $(\beta_n)$  is slowly oscillating by construction. Indeed, it suffices to check that for  $a_k \leq n/2 < b_k \leq n < a_{k+1}$ , the quotient  $\beta_m/\beta_n$  remains lower and upper bounded when  $n/2 \leq m \leq n$ . But for  $n/2 \leq m < b_k$ , we have

$$\frac{\beta_m}{\beta_n} = \frac{a_k/m}{n/a_{k+1}} = \frac{a_k a_{k+1}}{mn} = \frac{b_k^2}{mn},$$

which is  $\leq 2b_k^2/n^2 \leq 2$  and  $\geq b_k^2/n^2 \geq (n/2)^2/n^2 = 1/4$ ; and for  $b_k \leq m$ , we have

$$\frac{\beta_m}{\beta_n} = \frac{m/a_{k+1}}{n/a_{k+1}} = \frac{m}{n} \in [1/2, 1].$$

However, though  $(\beta_n)$  is bounded, since  $\beta_n \leq 1$  for  $a_k \leq n < b_k$  and, for  $b_k \leq n < a_{k+1}$ ,

$$\beta_n \leq \beta_{a_{k+1}-1} = \frac{1}{a_{k+1}} (a_{k+1} - 1) \leq 1,$$

it is not essentially decreasing, since

$$\frac{\beta_{a_{k+1}-1}}{\beta_{b_k}} = \frac{1}{\sqrt{a_k a_{k+1}}} (a_{k+1} - 1) \sim \sqrt{\frac{a_{k+1}}{a_k}} \xrightarrow{k \rightarrow \infty} \infty. \quad \square$$

By a slight modification (change the value of the constants in the definition of  $\beta_n$ ), we could obtain a sequence which is slowly oscillating, tends to zero, yet again not essentially decreasing.

Now, Theorem 3.3 admits the following variant.

**Theorem 6.2.** *Let  $(\beta_n)$  be a sequence of positive numbers which is bounded above and slowly oscillating. Then all symbols that extend analytically in a neighborhood of  $\mathbb{D}$  induce a bounded composition operator on  $H^2(\beta)$ .*

It is the case, for example, for finite Blaschke products. The proof follows that of Theorem 3.3, with the help of the following lemma.

**Lemma 6.3.** *Let  $(\beta_n)$  be a sequence of positive numbers which is bounded above and slowly oscillating. Let  $A = (a_{m,n})_{m,n}$  be the matrix of a bounded operator on  $\ell_2$ . Assume that, for constants  $C_1 < 1$ ,  $C_2 > 1$ , and  $c, b$ , we have:*

- 1)  $|a_{m,n}| \leq c e^{-bn}$  for  $m \leq C_1 n$ ;
- 2)  $|a_{m,n}| \leq c e^{-bm}$  for  $m \geq C_2 n$ .

Then the matrix  $\tilde{A} = \left( a_{m,n} \sqrt{\frac{\beta_m}{\beta_n}} \right)$  also defines a bounded operator on  $\ell_2$ .

*Sketch of proof.* The matrix  $\tilde{A}$  is Hilbert-Schmidt far from the diagonal since, with  $\lambda \geq \beta_n \geq \delta n^{-\alpha}$ , we have:

$$\sum_{m < C_1 n} |a_{m,n}|^2 \beta_m / \beta_n \leq \sum_{m < C_1 n} \lambda \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n \geq 1} n^{\alpha+1} e^{-bn} < \infty,$$

and

$$\sum_{m > C_2 n} |a_{m,n}|^2 \beta_m / \beta_n \leq \sum_{m > C_2 n} \lambda \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n \geq 1} n^\alpha \left( \sum_{m > C_2 n} e^{-bm} \right).$$

Since  $\beta_m / \beta_n$  remains bounded from above and below around the diagonal, the matrix  $\tilde{A}$  behaves like  $A$  near the diagonal.  $\square$

**Remark.** The proof shows that, instead of 1) and 2), it is enough to have:

$$\sum_{m < C_1 n} n^{\alpha+1} |a_{m,n}|^2 < \infty \quad \text{and} \quad \sum_{m > C_2 n} m^\alpha |a_{m,n}|^2 < \infty.$$

Moreover the proof also shows that when  $\beta$  is slowly oscillating, if we set  $E = \{(m, n); C_1 n \leq m \leq C_2 n\}$ , then the matrix  $(\sqrt{\beta_m / \beta_n} \mathbb{1}_E(m, n))$  is a Schur multiplier over *all* the bounded matrices, while Kacnel'son's theorem (Theorem 3.5) says that, if  $\gamma = (\gamma_n)$  is non-increasing, the matrix  $(\gamma_m / \gamma_n)$  is a Schur multiplier of all bounded *lower-triangular* matrices.

*Proof of Theorem 6.2.* We first prove that the assumptions of Lemma 6.3 are satisfied with  $a_{m,n} = \widehat{\varphi}^n(m)$ . This works nearly as in Lemma 3.4. For every symbol  $\varphi$ , let  $M(r) = \sup_{|z|=r} |\varphi(z)|$  and write  $M(1/e) = e^{-\delta}$ , with  $\delta > 0$ . Let  $A$  be the matrix of  $C_\varphi$ , with respect to the canonical basis of  $H^2$ . The Cauchy inequalities give, if  $m \leq (\delta/2)n$ :

$$|a_{m,n}| \leq [M(1/e)]^n e^m \leq e^{m-\delta n} \leq e^{-\delta/2n}$$

When  $\varphi$  is analytic in a neighborhood of  $\overline{D(0, R)}$  with  $R > 1$ , the Cauchy inequalities give, writing  $M(R) = e^\rho$  and  $R = e^\delta$ , for  $m \geq C_2 n$ , and a suitable constant  $C_2$  (take  $C_2 = (2\rho)/\delta$  for instance):

$$|a_{m,n}| \leq \frac{[M(R)]^n}{R^m} \leq e^{n\rho - \delta m} \leq e^{-(\delta/2)m}.$$

We now conclude with Lemma 6.3.  $\square$

## 6.2 Singular inner functions

Let  $a > 0$  and let  $I_a$  be the singular inner function defined by

$$(6.1) \quad I_a(z) = \exp\left(-a \frac{1+z}{1-z}\right) = \sum_{m=0}^{\infty} c_m(a) z^m.$$

If  $n$  is a positive integer, we have  $I_n(z) = [I_1(z)]^n$  and we write

$$I_n(z) = \sum_{m=0}^{\infty} a_{m,n} z^m, \quad \text{with } a_{m,n} = c_m(n).$$

We rely on the following lemma, familiar to experts in orthogonal polynomials and special functions, but maybe not so much as regards the uniformity, essential for our present purposes (see [16] or [12]).

**Lemma 6.4.** *It holds*

$$(6.2) \quad c_m(n) = c n^{1/4} m^{-3/4} \cos(2\sqrt{2nm} + \pi/4) + R_m(n) =: M_m(n) + R_m(n),$$

where  $c = \pi^{-1/2} 2^{1/4}$  and where  $|R_m(n)| \leq K \sqrt{n} m^{-5/4}$ , with  $K$  some numerical constant.

We use the following [16, p. 253] and [19, p. 198], where  $L_m^{(\alpha)}$  denotes the generalized Laguerre polynomial of degree  $m$  with parameter  $\alpha$ .

**Theorem 6.5.** *With the notation of (6.1), we have*

$$c_m(a) = e^{-a} L_m^{(-1)}(2a).$$

Moreover, we have, uniformly for  $0 < \varepsilon \leq x \leq M < \infty$ :

$$L_m^{(\alpha)}(x) = \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} m^{\alpha/2-1/4} \cos(2\sqrt{mx} - \alpha\pi/2 - \pi/4) + R_m(x),$$

where  $|R_m(x)| \leq K_\alpha m^{\alpha/2-3/4}$ , and  $K_\alpha > 0$  only depending on  $\alpha$ .

Now, using Theorem 6.5 with  $\alpha = -1$  and  $a = n$ , we get Lemma 6.4. Actually, to get Lemma 6.4, we need some uniformity with respect to  $a$  in Theorem 6.5, which is not given by the above statement of that theorem; but provided we change  $m^{-5/4}$  into  $\sqrt{n} m^{-5/4}$  in  $R_m$ , a careful examination of Fejér's proof of Theorem 6.5 shows that this uniformity holds. Alternatively, write  $c_m(a)$  as a Fourier coefficient

$$(6.3) \quad c_m(n) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp[i(n \cot x + 2mx)] dx$$

and use the previous van der Corput estimates.

Once this lemma is at our disposal, we can prove again the following theorem.

**Theorem 6.6.** *Assume that  $\beta$  is a non-increasing sequence and that the composition operator  $C_{I_1}$  maps  $H^2(\beta)$  to itself. Then  $\beta$  satisfies the  $\Delta_2$ -condition.*

*Proof.* Our assumption implies, with  $M = \|C_{I_1}\|^2$ :

$$(6.4) \quad \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m \leq M \beta_n.$$

We use Lemma 6.4 with  $C^{-2}n \leq m < n$ , where  $C > 1$  satisfies  $C\sqrt{2} < \pi/2$ , which is possible since  $2\sqrt{2} < \pi$ . The term  $R_m(n)$  is dominated by  $KC^{-1/2}m^{-3/4}$ . Let  $\theta_m = 2\sqrt{2}\sqrt{nm} + \pi/4$  be the argument of the number appearing in (6.2). We have

$$\theta_{m+1} - \theta_m = \frac{2\sqrt{2}\sqrt{n}}{\sqrt{m} + \sqrt{m+1}};$$

hence

$$\sqrt{2} \leq \theta_{m+1} - \theta_m \leq C\sqrt{2} < \frac{\pi}{2}.$$

The argument  $\theta_m$  then varies regularly, and there is a positive constant  $\delta$  and a subset  $E$  of integers in the interval  $[C^{-2}n, n]$  such that  $|E| \geq \delta n$  and

$$M_m(n) \geq 2\delta n^{1/4}m^{-3/4} \geq 2\delta n^{-1/2}$$

for all  $m \in E$ . Therefore, for  $n$  large enough, we have, for all  $m \in E$ :

$$|a_{m,n}| \geq 2\delta n^{-1/2} - K\sqrt{n}m^{-5/4} \geq 2\delta n^{-1/2} - KC^{5/2}n^{-3/4} \geq \delta n^{-1/2}.$$

With this information, (6.4) gives:

$$M\beta_n \geq \sum_{m \in E} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} |E| \beta_{\lfloor C^{-2}n \rfloor} \geq \delta^3 \beta_{\lfloor C^{-2}n \rfloor},$$

where  $\lfloor \cdot \rfloor$  stands for the integer part, and this proves the theorem.  $\square$

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