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Pascal Lefèvre, Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza. Boundedness of composition operators on general weighted Hardy spaces of analytic functions. 2020. hal-03029931v1

**HAL Id: hal-03029931**

**<https://univ-artois.hal.science/hal-03029931v1>**

Preprint submitted on 29 Nov 2020 (v1), last revised 14 Mar 2022 (v2)

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# Boundedness of composition operators on general weighted Hardy spaces of analytic functions

*Pascal Lefèvre, Daniel Li,  
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November 29, 2020

**Abstract.** We characterize the (essentially) decreasing sequences of positive numbers  $\beta = (\beta_n)$  for which all composition operators on  $H^2(\beta)$  are bounded, where  $H^2(\beta)$  is the space of analytic functions  $f$  in the unit disk such that  $\sum_{n=0}^{\infty} |c_n|^2 \beta_n < \infty$  if  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ . We also give conditions for the boundedness when  $\beta$  is not assumed essentially decreasing.

**MSC 2010** primary: 47B33 ; secondary: 30H10

**Key-words** automorphism of the unit disk; composition operator;  $\Delta_2$ -condition; multipliers; weighted Hardy space

## 1 Introduction

Let  $\beta = (\beta_n)_{n \geq 0}$  be a sequence of positive numbers such that

$$(1.1) \quad \liminf_{n \rightarrow \infty} \beta_n^{1/n} \geq 1.$$

The associated weighted Hardy space  $H^2(\beta)$  is the set of analytic functions  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  such that

$$(1.2) \quad \|f\|^2 := \sum_{n=0}^{\infty} |a_n|^2 \beta_n < \infty.$$

In view of (1.1),  $H^2(\beta)$  is a Hilbert space of analytic functions on  $\mathbb{D}$  with the canonical orthonormal basis

$$(1.3) \quad e_n^\beta(z) = \frac{1}{\sqrt{\beta_n}} z^n, \quad n \geq 0,$$

and the reproducing kernel  $K_w$  given for all  $w \in \mathbb{D}$  by

$$(1.4) \quad K_w(z) = \sum_{n=0}^{\infty} e_n^\beta(z) \overline{e_n^\beta(w)} = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \overline{w}^n z^n.$$

Indeed, we have:

$$\left| \sum_{k=0}^{\infty} a_k w^k \right| \leq \left( \sum_{k=0}^{\infty} \beta_k |a_k|^2 \right)^{1/2} \left( \sum_{k=0}^{\infty} \frac{1}{\beta_k} |w|^{2k} \right)^{1/2} < \infty,$$

thanks to (1.1).

Note that (1.1) is necessary for  $H^2(\beta)$  to consist of analytic functions in  $\mathbb{D}$ . Indeed the fact that  $\sum_{n \geq 1} \frac{1}{n\sqrt{\beta_n}} z^n$  belongs to  $H^2(\beta)$  and is analytic in  $\mathbb{D}$  implies (1.1).

When  $\beta_n \equiv 1$ , we recover the usual Hardy space  $H^2$ .

Note that  $H^2$  is continuously embedded in  $H^2(\beta)$  if and only if  $\beta$  is bounded above. In particular, this is the case when  $\beta$  is non-increasing.

Most of works with weighted Hardy spaces concern the case

$$(1.5) \quad \beta_n = \int_0^1 t^n d\sigma(t)$$

where  $\sigma$  is a positive measure on  $(0, 1)$ . More specifically the following definition is often used. Let  $G: (0, 1) \rightarrow \mathbb{R}_+$  be an integrable function and let  $H_G^2$  be the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that:

$$(1.6) \quad \|f\|_{H_G^2}^2 := \int_{\mathbb{D}} |f(z)|^2 G(1 - |z|^2) dA(z) < \infty.$$

Such weighted Bergman type spaces are used, for instance, in [11], [10] and in [14]. We have  $H_G^2 = H^2(\beta)$  with:

$$(1.7) \quad \beta_n = 2 \int_0^1 r^{2n+1} G(1 - r^2) dr = \int_0^1 t^n G(1 - t) dt,$$

and the sequence  $\beta = (\beta_n)_n$  is non-increasing.

Recall that a *symbol* is a (non constant) analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ , and the associated composition operator  $C_\varphi: H^2(\beta) \rightarrow H^2(\beta)$  is defined (formally) as:

$$(1.8) \quad C_\varphi(f) = f \circ \varphi.$$

An important question in the theory is to decide when  $C_\varphi$  is bounded on  $H^2(\beta)$ , i.e. when  $C_\varphi: H^2(\beta) \rightarrow H^2(\beta)$ .

When  $\beta_n \equiv 1$ , i.e. when  $H^2(\beta)$  is the usual Hardy space  $H^2$ , it is known ([18, pp. 13–17]) that all symbols generate bounded composition operators. But in Shapiro's presentation, the main point is the case  $\varphi(0) = 0$  and a subordination principle for subharmonic functions. The case of automorphisms is claimed simple, using an integral representation for the norm and some change of variable. When  $\beta$  is defined as in (1.5), one disposes of integral representations for the

norm in  $H^2(\beta)$ , and, as in the Hardy space case, this integral representation rather easily gives the boundedness of  $C_{T_a}$  on  $H^2$ , where

$$(1.9) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}$$

for  $a \in \mathbb{D}$ . But the above representation (1.5) is equivalent, by the Hausdorff moment theorem, to a high regularity of the sequence  $\beta$ , namely its *complete monotony*. When integral representations fail, we have to work with bare hands. If the symbol vanishes at the origin, Kacnel'son's theorem (see Theorem 3.6 below) gives a positive answer when  $\beta$  is essentially decreasing (see [2] or [13, Theorem 3.12]). Actually that follows from an older theorem of Goluzin [8] (see [5, Theorem 6.3]), which itself uses a refinement by Rogosinski of Littlewood's principle ([5, Theorem 6.2]). So that the main issue remains the boundedness of  $C_{T_a}$ . A polynomial minoration (see Definition 2.3 below) for  $\beta$  is necessary for any  $C_{T_a}$  to be bounded on  $H^2(\beta)$  (Proposition 2.5) and we showed in [13, end of Section 3] that for  $\beta_n = \exp(-\sqrt{n})$ ,  $C_{T_a}$  is never bounded on  $H^2(\beta)$ . But this polynomial minoration is not sufficient, as we will see in Theorem 1.1 below, which also evidences the basic role of the maps  $T_a$  in the question.

So, our goal in this paper is characterizing the sequences  $\beta$  for which all composition operators act boundedly on the space  $H^2(\beta)$ , i.e. send  $H^2(\beta)$  into itself. Eventually, we will obtain in Theorem 3.4 and Theorem 4.1 the following result, where the  $\Delta_2$ -condition is defined in (2.2).

**Theorem 1.1.** *Let  $\beta$  be an essentially decreasing sequence of positive numbers. The following assertions are equivalent:*

- 1) *all composition operators are bounded on  $H^2(\beta)$ ;*
- 2) *all maps  $T_a$ , for  $0 < a < 1$ , induce bounded composition operators  $C_{T_a}$  on  $H^2(\beta)$ ;*
- 3) *for some  $a \in (0, 1)$ , the map  $T_a$  induces a bounded composition operator  $C_{T_a}$  on  $H^2(\beta)$ ;*
- 4)  *$\beta$  satisfies the  $\Delta_2$ -condition.*

Note that, by definition of the norm of  $H^2(\beta)$ , all rotations  $R_\theta$ ,  $\theta \in \mathbb{R}$ , induce bounded composition operators on  $H^2(\beta)$  and send isometrically  $H^2(\beta)$  into itself.

However, we construct a weight  $\beta$  which is not essentially decreasing and for which all composition operators with symbol vanishing at 0 are bounded (Theorem 3.3), though no map  $T_a$  with  $0 < a < 1$  induces a bounded composition operator (Proposition 4.4).

For spaces of Bergman type  $A_G^2 := H_G^2$ , where  $\tilde{G}(r) = G(1 - r^2)$ , defined as the spaces of analytic functions in  $\mathbb{D}$  such that  $\int_{\mathbb{D}} |f(z)|^2 \tilde{G}(|z|) dA < \infty$ , for a positive non-increasing continuous function  $\tilde{G}$  on  $[0, 1)$ , Kriete and MacCluer studied in [11] some analogous problems. They proved, in particular [11,

Theorem 3] that, for:

$$\tilde{G}(r) = \exp \left( -B \frac{1}{(1-r)^\alpha} \right), \quad B > 0, \quad 0 < \alpha \leq 2,$$

and

$$\varphi(z) = z + t(1-z)^\beta, \quad 1 < \beta \leq 3, \quad 0 < t < 2^{1-\beta},$$

then  $C_\varphi$  is bounded on  $A_G^2$  if and only if  $\beta \geq \alpha + 1$ .

Here

$$\beta_n = \int_0^1 t^n e^{-B/(1-\sqrt{t})^\alpha} dt;$$

and, since  $\beta_n \approx \exp(-c n^{\alpha/(\alpha+1)})$ , the sequence  $(\beta_n)$  does not satisfy the  $\Delta_2$ -condition, accordingly to our Theorem 3.4 below.

## 2 Definitions, notation, and preliminary results

The open unit disk of  $\mathbb{C}$  is denoted  $\mathbb{D}$  and we write  $\mathbb{T}$  its boundary  $\partial\mathbb{D}$ .

We set  $e_n(z) = z^n$ ,  $n \geq 0$ .

As said in the introduction,  $H^2$  is continuously embedded in  $H^2(\beta)$  when  $\beta$  is non-increasing. In this paper, we need a slightly more general notion.

**Definition 2.1.** *A sequence of positive numbers  $\beta = (\beta_n)$  is said essentially decreasing if, for some constant  $C \geq 1$ , we have, for all  $m \geq n \geq 0$ :*

$$(2.1) \quad \beta_m \leq C \beta_n.$$

Note that saying that  $\beta$  is essentially decreasing means that the shift operator on  $H^2(\beta)$  is power bounded.

If  $\beta$  is essentially decreasing, and if we set:

$$\tilde{\beta}_n = \sup_{m \geq n} \beta_m,$$

the sequence  $\tilde{\beta} = (\tilde{\beta}_n)$  is non-increasing and we have  $\beta_n \leq \tilde{\beta}_n \leq C \beta_n$ . In particular, the space  $H^2(\beta)$  is isomorphic to  $H^2(\tilde{\beta})$  and  $H^2(\beta)$  is continuously embedded in  $H^2$ .

**Definition 2.2.** *The sequence of positive numbers  $\beta = (\beta_n)$  is said to satisfy the  $\Delta_2$ -condition if there is a positive constant  $\delta < 1$  such that, for all integers  $n \geq 0$ :*

$$(2.2) \quad \beta_{2n} \geq \delta \beta_n.$$

**Definition 2.3.** *The sequence of positive numbers  $\beta = (\beta_n)$  is said to have a polynomial minoration if there are positive constants  $\delta$  and  $\alpha$  such that, for all integers  $n \geq 1$ :*

$$(2.3) \quad \beta_n \geq \delta n^{-\alpha}.$$

That means that  $H^2(\beta)$  is continuously embedded in the weighted Bergman space  $\mathfrak{B}_\alpha^2$  of the analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that

$$\|f\|_{\mathfrak{B}_\alpha^2}^2 := (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty$$

since  $\mathfrak{B}_\alpha^2 = H^2(\gamma)$  with  $\gamma_n \approx n^{-\alpha}$ .

The following simple proposition links those notions.

**Proposition 2.4.** *Let  $\beta$  be an essentially decreasing sequence of positive numbers. Then if  $\beta$  satisfies the  $\Delta_2$ -condition, it has a polynomial minoration.*

*The converse does not hold.*

*Proof.* 1) Assume  $\beta_m \leq C \beta_n$  for  $m \geq n$  and  $\beta_{2^k} \geq e^{-A} \beta_1$ . Let now  $n$  be an integer  $\geq 2$ , and  $k \geq 1$  the smallest integer such that  $2^k \geq n$ , so that  $k \leq a \log n$  with  $a$  a positive constant. We get:

$$\beta_n \geq C^{-1} \beta_{2^k} \geq C^{-1} e^{-kA} \beta_1 \geq C^{-1} \beta_1 e^{-aA \log n} =: \rho n^{-\alpha},$$

with  $\rho = C^{-1} \beta_1$  and  $\alpha = aA$ .

2) Let  $\delta > 0$ . We set  $\beta_0 = \beta_1 = 1$  and for  $n \geq 2$ :

$$\beta_n = \frac{1}{(k!)^\delta} \quad \text{when } k! < n \leq (k+1)!.$$

The sequence  $\beta$  is non-increasing.

For  $n$  and  $k$  as above, we have:

$$\beta_n = \frac{1}{(k!)^\delta} \geq \frac{1}{n^\delta};$$

hence  $\beta$  has arbitrary polynomial minoration. However we have, for  $k \geq 2$ :

$$\frac{\beta_{2k!}}{\beta_{k!}} = \frac{(k!)^{-\delta}}{[(k-1)!]^{-\delta}} = \frac{1}{k^\delta} \xrightarrow[k \rightarrow \infty]{} 0,$$

so  $\beta$  fails to satisfy the  $\Delta_2$ -condition. □

For  $a \in \mathbb{D}$ , we define:

$$(2.4) \quad T_a(z) = \frac{a+z}{1+\bar{a}z}, \quad z \in \mathbb{D}.$$

Recall that  $T_a$  is an automorphism of  $\mathbb{D}$  and that  $T_a(0) = a$  and  $T_a(-a) = 0$ .

Though we do not need this, we may remark that  $(T_a)_{a \in (-1,1)}$  is a group and  $(T_a)_{a \in (0,1)}$  is a semigroup. It suffices to see that  $T_a \circ T_b = T_{a*b}$ , with:

$$(2.5) \quad a * b = \frac{a+b}{1+ab}.$$

**Proposition 2.5.** *Let  $a \in (0, 1)$  and assume that  $T_a$  induces a bounded composition operator on  $H^2(\beta)$ . Then  $\beta$  has a polynomial minoration.*

**Remarks.** 1) For example, when  $\beta_n = \exp[-c(\log(n+1))^2]$ , with  $c > 0$ , no  $T_a$  induces a bounded composition operator on  $H^2(\beta)$ , though all symbols  $\varphi$  with  $\varphi(0) = 0$  are bounded, since  $\beta$  is decreasing, as said by the forthcoming Proposition 3.2.

2) For the Dirichlet space  $\mathcal{D}^2$ , we have  $\beta_n = n+1$ , but all the maps  $T_a$  induce bounded composition operators on  $\mathcal{D}^2$  (see [13, Remark before Theorem 3.12]). In this case  $\beta$  has a polynomial minoration though it is not bounded above.

3) However, even for decreasing sequences, a polynomial minoration for  $\beta$  is not enough for some  $T_a$  to induce a bounded composition operator. Indeed, we saw in Proposition 2.4 an example of a decreasing sequence  $\beta$  with polynomial minoration, but not sharing the  $\Delta_2$ -condition, and we will see in Theorem 4.1 that the  $\Delta_2$ -condition is needed for having some  $T_a$  inducing a bounded composition operator.

4) In [7], Eva Gallardo-Gutiérrez and Jonathan Partington give estimates for the norm of  $C_{T_a}$ , with  $a \in (0, 1)$ , when  $C_{T_a}$  is bounded on  $H^2(\beta)$ . More precisely, they proved that if  $\beta$  is bounded above and  $C_{T_a}$  is bounded, then

$$\|C_{T_a}\| \geq \left(\frac{1+a}{1-a}\right)^\sigma,$$

where  $\sigma = \inf\{s \geq 0; (1-z)^{-s} \notin H^2(\beta)\}$ , and

$$\|C_{T_a}\| \leq \left(\frac{1+a}{1-a}\right)^\tau,$$

where  $\tau = \frac{1}{2} \sup \Re W(A)$ , with  $A$  the infinitesimal generator of the continuous semigroup  $(S_t)$  defined as  $S_t = C_{T_{\tanh t}}$ , namely  $(Af)(z) = f'(z)(1-z^2)$ , and  $W(A)$  its numerical range.

For  $\beta_n = 1/(n+1)^\nu$  with  $0 \leq \nu \leq 1$ , the two bounds coincide, so they get  $\|C_{T_a}\| = \left(\frac{1+a}{1-a}\right)^{(\nu+1)/2}$ .

*Proof of Proposition 2.5.* Since

$$\|K_x\|^2 = \sum_{n=0}^{\infty} \frac{x^{2n}}{\beta_n},$$

we have  $\|K_x\| \leq \|K_y\|$  for  $0 \leq x \leq y < 1$ .

We define by induction a sequence  $(u_n)_{n \geq 0}$  with:

$$u_0 = 0 \quad \text{and} \quad u_{n+1} = T_a(u_n).$$

Since  $T_a(1) = 1$  (recall that  $a \in (0, 1)$ ), we have:

$$1 - u_{n+1} = \int_{u_n}^1 T'_a(t) dt = \int_{u_n}^1 \frac{1-a^2}{(1+at)^2} dt;$$

hence

$$\frac{1-a}{1+a}(1-u_n) \leq 1-u_{n+1} \leq (1-a^2)(1-u_n).$$

Let  $0 < x < 1$ . We can find  $N \geq 0$  such that  $u_N \leq x < u_{N+1}$ . Then:

$$1-x \leq 1-u_N \leq (1-a^2)^N.$$

On the other hand, since  $C_{T_a}^* K_z = K_{T_a(z)}$  for all  $z \in \mathbb{D}$ , we have:

$$\|K_x\| \leq \|K_{u_{N+1}}\| \leq \|C_{T_a}\| \|K_{u_N}\| \leq \|C_{T_a}\|^{N+1} \|K_{u_0}\| = \|C_{T_a}\|^{N+1}.$$

Let  $s \geq 0$  such that  $(1-a^2)^{-s} = \|C_{T_a}\|$ . We obtain:

$$(2.6) \quad \|K_x\| \leq \|C_{T_a}\| \frac{1}{(1-x)^s}.$$

But

$$\|K_x\|^2 = \sum_{k=0}^{\infty} \frac{x^{2k}}{\beta_k};$$

so we get, for any  $k \geq 2$ :

$$\frac{x^{2k}}{\beta_k} \leq \|C_{T_a}\|^2 \frac{1}{(1-x)^{2s}}.$$

Taking  $x = 1 - \frac{1}{k}$ , we obtain  $\beta_k \geq C k^{-2s}$ . □

**Remark.** We saw in the proof of Proposition 2.5 that if  $C_{T_a}$  is bounded on  $H^2(\beta)$  for some  $a \in (0, 1)$ , then the reproducing kernels  $K_w$  have, by (2.6), a *slow growth*:

$$(2.7) \quad \|K_w\| \leq \frac{C}{(1-|w|)^s}$$

for positive constants  $C$  and  $s$ . Actually, we have the following equivalence.

**Proposition 2.6.** *The sequence  $\beta$  has a polynomial minoration if and only if the reproducing kernels  $K_w$  of  $H^2(\beta)$  have a slow growth.*

*Proof.* The sufficiency is easy and seen at the end of the proof of Proposition 2.5. For the necessity, we only have to see that:

$$\|K_w\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{|w|^{2n}}{\beta_n} \leq \frac{1}{\beta_0} + \delta^{-1} \sum_{n=1}^{\infty} n^\alpha |w|^{2n} \leq \frac{C}{(1-|w|^2)^{\alpha+1}}. \quad \square$$



### 3 Boundedness of composition operators

We study in this section conditions ensuring that all composition operators on  $H^2(\beta)$  are bounded.

#### 3.1 Conditions on the weight

We begin with this simple observation.

**Proposition 3.1.** *If all composition operators, and even if all composition operators with symbol vanishing at 0, are bounded on  $H^2(\beta)$ , then the sequence  $\beta$  is bounded above.*

*Proof.* If  $\beta$  is not bounded above, there is a subsequence  $(\beta_{n_k})$ , with  $n_k \geq 1$ , such that  $\beta_{n_k} \geq 4^k$ . Then  $\varphi(z) = \sum_{k=1}^{\infty} 2^{-k} z^{n_k}$  defines a symbol, since  $|\varphi(z)| < 1$ , and  $\varphi(0) = 0$ ; but:

$$\|C_{\varphi}(e_1)\|^2 = \|\varphi\|^2 = \sum_{k=1}^{\infty} 4^{-k} \beta_{n_k} = \infty. \quad \square$$

For symbols vanishing at 0, we have the following characterization.

**Proposition 3.2.** *The following assertions are equivalent:*

1) *all symbols  $\varphi$  such that  $\varphi(0) = 0$  induce bounded composition operators  $C_{\varphi}$  on  $H^2(\beta)$  and*

$$(3.1) \quad \sup_{\varphi(0)=0} \|C_{\varphi}\| < \infty;$$

2)  *$\beta$  is an essentially decreasing sequence.*

Of course, by the uniform boundedness principle, (3.1) is equivalent to:

$$\sup_{\varphi(0)=0} \|f \circ \varphi\| < \infty \quad \text{for all } f \in H^2(\beta).$$

*Proof.* 2)  $\Rightarrow$  1) We may assume that  $\beta$  is non-increasing. Then the Goluzin-Rogosinski theorem ([5, Theorem 6.3]) gives the result; in fact, writing  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  and  $(C_{\varphi}f)(z) = \sum_{n=0}^{\infty} d_n z^n$ , it says that:

$$\|C_{\varphi}f\|^2 = |d_0|^2 \beta_0 + \sum_{n=1}^{\infty} |d_n|^2 \beta_n \leq |c_0|^2 \beta_0 + \sum_{n=1}^{\infty} |c_n|^2 \beta_n = \|f\|^2,$$

leading to  $C_{\varphi}$  bounded and  $\|C_{\varphi}\| \leq 1$ . Alternatively, we can use a result of Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]). This result was also proved by C. Cowen [4, Corollary of Theorem 7].

1)  $\Rightarrow$  2) Set  $M = \sup_{\varphi(0)=0} \|C_{\varphi}\|$ . Let  $m > n$ , and take:

$$\varphi(z) = z \left( \frac{1 + z^{m-n}}{2} \right)^{1/n}.$$

Then  $\varphi(0) = 0$  and  $[\varphi(z)]^n = \frac{z^n + z^m}{2}$ ; hence

$$\frac{1}{4}(\beta_n + \beta_m) = \|\varphi^n\|^2 = \|C_\varphi(e_n)\|^2 \leq \|C_\varphi\|^2 \|e_n\|^2 \leq M^2 \beta_n,$$

so  $\beta$  is essentially decreasing.  $\square$

For example, let  $(\beta_n)$  such that  $\beta_{2k+2}/\beta_{2k+1} \xrightarrow[k \rightarrow \infty]{} \infty$  (for instance  $\beta_{2k} = 1$  and  $\beta_{2k+1} = 1/(k+1)$ ); if  $\varphi(z) = z^2$ , then  $\|C_\varphi(z^{2n+1})\|^2 = \|z^{2(2n+1)}\|^2 = \beta_{2(2n+1)}$ ; since  $\|z^{2n+1}\|^2 = \beta_{2n+1}$ ,  $C_\varphi$  is not bounded on  $H^2(\beta)$ .

A more interesting example is the following. For  $0 < r < 1$ , let  $\beta_n = \pi n r^{2n}$ . This sequence is eventually decreasing, so it is essentially decreasing. The square of the norm  $\|f\|_{H^2(\beta)}^2$  is the area of the part of the Riemann surface on which  $r\mathbb{D}$  is mapped by  $f$ . E. Reich [17], generalizing Goluzin's result [8] (see [5, Theorem 6.3]), proved that for all symbols  $\varphi$  such that  $\varphi(0) = 0$ , the composition operator  $C_\varphi$  is bounded on  $H^2(\beta)$  and

$$\|C_\varphi\| \leq \sup_{n \geq 1} \sqrt{n} r^{n-1} \leq \frac{1}{\sqrt{2e}} \frac{1}{r \sqrt{\log(1/r)}}.$$

For  $0 < r < 1/\sqrt{2}$ , Goluzin's theorem asserts that  $\|C_\varphi\| \leq 1$ .

Note that this sequence  $\beta$  does not satisfy the  $\Delta_2$ -condition since  $\beta_{2n}/\beta_n = 2r^{2n}$ , Theorem 4.1 below states that no composition operator  $C_{T_a}$  is bounded.

However that the weight  $\beta$  is essentially decreasing is not necessary for the boundedness of all composition operators  $C_\varphi$ , with symbol  $\varphi$  vanishing at 0. as stated by the following theorem, whose proof will be given in Section 3.3.

**Theorem 3.3.** *There exists a bounded sequence  $\beta$ , with a polynomial minoration, but which is not essentially decreasing, for which every composition operator with symbol vanishing at 0 is bounded on  $H^2(\beta)$ .*

It should be noted that for this weight, the composition operators are not all bounded, as we will see in Proposition 4.4.

### 3.2 Boundedness of composition operators I

We now have one of the the main results of this section.

**Theorem 3.4.** *Let  $H^2(\beta)$  be a weighted Hardy space with  $\beta = (\beta_n)$  essentially decreasing and satisfying the  $\Delta_2$ -condition. Then all composition operators on  $H^2(\beta)$  are bounded.*

For the proof, we need a lemma.

**Lemma 3.5.** For  $0 < a < 1$ , we write:

$$(3.2) \quad (T_a z)^n = \sum_{m=0}^{\infty} a_{m,n} z^m.$$

Then, for every  $\alpha > 0$ , there exists a positive constant  $\rho$ , depending on  $a$ , such that:

$$(3.3) \quad |a_{m,n}| \leq e^{-\alpha(\rho n - m)}.$$

Note that this estimate is interesting only when  $m < \rho n$  since we know that  $|a_{m,n}| \leq \|T_a^n\|_{\infty} = 1$  for all  $m$  and  $n$ .

*Proof.* For  $0 < r < 1$ , let:

$$M(r) = \sup_{|z|=r} |T_a(z)| = \sup_{|z|=r} \left| \frac{z+a}{1+az} \right|.$$

We have  $M(r) < 1$ , so we can write  $M(r) = r^{\rho}$ , for some positive  $\rho = \rho(a)$ .

The Cauchy inequalities give:

$$|a_{m,n}| \leq \frac{[M(r)]^n}{r^m} = r^{\rho n - m},$$

and we obtain the result by taking  $r = e^{-\alpha}$ .  $\square$

*Proof of Theorem 3.4.* We may, and do, assume that  $\beta$  is non-increasing.

Proposition 3.2 gives the result when  $\varphi(0) = 0$ .

It remains to show that all  $C_{T_a}$ ,  $a \in \mathbb{D}$ , are bounded. Indeed, if  $a = \varphi(0)$  and  $\psi = T_{-a} \circ \varphi$ , then  $\psi(0) = 0$  and  $\varphi = T_a \circ \psi$ , so  $C_{\varphi} = C_{\psi} \circ C_{T_a}$ . Moreover, we have only to show that when  $a \in [0, 1)$ . Indeed, if  $a \in \mathbb{D}$  and  $a = |a|e^{i\theta}$ , we have  $T_a = R_{\theta} \circ T_{|a|} \circ R_{-\theta}$ , so  $C_{T_a} = C_{R_{-\theta}} \circ C_{T_{|a|}} \circ C_{R_{\theta}}$ .

We consider the matrices

$$A = (a_{m,n})_{m,n \geq 0} \quad \text{and} \quad A_{\beta} = \left( \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} \right)_{m,n \geq 0}.$$

Since  $C_{T_a} e_n = T_a^n$ , the formula (3.2) shows that  $A$  is the matrix of  $C_{T_a}$  in  $H^2$  with respect to the basis  $(e_n)_{n \geq 0}$ . On the other hand,  $A_{\beta}$  is the matrix of  $C_{T_a}$  in  $H^2(\beta)$  with respect to the basis  $(e_n^{\beta})_{n \geq 0}$ . We note that  $A_{\beta} = BAB^{-1}$ , where  $B$  is the diagonal matrix with values  $\sqrt{\beta_0}, \sqrt{\beta_1}, \dots$  on the diagonal.

Since  $C_{T_a}$  is a bounded composition operators on  $H^2$ , the matrix  $A$  defines a bounded operator on  $\ell_2$ . We have to show that  $A_{\beta}$  also, i.e.  $\|A_{\beta}\| < \infty$ .

For that purpose, we split  $A$  and  $A_{\beta}$  into several sub-matrices.

Let  $N$  be an integer such that  $N \geq 2/\rho$ , where  $\rho$  is defined in Lemma 3.5 (actually, the proof of that lemma shows that we can take  $\rho$  such that  $1/\rho$  is

an integer, so we could take  $N = 2/\rho$ ). Let  $I_0 = [0, N[$   $J_0 = [N, +\infty[$  and for  $k = 1, 2, \dots$ :

$$I_k = [N^k, N^{k+1}[ \quad \text{and} \quad J_k = [N^{k+1}, +\infty[.$$

We define the matrices  $D_\beta$  and  $R_\beta$ , whose entries are respectively:

$$d_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty} (I_k \times I_k) \\ 0 & \text{elsewhere;} \end{cases}$$

and

$$r_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty} (I_k \times I_{k+1}) \\ 0 & \text{elsewhere.} \end{cases}$$

We also define the matrix  $S_\beta$  with entries:

$$s_{m,n} = \begin{cases} \sqrt{\frac{\beta_m}{\beta_n}} a_{m,n} & \text{if } (m,n) \in \bigcup_{k=0}^{\infty} (J_k \times I_k) \\ 0 & \text{elsewhere.} \end{cases}$$

Matrices  $D$ ,  $R$ , and  $S$  are constructed in the same way from  $A$  and we set  $U = A - (D + R + S)$ .

Now, let  $H_k$  be the subspace of the sequences  $(x_n)_{n \geq 0}$  in  $\ell_2$  such that  $x_n = 0$  for  $n \notin I_k$ , i.e.  $H_k = \text{span}\{e_k; k \in I_k\}$ , and let  $P_k$  be (the matrix of) the orthogonal projection of  $\ell_2$  with range  $H_k$ . We have:

$$D = \sum_{k=0}^{\infty} P_k A P_k \quad \text{and} \quad R = \sum_{k=0}^{\infty} P_k A P_{k+1},$$

where  $D_k = P_k A P_k$  is the matrix with entries  $a_{m,n}$  when  $(m,n) \in I_k \times I_k$  and 0 elsewhere, and  $R_k = P_k A P_{k+1}$  the matrix with entries  $a_{m,n}$  when  $(m,n) \in I_k \times I_{k+1}$  and 0 elsewhere.

$$\left( \begin{array}{c|c|c|c} \boxed{D_0} & \boxed{R_0} & & \\ \hline & \boxed{D_1} & \boxed{R_1} & \\ \hline S_0 & \boxed{S_1} & \boxed{D_2} & \boxed{R_2} \\ \hline & & \boxed{S_2} & \end{array} \right) U$$

Since the subspaces  $H_k$  are orthogonal, the matrices  $D$  and  $R$  induce bounded operators on  $\ell_2$ , and

$$(3.4) \quad \|D\| \leq \|A\|, \quad \|R\| \leq \|A\|.$$

Now, for  $k \geq 1$ , let  $B_k$  be the diagonal matrix whose entries are  $b_{m,m} = \sqrt{\beta_m}$  if  $m \in I_k$  and  $b_{m,n} = 0$  otherwise.

Then  $P_k D_\beta P_k = P_k B_k D B_k^{-1} P_k$ , so

$$\|P_k D_\beta P_k\| \leq \|B_k\| \|B_k^{-1}\| \|D\| \leq \max_{j \in I_k} \sqrt{\beta_j} \max_{j \in I_k} \frac{1}{\sqrt{\beta_j}} \|A\|.$$

But the weight  $\beta$  satisfies the  $\Delta_2$ -condition:  $\beta_{2l} \geq \delta_0 \beta_l$ , and it follows that for every  $l \geq 1$ :

$$\beta_{Nl} \geq \delta^2 \beta_l,$$

for some other constant  $\delta$ , chosen small enough to have  $\|P_0 D_\beta P_0\| \leq \delta^{-1} \|A\|$ . Since  $\beta$  is non-increasing, we have  $\beta_j \geq \delta^2 \beta_{N^k}$  for  $N^k \leq j \leq N^{k+1}$ . In particular  $\max_{j \in I_k} \sqrt{\beta_j} \leq \delta^{-1} \min_{j \in I_k} \sqrt{\beta_j}$  and

$$\|P_k D_\beta P_k\| \leq \delta^{-1} \|A\|.$$

Hence, by orthogonality of the subspaces  $H_k$ :

$$(3.5) \quad \|D_\beta\| \leq \delta^{-1} \|D\|.$$

In the same way, we have  $P_k R_\beta P_k = P_k B_k D B_{k+1}^{-1} P_k$ , so:

$$\|P_k R_\beta P_k\| \leq \max_{j \in I_k} \sqrt{\beta_j} \max_{j \in I_{k+1}} \frac{1}{\sqrt{\beta_j}} \|A\|,$$

and, since  $n \leq N^{k+2} \leq N^2 m$  when  $(m, n) \in I_k \times I_{k+1}$ , we get  $\max_{j \in I_k} \sqrt{\beta_j} \leq \delta^{-2} \min_{j \in I_{k+1}} \sqrt{\beta_j}$ , so

$$(3.6) \quad \|R_\beta\| \leq \delta^{-2} \|R\|.$$

Next, consider  $U = A - (D + R + S)$ ; we can compute its Hilbert-Schmidt norm using Lemma 3.5. Note that  $u_{m,n} \neq 0$  only (if it happens) when  $m \in I_k$  for some  $k \geq 0$  and  $n \geq \max I_{k+1} > N^{k+2} > Nm$ , since  $m \in I_k$ , so only when  $m < \rho n/2$ . We have, since then  $u_{m,n} = a_{m,n}$ :

$$\|U\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m < \rho n/2} |a_{m,n}|^2 \leq \sum_{n=0}^{\infty} \sum_{m < \rho n/2} e^{-2\alpha(\rho n - m)} \leq \sum_{n=0}^{\infty} \frac{\rho n}{2} e^{-\alpha \rho n} < \infty.$$

Consequently, with (3.4), we have  $\|S\| \leq \|A\| + \|D\| + \|R\| + \|U\| < \infty$ .

Now, since  $S$  is a lower-triangular matrix and  $\beta$  is non-increasing, we can use the following result of V. È Kacnel'son ([9]; see also [2], [3, Corollary 2.2], or [13, Theorem 3.12]), with  $\gamma_j = 1/\sqrt{\beta_j}$ .

**Theorem 3.6** (V. È Kacnel'son). *Let  $H$  be a separable complex Hilbert space and  $(e_i)_{i \geq 0}$  a fixed orthonormal basis of  $H$ . Let  $M: H \rightarrow H$  be a bounded linear operator. We assume that the matrix of  $M$  with respect to this basis is lower-triangular:  $\langle Me_j | e_i \rangle = 0$  for  $i < j$ .*

*Let  $(\gamma_j)_{j \geq 0}$  be a non-decreasing sequence of positive real numbers and  $\Gamma$  the (possibly unbounded) diagonal operator such that  $\Gamma(e_j) = \gamma_j e_j$ ,  $j \geq 0$ . Then the operator  $\Gamma^{-1} M \Gamma: H \rightarrow H$  is bounded and moreover:*

$$\|\Gamma^{-1} M \Gamma\| \leq \|M\|.$$

We get that  $S_\beta$  defines a bounded operator and  $\|S_\beta\| \leq \|S\|$ .

Further, Proposition 2.4 says that  $\beta$  has a polynomial minoration:

$$\beta_n \geq c n^{-\sigma}$$

for positive constants  $c$  and  $\sigma$ . Then, if  $U_\beta = A_\beta - (D_\beta + R_\beta + S_\beta)$ , we have:

$$\|U_\beta\|_{HS}^2 \leq \sum_{n=0}^{\infty} \sum_{m < \rho n/2} \frac{|a_{m,n}|^2}{\beta_n} \leq \sum_{n=0}^{\infty} \frac{\rho n}{2} e^{-\alpha \rho n} \frac{n^\sigma}{c} < \infty.$$

Putting this together with (3.5) and (3.6), we finally obtain that  $A_\beta = S_\beta + D_\beta + R_\beta + U_\beta$  is the matrix of a bounded operator, and that ends the proof of Theorem 3.4.  $\square$

### 3.3 Boundedness of composition operators II

In this section, we give a sufficient condition for the boundedness of composition operators with symbol vanishing at 0.

**Theorem 3.7.** *Let  $\beta = (\beta_n)_{n=0}^{\infty}$  be a bounded sequence of positive numbers with a polynomial minoration. Assume that:*

$$(3.7) \quad \text{For every } \delta > 0, \text{ there exists } C = C(\delta) \text{ such that} \\ \beta_m \leq C \beta_n \quad \text{whenever } m > (1 + \delta) n.$$

*Then, for all symbols  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  vanishing at 0, the composition operator  $C_\varphi$  is bounded on  $H^2(\beta)$ .*

To prove Theorem 3.7, we need several lemmas.

**Lemma 3.8.** *Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  be an analytic self-map such that  $\varphi(0) = 0$  and  $|\varphi'(0)| < 1$ . Then there exists  $\rho > 0$  such that*

$$|\widehat{\varphi^n}(m)| \leq \exp \left( -\frac{1}{2} [(1 + \rho) n - m] \right).$$

*Proof.* It is the same as that of Lemma 3.5. Since  $\varphi(0) = 0$ , we can write  $\varphi(z) = z \varphi_1(z)$ . Since  $|\varphi'(0)| < 1$ , we have  $\varphi_1: \mathbb{D} \rightarrow \mathbb{D}$ . Let  $M(r) = \sup_{|z|=r} |\varphi_1(z)|$ . The Cauchy inequalities say that  $|\widehat{\varphi_1^n}(m)| \leq [M(r)]^n / r^m$ . We have  $M(r) < 1$ , so there exists a positive number  $\rho = \rho(r)$  such that  $M(r) = r^\rho$ . We get:

$$|\widehat{\varphi^n}(m)| = |\widehat{\varphi_1^n}(m-n)| \leq \frac{r^\rho}{r^{m-n}} = r^{(1+\rho)n-m},$$

and the result follows, by taking  $r = e^{-1/2}$ .  $\square$

The next lemma is a variant of Theorem 3.6, with the same proof.

**Lemma 3.9.** *Let  $A: \ell_2 \rightarrow \ell_2$  be a bounded operator represented by the matrix  $(a_{m,n})_{m,n}$ , i.e.  $a_{m,n} = \langle A e_n, e_m \rangle$ .*

*Let  $(d_n)$  be a sequence of positive numbers such that, for every  $m$  and  $n$ :*

$$(3.8) \quad d_m < d_n \implies a_{m,n} = 0.$$

*Then,  $D$  being the diagonal operator with entries  $d_n$ , we have:*

$$\|D^{-1}AD\| \leq \|A\|.$$

For convenience of the reader, we reproduce the proof.

*Proof.* Let  $\mathbb{C}_0$  be the right-half plane  $\mathbb{C}_0 = \{z \in \mathbb{C}; \Re z > 0\}$ . We set  $H_N = \text{span}\{e_n; n \leq N\}$  and

$$A_N = P_N A J_N,$$

where  $P_N$  is the orthogonal projection from  $H$  onto  $H_N$  and  $J_N$  the canonical injection from  $H_N$  into  $H$ . We consider, for  $z \in \overline{\mathbb{C}_0}$ :

$$A_N(z) = D^{-z} A_N D^z: H_N \rightarrow H_N,$$

where  $D^z(e_n) = d_n^z e_n$ .

If  $(a_{m,n}(z))_{m,n}$  is the matrix of  $A_N(z)$  on the basis  $\{e_n; n \leq N\}$  of  $H_N$ , we clearly have:

$$a_{m,n}(z) = a_{m,n}(d_n/d_m)^z.$$

In particular, we have, thanks to (3.8):

$$a_{m,n}(z) = 0 \quad \text{if } d_m < d_n,$$

and

$$|a_{m,n}(z)| \leq \sup_{k,l} |a_{k,l}| := M, \quad \text{for all } z \in \overline{\mathbb{C}_0}.$$

Since  $\|A_N(z)\|^2 \leq \|A_N(z)\|_{HS}^2 = \sum_{m,n \leq N} |a_{m,n}^N(z)|^2 \leq (N+1)^2 M^2$ , we get:

$$\|A_N(z)\| \leq (N+1) M \quad \text{for all } z \in \overline{\mathbb{C}_0}.$$

Let us consider the function  $u: \overline{\mathbb{C}_0} \rightarrow \overline{\mathbb{C}_0}$  defined by:

$$(3.9) \quad u_N(z) = \|A_N(z)\|.$$

This function  $u_N$  is continuous on  $\overline{\mathbb{C}_0}$ , bounded above by  $(N+1)M$ , and subharmonic in  $\mathbb{C}_0$ . Moreover, thanks to (3.8), the maximum principle gives:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) = \sup_{\partial \mathbb{C}_0} u_N(z).$$

Since  $\|D^z\| = \|D^{-z}\| = 1$  for  $z \in \partial \mathbb{C}_0$ , we have  $\|A_N(z)\| \leq \|A_N\|$  for  $z \in \partial \mathbb{C}_0$ , and we get:

$$\sup_{\overline{\mathbb{C}_0}} u_N(z) \leq \|A_N\| \leq \|A\|.$$

In particular  $u_N(1) \leq \|A\|$ , and, letting  $N$  going to infinity, we get  $\|D^{-1}AD\| \leq \|A\|$ .  $\square$

*Proof of Theorem 3.7.* First, if  $|\varphi'(0)| = 1$ , we have  $\varphi(z) = \alpha z$  for some  $\alpha$  with  $|\alpha| = 1$ , and the result is trivial.

So, we assume that  $|\varphi'(0)| < 1$ . Then, by Lemma 3.8, there exists  $\rho > 0$  such that, for all  $m, n$ :

$$|\widehat{\varphi^n}(m)| \leq \exp\left(-\frac{1}{2}[(1+\rho)n - m]\right).$$

Since  $\varphi(0) = 0$ , we also know that  $\widehat{\varphi^n}(m) = 0$  if  $m < n$ .

Take  $\delta = \rho/2$  and use property (3.7): there exists  $C > 0$  such that:

$$\frac{\beta_m}{\beta_n} \leq C \quad \text{when } m \geq (1+\delta)n.$$

Define now a new sequence  $\gamma = (\gamma_n)$  as:

$$\gamma_n = \max\left\{\beta_n, \sup_{m > (1+\delta)n} \beta_m\right\}.$$

We have:

- 1)  $\beta_n \leq \gamma_n \leq C\beta_n$ ;
- 2)  $\gamma_m \leq \gamma_n$  if  $m \geq (1+\delta)n$ .

Item 1) implies that  $H^2(\gamma) = H^2(\beta)$ , and we are reduced to prove that  $C_\varphi: H^2(\gamma) \rightarrow H^2(\gamma)$  is bounded.

Let  $A = (a_{m,n}) = (\widehat{\varphi^n}(m))$ . We have to prove that

$$B = (\gamma_m^{1/2} \gamma_n^{-1/2} a_{m,n})_{m,n}$$

represents a bounded operator on  $\ell_2$ .



Define the matrix

$$A_1 = (a_{m,n} \mathbb{1}_{\{(m,n); m \leq (1+\delta)n\}})_{m,n}$$

and set  $A_2 = A - A_1$ . Define analogously  $B_1$  and  $B_2 = B - B_1$ .

Then  $A_1$  is a Hilbert-Schmidt operator, because (recall that  $a_{m,n} = 0$  if  $m < n$ )

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \exp(-[(1+\rho)n - m]) \\ &\leq \sum_{n=1}^{\infty} (\delta n + 1) \exp(-\delta n) < \infty. \end{aligned}$$

Now,  $\beta$  is bounded above and has a polynomial minoration, so, for some positive constants  $C_1$ ,  $C_2$ , and  $\alpha$ , we have:

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \frac{\gamma_m}{\gamma_n} |a_{m,n}|^2 &\leq \sum_{n=1}^{\infty} \sum_{m=n}^{(1+\delta)n} \frac{C_1}{n^{-\alpha}} \exp(-\delta n) \\ &\leq \sum_{n=1}^{\infty} C_2 n^{\alpha+1} \exp(-\delta n) < \infty, \end{aligned}$$

meaning that  $B_1$  is a Hilbert-Schmidt operator.

Since  $A$  is bounded, it follows that  $A_2 = A - A_1$  is bounded. Remark that, writing  $A_2 = (\alpha_{m,n})_{m,n}$ , we have, with  $d_n = 1/\sqrt{\gamma_n}$ :

$$d_m < d_n \implies \gamma_m > \gamma_n \implies m < (1+\delta)n \implies \alpha_{m,n} = 0.$$

Hence we can apply Lemma 3.9 to the matrix  $A_2$ , and it ensues that  $B_2$  is bounded, and therefore that  $B = B_1 + B_2$  is bounded as well, as wanted.  $\square$

We now can prove Theorem 3.3, as a corollary of Theorem 3.7.

*Proof of Theorem 3.3.* Define  $\beta_n = 1$  for  $n \leq 3!$ , and, for  $k \geq 3$ :

$$\left\{ \begin{array}{ll} \beta_n = \frac{1}{k!} & \text{for } k! < n \leq (k+1)! - 2 \text{ and for } n = (k+1)! \\ \beta_n = \frac{1}{(k+1)!} & \text{for } n = (k+1)! - 1. \end{array} \right.$$

Note that, for  $m > n$ , we have  $\beta_m > \beta_n$  only if  $n = (k+1)! - 1$  and  $m = (k+1)! = n + 1$ , for some  $k \geq 3$ .

However,  $\beta$  is not essentially decreasing since, for every  $k \geq 3$ , we have  $\beta_{n+1}/\beta_n = k + 1$  if  $n = (k+1)! - 1$ .

The sequence  $\beta$  has a polynomial minoration because  $\beta_n \geq 1/(2n)$  for all  $n \geq 1$ . In fact, for  $k \geq 3$ , we have  $\beta_n \geq (k+1)/n \geq 1/n$  if  $k! < n \leq (k+1)! - 2$  or if  $n = (k+1)!$ ; and for  $n = n = (k+1)! - 1$ , we have  $n \beta_n = [(k+1)! - 1]/(k+1)! \geq 1/2$ .

Now, it remains to check (3.7) in order to apply Theorem 3.7 and finish the proof of Theorem 3.3. Note first that we have  $\beta_m/\beta_n \leq 1$  if  $m \geq n + 2$ . Next, for given  $\delta > 0$ , there exists an integer  $N$  such that  $(1 + \delta)n \geq n + 2$  for every  $n \geq N$ , so  $\beta_m/\beta_n \leq 1$  if  $m \geq (1 + \delta)n$  and  $n \geq N$ . It suffices to take  $C = \max_{1 \leq n \leq N} \beta_{n+1}/\beta_n$  to obtain (3.7).  $\square$

## 4 Necessity of the $\Delta_2$ -condition

In this section, we will show that, for essentially decreasing sequences  $\beta$ , the  $\Delta_2$ -condition is necessary for having boundedness of composition operators on  $H^2(\beta)$ .

**Theorem 4.1.** *Let  $\beta$  be an essentially decreasing sequence and assume that, for some  $a \in (0, 1)$ ,  $T_a$  induces a bounded composition operator on  $H^2(\beta)$ . Then  $\beta$  satisfies the  $\Delta_2$ -condition.*

In order to prove Theorem 4.1, we need several preliminary lemmas. The first one is standard, but we give it for convenience.

**Lemma 4.2.** *Let  $a \in (0, 1)$  and let*

$$P_{-a}(x) = \frac{1 - a^2}{1 + 2a \cos x + a^2}$$

*be the Poisson kernel at the point  $-a$ . Then, for all  $x \in [-\pi, \pi]$ :*

$$(4.1) \quad T_a(e^{ix}) = \exp[i h_a(x)],$$

*where*

$$(4.2) \quad h_a(x) = \int_0^x P_{-a}(t) dt.$$

*Proof.* For  $t \in [-\pi, \pi]$ , write:

$$\psi(t) := \frac{e^{it} + a}{1 + a e^{it}} = \exp(i u(t)),$$

with  $u$  a real-valued,  $C^1$  function on  $[-\pi, \pi]$  such that  $u(0) = 0$ . This is possible since  $|\psi(e^{it})| = 1$  and  $\psi(0) = 1$ . Differentiating both sides with respect to  $t$ , we get:

$$i e^{it} \frac{1 - a^2}{(1 + a e^{it})^2} = i u'(t) \frac{e^{it} + a}{1 + a e^{it}}.$$

This implies

$$u'(t) = \frac{1 - a^2}{|1 + a e^{it}|^2} = P_{-a}(t),$$

and the result follows since  $u(0) = 0 = h_a(0)$ .  $\square$

## 4.1 Main lemma

In this section, we state and prove the following lemma. The proof of this lemma is delicate, uses Lemma 4.2 and a van der Corput type estimate, inspired from [21, pp. 72–73]. We thank R. Zarouf [22] for interesting recent informations in this respect, related to his joint work with O. Szehr on the Schäffer problem (see [20]; see also the forthcoming thesis of K. Fouchet [6] and paper [1]). A similar estimate on Taylor coefficients of high powers of Blaschke factors is needed by these authors, even if they were primarily concerned with upper bounds.

Recall that we have set:

$$(4.3) \quad [T_a(z)]^n = \sum_{m=0}^{\infty} a_{m,n} z^m.$$

**Lemma 4.3.** *Let  $a \in (0, 1)$ . We set:*

$$(4.4) \quad \tau = \frac{1+a}{1-a} > 1$$

*and write:*

$$(4.5) \quad \tau^{-1} = 1 - 3\mu,$$

*with  $\mu = \mu_a \in (0, 1/3)$ . For every fixed positive integer  $n$ , let:*

$$(4.6) \quad J_n = [(1 - 2\mu)n, (1 - \mu)n].$$

*Then, there exists  $\delta = \delta_a > 0$  such that, for every  $n$  large enough, there exists a set of indices  $E \subseteq J_n$  with cardinality  $|E| \geq \delta n$  and such that:*

$$(4.7) \quad m \in E \implies |a_{m,n}| \geq \delta n^{-1/2}.$$

As a corollary of this lemma, we have the following result.

**Proposition 4.4.** *For the weight  $\beta$  constructed in the proof of Theorem 3.3, no automorphism  $T_a$  with  $0 < a < 1$  can be bounded.*

*Proof.* When  $T_a$  is bounded, we have, with  $M = \|C_{T_a}\|$ , for every  $n$ :

$$(4.8) \quad \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|T_a^n\|^2 = \|C_{T_a}(z^n)\|^2 \leq \|C_{T_a}\|^2 \|z^n\|^2 = M^2 \beta_n.$$

For the weight  $\beta$  constructed in the proof of Theorem 3.3, we are going to see that this condition is not satisfied for  $n = (k+1)! + 1 := n_k$ .

In fact, by Lemma 4.3, there exist constants  $0 < \alpha_1 < \alpha_2 < 1$  and  $\gamma, \delta > 0$ , depending only on  $a$ , and a set  $E_k$  of integers such that  $E_k \subseteq [\alpha_1 n_k, \alpha_2 n_k]$  and  $|E_k| \geq \gamma n_k$  for which

$$|a_{m,n}| \geq \delta n_k^{-1/2} \quad \text{for } m \in E_k.$$

For  $k$  large enough, we have  $[\alpha_1 n_k, \alpha_2 n_k] \subseteq [k! + 1, (k+1)! - 1] \setminus \{(k+1)! - 1\}$ ; hence the left-hand side of (4.8) is larger than

$$\frac{\delta^2}{k!} \sum_{m \in E_k} \frac{1}{n_k} \geq \frac{\delta^2}{k!} \gamma.$$

Since the right-hand side of (4.8) is  $M^2/(k+1)!$ , that is much smaller,  $T_a$  cannot be bounded.  $\square$

To prove Lemma 4.3, we will use a variant of [21, Lemma 4.6 p. 72] on the stationary phase method. It is observed in [21, p. 90] that some error term  $O(\lambda_2^{-4/5} \lambda_3^{1/5})$  to come can be replaced by  $O(\lambda_2^{-1} \lambda_3^{1/3})$ , but this refinement is not needed here. However, a careful reading of the proof in [21, p. 72] gives the version below, which allows the derivative  $F'$  of  $F$  to vanish at some point, as occurs in our situation. For sake of completeness, we will give a proof, however postponed.

**Theorem 4.5** (Stationary phase). *Let  $F$  be real function on the interval  $[A, B]$ , with continuous derivatives up to the third order and  $F'' > 0$  throughout  $]A, B[$ . Let  $c$  be the unique point in  $]A, B[$  where  $F'(c) = 0$ . Assume that, for some positive numbers  $\lambda_2, \lambda_3$ , and  $\eta$ , the following assertions hold:*

- 1)  $[c - \eta, c + \eta] \subseteq [A, B]$ ;
- 2)  $F''(x) \geq \lambda_2$  for all  $x \in [c - \eta, c + \eta]$ ;
- 3)  $|F'''(x)| \leq \lambda_3$  for all  $x \in [A, B]$ .

Then:

$$(4.9) \quad \int_A^B e^{iF(x)} dx = \sqrt{2\pi} \frac{e^{i(F(c) + \pi/4)}}{|F''(c)|^{1/2}} + O\left(\frac{1}{\eta\lambda_2} + \eta^4\lambda_3\right),$$

where the  $O$  involves an absolute constant.

*Proof of Lemma 4.3.* We turn to the problem of bounding  $a_{m,n}$  from below, in the case  $m \in J$ , and only in that case. Since  $\inf_{[0, \pi]} P_{-a} = \tau^{-1} < \sup_{[0, \pi]} P_{-a} = \tau$ , there exists a unique point  $x_m = x_{m,n} \in [0, \pi]$  such that

$$nP_{-a}(x_m) - m = n \frac{(1 - a^2)}{1 + 2a \cos x_m + a^2} - m = 0,$$

or else:

$$(4.10) \quad \cos x_m = \frac{n}{m} \frac{1 - a^2}{2a} - \frac{1 + a^2}{2a}.$$

The point is that if  $m \in J$ ,  $x_m$  can approach neither 0 nor  $\pi$ , so that  $\sin x_m \geq \delta_a > 0$ ; in fact, otherwise  $m/n$  approaches  $\tau^{\pm 1}$ , which is impossible by

definition of  $J$ . With  $h_a$  the function of Lemma 4.2, the Fourier formulas give, since  $a_{m,n}$  is real, or since  $h_a(x) - mx$  is odd:

$$2\pi a_{m,n} = \int_{-\pi}^{\pi} \exp i[nh_a(x) - mx] dx = 2 \Re I_{m,n},$$

where

$$(4.11) \quad I_{m,n} = \int_0^{\pi} \exp i[nh_a(x) - mx] dx.$$

Write:

$$(4.12) \quad I_{m,n} = \int_0^{\pi} \exp[i F(x)] dx,$$

with:

$$(4.13) \quad F(x) = nh_a(x) - mx = n \int_0^x P_{-a}(t) dt - mx.$$

We have:

$$(4.14) \quad F'(x) = n P_{-a}(x) - m.$$

We will now proceed in two steps, first giving good lower bounds for  $|I_{m,n}|$ , then showing that the argument of  $I_{m,n}$  is often far from  $\pi/2 \bmod \pi$ . Then, we will be done.

**First step.** We will prove that:

$$(4.15) \quad I_{m,n} = \sqrt{2\pi} n^{-1/2} \frac{e^{i(F(x_m) + \pi/4)}}{\sqrt{|h_a''(x_m)|}} + O(n^{-3/5}),$$

where the  $O$  only depends on  $a$ .

Note that  $3/5 > 1/2$  and  $F'' = n h_a''$ .

To get (4.15), we will show that Theorem 4.5 is applicable with:

$$[A, B] = [0, \pi], \quad c = x_m, \quad \lambda_2 = \kappa_0 n, \quad \lambda_3 = C_0 n, \quad \eta = (\lambda_2 \lambda_3)^{-1/5}.$$

The parameter  $\eta$  is chosen in order to make both error terms in Theorem 4.5 equal:  $\frac{1}{\eta \lambda_2} = \eta^4 \lambda_3$ ; so:

$$\eta = \kappa n^{-2/5}$$

and

$$(4.16) \quad \frac{1}{\eta \lambda_2} + \eta^4 \lambda_3 = \tilde{\kappa} n^{-3/5} = O(n^{-3/5})$$

(with  $\kappa = (\kappa_0 C_0)^{-1/5}$  and  $\tilde{\kappa} = 2/\kappa_0 \kappa$ ).

The slight technical difficulty encountered here is that  $F''(x)$  vanishes at 0 and  $\pi$ . But Theorem 4.5 covers this case. We have

$$F''(x) = nP'_{-a}(x) = 2a(1-a^2) \frac{\sin x}{(1+2a\cos x+a^2)^2} n,$$

and there are some positive  $\kappa_0$  and  $\sigma$  such that

$$(4.17) \quad F''(x) \geq \kappa_0 n = \lambda_2 \quad \text{for } x \in [\sigma, \pi - \sigma].$$

Now (for  $n$  large enough),  $[x_m - \eta, x_m + \eta] \subseteq [\sigma, \pi - \sigma]$ . Hence the assumptions 1) and 2) of Theorem 4.5 are satisfied.

Finally, since  $F = nh_a$ , and  $h_a$  is  $\mathcal{C}^\infty$  on  $\mathbb{R}$ , we have, for all  $x \in [0, \pi]$ :

$$|F'''(x)| \leq C_0 n = \lambda_3,$$

and assertion 3) of Theorem 4.5 holds.

With (4.16) this ends the proof of (4.15), once we remarked that  $nh_a''(x_m) = F''(x_m)$ .

Note that, since  $|h_a''(x_m)| \leq M_a$ , we get that  $|I_{m,n}| \geq \delta n^{-1/2}$  when  $m \in J$ .

**Second step.** The mean-value theorem gives, for  $m \in J$ :

$$(4.18) \quad |\sin x_m| \geq \delta \quad \text{and} \quad x_{m+1} - x_m \approx \cos x_m - \cos x_{m+1}.$$

We also have, for  $x \in \mathcal{J} = [1 - 2\mu, 1 - \mu]$ , with another constant  $\delta$ :

$$(4.19) \quad \delta \leq P'_{-a}(x) = 2a(1-a^2) \frac{\sin x}{(1+2a\cos x+a^2)^2} \leq \delta^{-1}.$$

We now claim that

$$(4.20) \quad x_{m+1} - x_m \approx n^{-1} \quad \text{for } m \in J.$$

Indeed, since  $m \in J$ , we have, using (4.10):

$$\cos x_m - \cos x_{m+1} = \frac{1-a^2}{2a} \frac{n}{m(m+1)} \approx \frac{n}{m^2} \approx n^{-1}.$$

In view of (4.18), this proves (4.20).

Now, according to (4.15), when  $m \in J$ , the main term in  $I_{m,n}$  is

$$A_{m,n} := n^{-1/2} \frac{\sqrt{2\pi}}{\sqrt{|h_a''(x_m)|}} e^{i(F(x_m) + \pi/4)},$$

and its argument  $\theta_m$  is  $F(x_m) + \pi/4$ . Going from  $m$  to  $m+1$ , the variation  $F(x_{m+1}) - F(x_m)$  of this argument is

$$\theta_{m+1} - \theta_m = n \int_{x_m}^{x_{m+1}} \left( P_{-a}(t) - \frac{m}{n} \right) dt = n \int_{x_m}^{x_{m+1}} \left( P_{-a}(t) - P_{-a}(x_m) \right) dt.$$

But, due to (4.19), this implies:

$$\theta_{m+1} - \theta_m \approx n \int_{x_m}^{x_{m+1}} (t - x_m) dt \approx n (x_{m+1} - x_m)^2 \approx n^{-1}.$$

The variation of  $\theta_m$  is thus regular, like that of the argument of  $n$ -th roots of unity. As a consequence, for a positive proportion  $E$  of the indices  $m \in J$ , the argument  $\theta_m$  will belong to a subarc of  $\mathbb{T}$  which lies  $\delta$ -apart from  $\pm\pi/2$ , implying  $\cos\theta_m \geq \delta$ , or else:

$$|E| \geq \delta n,$$

and, for all  $m \in E$ :

$$\Re A_{m,n} \geq \delta |A_{m,n}| \geq \delta n^{-1/2}.$$

It follows that, for  $m \in E$ :

$$\Re I_{m,n} \geq \delta n^{-1/2} - C n^{-3/5} \geq \tilde{\delta} n^{-1/2}.$$

Since  $a_{m,n} = \pi^{-1} \Re I_{m,n}$ , that ends the proof of Lemma 4.3.  $\square$

## 4.2 Proof of Theorem 4.5

The following lemma can be found in [15, Lemma 1, page 47].

**Lemma 4.6.** *Let  $F: [u, v] \rightarrow \mathbb{R}$ , with  $u < v$ , be a  $C^2$ -function with  $F'' > 0$ , and  $F'$  not vanishing on  $[u, v]$ . Let*

$$J = \int_u^v e^{iF(x)} dx.$$

*Then:*

- a) *if  $F' > 0$  on  $[u, v]$ , then  $|J| \leq \frac{2}{F'(u)}$ ;*
- b) *If  $F' < 0$  on  $[u, v]$ , then  $|J| \leq \frac{2}{|F'(v)|}$ .*

Write now the integral  $I$  of Theorem 4.5 on  $[A, B]$  as  $I = I_1 + I_2 + I_3$  with:

$$I_1 = \int_A^{c-\eta} e^{iF(x)} dx, \quad I_2 = \int_{c-\eta}^{c+\eta} e^{iF(x)} dx, \quad I_3 = \int_{c+\eta}^B e^{iF(x)} dx.$$

Lemma 4.6 with  $u = A$  and  $v = c - \eta$  implies:

$$(4.21) \quad |I_1| \leq \frac{2}{|F'(c-\eta)|} \leq \frac{2}{\eta \lambda_2},$$

where, for the last inequality, we just have to write

$$|F'(c-\eta)| = F'(c) - F'(c-\eta) = \eta F''(\xi)$$

for some  $\xi \in [c-\eta, c]$  so that  $F''(\xi) \geq \lambda_2$ .

Similarly, Lemma 4.6 with  $u = c + \eta$  and  $v = B$  implies

$$(4.22) \quad |I_3| \leq \frac{2}{F'(c + \eta)} \leq \frac{2}{\eta \lambda_2}.$$

We can now estimate  $I_2$ . The Taylor formula shows that

$$F(x) = F(c) + \frac{(x - c)^2}{2} F''(c) + R,$$

with

$$|R| \leq \frac{|x - c|^3}{6} \lambda_3.$$

Hence

$$I_2 = e^{iF(c)} \int_0^\eta 2 \exp\left(\frac{i}{2} x^2 F''(c)\right) dx + S$$

with

$$|S| \leq \lambda_3 \int_0^\eta \frac{x^3}{3} dx = \frac{\eta^4}{12} \lambda_3.$$

Finally, set

$$K = \int_0^\eta 2 \exp\left(\frac{i}{2} x^2 F''(c)\right) dx.$$

We make the change of variable  $x = \sqrt{\frac{2}{F''(c)}} \sqrt{t}$ . Recall that  $\int_0^\infty \frac{e^{it}}{\sqrt{t}} dt = \sqrt{\pi} e^{i\pi/4}$  is the classical Fresnel integral, and that an integration by parts gives, for  $m > 0$ :

$$\left| \int_m^\infty \frac{e^{it}}{\sqrt{t}} dt \right| \leq \frac{2}{\sqrt{m}}.$$

Therefore, with  $m = \frac{\eta^2}{2} F''(c)$ :

$$K = \sqrt{\frac{2}{F''(c)}} \int_0^m \frac{e^{it}}{\sqrt{t}} dt = \sqrt{\frac{2\pi}{F''(c)}} e^{i\pi/4} + R_m,$$

with

$$|R_m| \leq C \sqrt{\frac{1}{F''(c)}} \frac{1}{\sqrt{m}} \leq \frac{C}{\eta \lambda_2}.$$

All in all, we proved that

$$(4.23) \quad I_2 = \sqrt{\frac{2\pi}{F''(c)}} \exp[i(F(c) + \pi/4)] + O\left(\frac{1}{\eta \lambda_2} + \eta^4 \lambda_3\right).$$

and the same estimate holds for  $I$ , thanks to (4.21) and (4.22).

We have hence proved Theorem 4.5. □



### 4.3 Proof of Theorem 4.1

We may, and do, assume that  $\beta$  is non-increasing.

Set  $M = \|C_{T_a}\|$ . We have:

$$\sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|T_a^n\|^2 = \|C_{T_a}(z^n)\|^2 \leq \|C_{T_a}\|^2 \|z^n\|^2 = M^2 \beta_n;$$

so, by Lemma 4.3, since  $\beta$  is non-increasing and  $E \subseteq J = [(1-2\mu)n, (1-\mu)n]$ :

$$M^2 \beta_n \geq \sum_{m \in E} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} \sum_{m \in E} \beta_m \geq \delta^2 n^{-1} |E| \beta_{(1-\mu)n},$$

where we have set, for  $x$  not an integer,  $\beta_x = \beta_k$  with  $k$  the least integer greater than  $x$ .

That implies, for all integers  $n \geq 1$ :

$$\beta_n \geq (\delta^2/M^2) n^{-1} (\delta n) \beta_{(1-\mu)n} \geq c \beta_{(1-\mu)n}.$$

Let  $r \geq 1$  such that  $(1-\mu)^r \leq 1/2$ ; we have:

$$\beta_n \geq c^r \beta_{(1-\mu)^r n} \geq c^r \beta_{n/2},$$

so  $\beta$  satisfies the  $\Delta_2$ -condition.  $\square$

## 5 Some results on multipliers

The set  $\mathcal{M}(H^2(\beta))$  of multipliers of  $H^2(\beta)$  is by definition the vector space of functions  $h$  analytic on  $\mathbb{D}$  and such that  $hf \in H^2(\beta)$  for all  $f \in H^2(\beta)$ . When  $h \in \mathcal{M}(H^2(\beta))$ , the operator  $M_h$  of multiplication by  $h$  is bounded on  $H^2(\beta)$  by the closed graph theorem. The space  $\mathcal{M}(H^2(\beta))$  equipped with the operator norm is a Banach space. We note the obvious property:

$$(5.1) \quad \mathcal{M}(H^2(\beta)) \hookrightarrow H^\infty \quad \text{contractively.}$$

Indeed, if  $h \in \mathcal{M}(H^2(\beta))$ , we easily get for all  $w \in \mathbb{D}$ :

$$M_h^*(K_w) = \overline{h(w)} K_w;$$

so that taking norms and simplifying, we are left with  $|h(w)| \leq \|M_h\|$ , showing that  $h \in H^\infty$  with  $\|h\|_\infty \leq \|M_h\|$ .

**Proposition 5.1.** *We have  $\mathcal{M}(H^2(\beta)) = H^\infty$  isomorphically if and only if  $\beta$  is essentially decreasing.*

*Proof.* The sufficient condition is proved in [13, beginning of the proof of Proposition 3.16]. For the necessity, we have  $\|M_h\| \approx \|h\|_\infty$  for every  $h \in H^\infty$ . Now, for  $m > n$  (recall that  $e_n(z) = z^n$ ):

$$e_m(z) = z^{m-n} z^n = (M_{e_{m-n}} e_n)(z);$$

so, since  $\|M_{e_{m-n}}\| \leq C \|e_{m-n}\|_\infty = C$  for some positive constant  $C$ :

$$\beta_m = \|e_m\|^2 \leq C^2 \|e_n\|^2 = C^2 \beta_n. \quad \square$$

In [13, Section 3.6], we gave the following notion of *admissible* Hilbert space of analytic functions.

**Definition 5.2.** A Hilbert space  $H$  of analytic functions on  $\mathbb{D}$ , containing the constants, and with reproducing kernels  $K_a$ ,  $a \in \mathbb{D}$ , is said *admissible* if:

- (i)  $H^2$  is continuously embedded in  $H$ ;
- (ii)  $\mathcal{M}(H) = H^\infty$ ;
- (iii) the automorphisms of  $\mathbb{D}$  induce bounded composition operators on  $H$ ;
- (iv)  $\frac{\|K_a\|_H}{\|K_b\|_H} \leq h\left(\frac{1-|b|}{1-|a|}\right)$  for  $a, b \in \mathbb{D}$ , where  $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an non-decreasing function.

We proved in that paper that every weighted Hilbert space  $H^2(\beta)$  with  $\beta$  non-increasing is admissible, under the additional hypothesis that the automorphisms of  $\mathbb{D}$  induce bounded composition operators. In view of Theorem 3.4, we get the following result.

**Proposition 5.3.** Let  $\beta$  be essentially decreasing that satisfies the  $\Delta_2$ -condition. Then  $H^2(\beta)$  is admissible.

Let us give a different proof.

*Proof.* Because  $\beta$  is essentially decreasing, item (i) holds, as well as item (ii), by Proposition 5.1. Item (iii) is Theorem 3.4. It remains to show (iv). We may assume that  $\beta$  is non-increasing.

Let  $0 < s < r < 1$ .

Without loss of generality, we may assume that  $r, s \geq 1/2$ . It is enough to prove:

$$(5.2) \quad \|K_r\|^2 \leq C \|K_{r^2}\|^2$$

for some constant  $C > 1$ . Indeed, iteration of (5.2) gives:

$$\|K_r\|^2 \leq C^k \|K_{r^{2^k}}\|^2$$

and if  $k$  is the smallest integer such that  $r^{2^k} \leq s$ , we have  $2^{k-1} \log r > \log s$  and  $2^k \leq D \frac{1-s}{1-r}$  where  $D$  is a numerical constant. Writing  $C = 2^\alpha$  with  $\alpha > 1$ , we obtain:

$$\left(\frac{\|K_r\|}{\|K_s\|}\right)^2 \leq C^k = (2^k)^\alpha \leq D^\alpha \left(\frac{1-s}{1-r}\right)^\alpha.$$

To prove (5.2), we pick some  $M > 1$  such that  $\beta_{2n} \geq M^{-1}\beta_n$  for all  $n \geq 1$  and write  $t = r^2$ . We have:

$$\|K_r\|^2 = \frac{1}{\beta_0} + \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_{2n}} + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\beta_{2n-1}},$$

implying, since  $\beta_{2n-1} \geq \beta_{2n} \geq M^{-1}\beta_n$  and  $t^{2n-1} \leq 4t^{2n}$ :

$$\|K_r\|^2 \leq \frac{1}{\beta_0} + M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} + 4M \sum_{n=1}^{\infty} \frac{t^{2n}}{\beta_n} \leq 5M\|K_t\|^2. \quad \square$$

The notion of admissible Hilbert space  $H$  is useful for the set of conditional multipliers:

$$\mathcal{M}(H, \varphi) = \{w \in H; w(f \circ \varphi) \in H \text{ for all } f \in H\}.$$

As a corollary of [13, Theorem 3.18] we get:

**Corollary 5.4.** *Let  $\beta$  be essentially decreasing and satisfying the  $\Delta_2$ -condition. Then:*

- 1)  $\mathcal{M}(H^2, \varphi) \subseteq \mathcal{M}(H^2(\beta), \varphi)$ ;
- 2)  $\mathcal{M}(H^2(\beta), \varphi) = H^2(\beta)$  if and only if  $\|\varphi\|_{\infty} < 1$ ;
- 3)  $\mathcal{M}(H^2(\beta), \varphi) = H^{\infty}$  if and only if  $\varphi$  is a finite Blaschke product.

## 6 Miscellaneous remarks

Some of the results of this paper can slightly be improved.

### 6.1 Conditions on the weight $\beta$

First, we say that a sequence  $(\beta_n)$  of positive numbers is *slowly oscillating* if there if a function  $\rho: (0, \infty) \rightarrow (0, \infty)$  that is bounded on each compact subset of  $(0, \infty)$  for which:

$$\frac{\beta_m}{\beta_n} \leq \rho\left(\frac{m}{n}\right).$$

This clearly amounts to say that, for some positive constants  $C_1 < C_2$ , we have:

$$C_1 \leq \frac{\beta_m}{\beta_n} \leq C_2 \quad \text{when } n/2 \leq m \leq 2n.$$

Every essentially decreasing sequence with the  $\Delta_2$ -condition is slowly oscillating.

**Proposition 6.1.** *The following holds:*

- 1) *every slowly oscillating sequence has a polynomial minoration;*
- 2) *there are bounded sequences which are slowly oscillating, but not essentially decreasing.*

*Proof.* 1) is clear, because if  $2^j \leq n < 2^{j+1}$ , then

$$\beta_n \geq C^{-1}\beta_{2^j} \geq C^{-j-1}\beta_1 \geq C^{-1}\beta_1 n^{-\alpha},$$

with  $\alpha = \log C / \log 2$ .

2) We define  $\beta_n$  as follows. Let  $(a_k)$  be an increasing sequence of positive square integers such that  $\lim_{k \rightarrow \infty} a_{k+1}/a_k = \infty$ , for example  $a_k = 4^{k^2}$ , and let  $b_k = \sqrt{a_k a_{k+1}}$ ; with our choice, this is an integer and we clearly have  $a_k < b_k < a_{k+1}$ . We set:

$$\beta_n = \begin{cases} a_k/n & \text{for } a_k \leq n < b_k \\ (a_k/b_k^2) n = (1/a_{k+1}) n & \text{for } b_k \leq n < a_{k+1} . \end{cases}$$

This sequence  $(\beta_n)$  is slowly oscillating by construction. Indeed, it suffices to check that for  $a_k \leq n/2 < b_k \leq n < a_{k+1}$ , the quotient  $\beta_m/\beta_n$  remains lower and upper bounded when  $n/2 \leq m \leq n$ . But for  $n/2 \leq m < b_k$ , we have

$$\frac{\beta_m}{\beta_n} = \frac{a_k/m}{n/a_{k+1}} = \frac{a_k a_{k+1}}{mn} = \frac{b_k^2}{mn} ,$$

which is  $\leq 2b_k^2/n^2 \leq 2$  and  $\geq b_k^2/n^2 \geq (n/2)^2/n^2 = 1/4$ ; and for  $b_k \leq m$ , we have

$$\frac{\beta_m}{\beta_n} = \frac{m/a_{k+1}}{n/a_{k+1}} = \frac{m}{n} \in [1/2, 1] .$$

However, though  $(\beta_n)$  is bounded, since  $\beta_n \leq 1$  for  $a_k \leq n < b_k$  and, for  $b_k \leq n < a_{k+1}$ ,

$$\beta_n \leq \beta_{a_{k+1}-1} = \frac{1}{a_{k+1}} (a_{k+1} - 1) \leq 1 ,$$

it is not essentially decreasing, since

$$\frac{\beta_{a_{k+1}-1}}{\beta_{b_k}} = \frac{1}{\sqrt{a_k a_{k+1}}} (a_{k+1} - 1) \sim \sqrt{\frac{a_{k+1}}{a_k}} \xrightarrow{k \rightarrow \infty} \infty . \quad \square$$

By a slight modification (change the value of the constants in the definition of  $\beta_n$ ), we could obtain a sequence which is slowly oscillating, tends to zero, yet again not essentially decreasing.

Now, Theorem 3.4 admits the following variant.

**Theorem 6.2.** *Let  $(\beta_n)$  be a sequence of positive numbers which is bounded above and slowly oscillating. Then all symbols that extend analytically in a neighborhood of  $\mathbb{D}$  induce a bounded composition operator on  $H^2(\beta)$ .*

It is the case, for example, for finite Blaschke products.

The proof follows that of Theorem 3.4, with the help of the following lemma.

**Lemma 6.3.** *Let  $(\beta_n)$  be a sequence of positive numbers which is bounded above and slowly oscillating. Let  $A = (a_{m,n})_{m,n}$  be the matrix of a bounded operator on  $\ell_2$ . Assume that, for constants  $C > 1$ , and  $a, b$ , we have:*

- 1)  $|a_{m,n}| \leq a e^{-bn}$  for  $m \leq C^{-1}n$ ;
- 2)  $|a_{m,n}| \leq a e^{-bm}$  for  $m \geq Cn$ .

*Then the matrix  $\tilde{A} = \left( a_{m,n} \sqrt{\frac{\beta_m}{\beta_n}} \right)$  also defines a bounded operator on  $\ell_2$ .*

*Sketch of proof.* The matrix  $\tilde{A}$  is Hilbert-Schmidt far the diagonal since, with  $\lambda \geq \beta_n \geq \delta n^{-\alpha}$ , we have:

$$\sum_{m < C^{-1}n} |a_{m,n}|^2 \beta_m / \beta_n \leq \sum_{m < C^{-1}n} \lambda \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n \geq 1} n^{\alpha+1} e^{-bn} < \infty,$$

and

$$\sum_{m > Cn} |a_{m,n}|^2 \beta_m / \beta_n \leq \sum_{m > Cn} \lambda \delta^{-1} n^\alpha |a_{m,n}|^2 \lesssim \sum_{n \geq 1} n^\alpha \left( \sum_{m > Cn} e^{-bm} \right).$$

Since  $\beta_m / \beta_n$  remains bounded from above and below around the diagonal, the matrix  $\tilde{A}$  behaves like  $A$  near the diagonal.  $\square$

**Remark.** The proof shows that, instead of 1) and 2), it is enough to have:

$$\sum_{m < C^{-1}n} n^{\alpha+1} |a_{m,n}|^2 < \infty \quad \text{and} \quad \sum_{m > Cn} m^\alpha |a_{m,n}|^2 < \infty.$$

Moreover the proof also shows that when  $\beta$  is slowly oscillating, if we set  $E = \{(m, n); C^{-1}n \leq m \leq Cn\}$ , then the matrix  $(\sqrt{\beta_m / \beta_n} \mathbf{1}_E(m, n))$  is a Schur multiplier over *all* the bounded matrices, while Theorem 3.6 says that, if  $\gamma = (\gamma_n)$  is non-increasing, the matrix  $(\gamma_m / \gamma_n)$  is a Schur multiplier of all bounded *lower-triangular* matrices.

*Proof of Theorem 6.2.* For every symbol  $\varphi$ , let  $M(r) = \sup_{|z|=r} |\varphi(z)|$  and write  $M(1/e) = e^{-\delta}$ , with  $\delta > 0$ . Let  $A$  be the matrix of  $C_\varphi$ , with respect to the canonical basis of  $H^2$ . The Cauchy inequalities give, if  $m \leq (\delta/2)n$ :

$$|a_{m,n}| \leq [M(1/e)]^n e^m \leq e^{m-\delta n} \leq e^{-\delta/2n}$$

When  $\varphi$  is analytic in a neighborhood of  $\overline{D(0, R)}$  with  $R > 1$ , the Cauchy inequalities give, writing  $M(R) = e^\rho$  and  $R = e^\delta$ , for  $m \geq Cn$ , and a suitable constant  $C$  (take  $C = 2\rho^\delta$  for instance):

$$|a_{m,n}| \leq \frac{[M(R)]^n}{R^m} \leq e^{n\rho - \delta m} \leq e^{-(\delta/2)m}.$$

We conclude with Lemma 6.3.  $\square$

The proof of Theorem 4.1 actually gives a little bit more than its statement.

Let  $(\beta_n)$  be a sequence of positive numbers with limit zero. We say that  $(\beta_n)$  is *mean- $\Delta_2$*  if, denoting by  $(\beta_n^*)$  its non-increasing rearrangement, there is a positive constant  $\delta$  such that:

$$(6.1) \quad \beta_{2n} \geq \delta \frac{1}{n} \sum_{j=1}^n \beta_j^*.$$

This condition is less restrictive than being slowly oscillating. Then, with the same proof, we have this variant of Theorem 4.1.

**Theorem 6.4.** *Let  $\beta = (\beta_n)$  be a bounded sequence of positive numbers. If some automorphism  $T_a$  induces a bounded composition operator on  $H^2(\beta)$ , then  $\beta$  is mean- $\Delta_2$ .*

*Sketch of proof.* We proved that, for some  $E \subseteq [n/2, n]$  with  $|E| \gtrsim n$ , we have  $\beta_{2n} \geq \delta \beta_j$  for  $j \in E$ . Since

$$\sum_{j=1}^n \beta_j^* = \max_{|E|=n} \sum_{j \in E} \beta_j,$$

we get (6.1). □

## 6.2 Singular inner functions

Let  $a > 0$  and let  $I_a$  be the singular inner function defined by

$$(6.2) \quad I_a(z) = \exp \left( -a \frac{1+z}{1-z} \right) = \sum_{m=0}^{\infty} c_m(a) z^m.$$

If  $n$  is a positive integer, we have  $I_n(z) = [I_1(z)]^n$  and we write

$$I_n(z) = \sum_{m=0}^{\infty} a_{m,n} z^m, \quad \text{with } a_{m,n} = c_m(n).$$

We rely on the following lemma, familiar to experts in orthogonal polynomials and special functions, but maybe not so much as regards the uniformity, essential for our present purposes (see [16] or [12]).

**Lemma 6.5.** *It holds*

$$(6.3) \quad c_m(n) = c n^{1/4} m^{-3/4} \cos(2\sqrt{2nm} + \pi/4) + R_m(n) =: M_m(n) + R_m(n),$$

where  $c = \pi^{-1/2} 2^{1/4}$  and where  $|R_m(n)| \leq K \sqrt{n} m^{-5/4}$ , with  $K$  some numerical constant.

We use the following [16, p. 253] and [19, p. 198], where  $L_m^{(\alpha)}$  denotes the generalized Laguerre polynomial of degree  $m$  with parameter  $\alpha$ .

**Theorem 6.6.** *With the notation of (6.2), we have*

$$c_m(a) = e^{-a} L_m^{(-1)}(2a).$$

Moreover, we have, uniformly for  $0 < \varepsilon \leq x \leq M < \infty$ :

$$L_m^{(\alpha)}(x) = \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} m^{\alpha/2-1/4} \cos(2\sqrt{mx} - \alpha\pi/2 - \pi/4) + R_m(x),$$

where  $|R_m(x)| \leq K_\alpha m^{\alpha/2-3/4}$ , and  $K_\alpha > 0$  only depending on  $\alpha$ .

Now, using Theorem 6.6 with  $\alpha = -1$  and  $a = n$ , we get Lemma 6.5. Actually, to get Lemma 6.5, we need some uniformity with respect to  $a$  in Theorem 6.6, which is not given by the above statement of that theorem; but provided we change  $m^{-5/4}$  into  $\sqrt{n} m^{-5/4}$  in  $R_m$ , a careful examination of Fejér's proof of Theorem 6.6 shows that this uniformity holds. Alternatively, write  $c_m(a)$  as a Fourier coefficient

$$(6.4) \quad c_m(n) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp [i(n \cot x + 2mx)] dx$$

and use the previous van der Corput estimates.

Once this lemma is at our disposal, we can prove again the following theorem.

**Theorem 6.7.** *Assume that  $\beta$  is a non-increasing sequence and that the composition operator  $C_{I_1}$  maps  $H^2(\beta)$  to itself. Then  $\beta$  satisfies the  $\Delta_2$ -condition.*

*Proof.* Our assumption implies, with  $M = \|C_{I_1}\|^2$ :

$$(6.5) \quad \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m \leq M \beta_n.$$

We use Lemma 6.5 with  $C^{-2}n \leq m < n$ , where  $C > 1$  satisfies  $C\sqrt{2} < \pi/2$ , which is possible since  $2\sqrt{2} < \pi$ . The term  $R_m(n)$  is dominated by  $KC^{-1/2}m^{-3/4}$ . Let  $\theta_m = 2\sqrt{2}\sqrt{nm} + \pi/4$  be the argument of the number appearing in (6.3). We have

$$\theta_{m+1} - \theta_m = \frac{2\sqrt{2}\sqrt{n}}{\sqrt{m} + \sqrt{m+1}};$$

hence

$$\sqrt{2} \leq \theta_{m+1} - \theta_m \leq C\sqrt{2} < \frac{\pi}{2}.$$

The argument  $\theta_m$  then varies not too fast, and there is a positive constant  $\delta$  and a subset  $E$  of integers in the interval  $[C^{-2}n, n]$  such that  $|E| \geq \delta n$  and

$$M_m(n) \geq 2\delta n^{1/4}m^{-3/4} \geq 2\delta n^{-1/2}$$

for all  $m \in E$ . Therefore, for  $n$  large enough, we have, for all  $m \in E$ :

$$|a_{m,n}| \geq 2\delta n^{-1/2} - K\sqrt{n} m^{-5/4} \geq 2\delta n^{-1/2} - KC^{5/2}n^{-3/4} \geq \delta n^{-1/2}.$$

With this information, (6.5) gives:

$$M\beta_n \geq \sum_{m \in E} |a_{m,n}|^2 \beta_m \geq \delta^2 n^{-1} |E| \beta_{\lfloor C^{-2}n \rfloor} \geq \delta^3 \beta_{\lfloor C^{-2}n \rfloor},$$

where  $\lfloor \cdot \rfloor$  stands for the integer part, and this proves the theorem.  $\square$

**Acknowledgements.** We warmly thank R. Zarouf for useful discussions and informations.

L. Rodríguez-Piazza is partially supported by the project PGC2018-094215-B-I00 (Spanish Ministerio de Ciencia, Innovación y Universidades, and FEDER funds). Parts of this paper was made when he visited the Université d'Artois in Lens and the Université de Lille in January 2020. It is his pleasure to thank all his colleagues in these universities for their warm welcome.

The third-named author was partly supported by the Labex CEMPI (ANR-LABX-0007-01).

This work is also partially supported by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front).

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