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► **To cite this version:**

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza. An extremal composition operator on the Hardy space of the bidisk with small approximation numbers. 2019. hal-01987504

HAL Id: hal-01987504

<https://univ-artois.hal.science/hal-01987504>

Preprint submitted on 21 Jan 2019

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An extremal composition operator on the Hardy space of the bidisk with small approximation numbers

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

January 21, 2019

Abstract. We construct an analytic self-map Φ of the bidisk \mathbb{D}^2 whose image touches the distinguished boundary, but whose approximation numbers of the associated composition operator on $H^2(\mathbb{D}^2)$ are small in the sense that $\limsup_{n \rightarrow \infty} [a_n(C_\Phi)]^{1/n} < 1$.

MSC 2010 primary: 47B33 ; secondary: 32A35 ; 46B28

Key-words approximation numbers ; bidisk ; composition operator ; cusp map ; distinguished boundary ; Hardy space

1 Introduction

For composition operators $C_\Phi: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ on the Hardy space of the unit disk, the decay of their approximation numbers $a_n(C_\Phi)$ cannot be arbitrarily fast, and actually cannot supersede a geometric speed ([16]; see also [10, Theorem 3.1]): there exists a positive constant c such that:

$$a_n(C_\Phi) \gtrsim e^{-cn}, \quad n = 1, 2, \dots$$

It is easy to see that this speed occurs when $\|\Phi\|_\infty < 1$, and we proved in [10, Theorem 3.4] that a geometrical speed only takes place in this case; in other words:

$$(1.1) \quad \|\Phi\|_\infty = 1 \iff \lim_{n \rightarrow \infty} [a_n(C_\Phi)]^{1/n} = 1.$$

This leads to the introduction, for an operator T between Banach spaces, of the parameters:

$$(1.2) \quad \beta^-(T) = \liminf_{n \rightarrow \infty} [a_n(T)]^{1/n} \quad \text{and} \quad \beta^+(T) = \limsup_{n \rightarrow \infty} [a_n(T)]^{1/n},$$

where $a_n(T)$ is the n -th approximation number of T . When $[a_n(T)]^{1/n}$ actually has a limit, i.e. when $\beta^-(T) = \beta^+(T)$, we write it $\beta(T)$.

What is proved in [10, Theorem 3.4] is that $\beta(C_\Phi) = 1$ if and only if $\|\Phi\|_\infty = 1$. Later, in [12], we gave, when $\|\Phi\|_\infty < 1$, a formula for this parameter in terms of the Green capacity of $\Phi(\mathbb{D})$, which allowed us to recover (1.1).

More generally, for $N \geq 1$, we introduce:

$$(1.3) \quad \beta_N^-(T) = \liminf_{n \rightarrow \infty} [a_{nN}(T)]^{1/n} \quad \text{and} \quad \beta_N^+(T) = \limsup_{n \rightarrow \infty} [a_{nN}(T)]^{1/n},$$

and:

$$(1.4) \quad \beta_N(T) = \lim_{n \rightarrow \infty} [a_{nN}(T)]^{1/n}$$

when the limit exists. It is clear that $0 \leq \beta_N^\pm(T) \leq 1$, and it is interesting to know when the extreme cases $\beta_N^\pm(T) = 0$ or $\beta_N^\pm(T) = 1$ occur. For example:

$$\begin{aligned} \beta_N^-(T) > 0 &\iff a_{nN}(T) \gtrsim e^{-\tau n}, \quad \text{with } \tau > 0 \\ \beta_N^-(T) = 1 &\iff a_{nN}(T) \gtrsim e^{-n\varepsilon_n}, \quad \text{with } \varepsilon_n \rightarrow 0. \end{aligned}$$

It is coined in [1] (see also [13] and [14]) that $\beta_N^\pm(C_\Phi)$ are the suitable parameters for the composition operators on $H^2(\mathbb{D}^N)$, and it is proved, for any $N \geq 1$, that $\beta_N^-(C_\Phi) > 0$, as soon as Φ is non degenerate (i.e. the Jacobian J_Φ is not identically 0) and the operator C_Φ is bounded on $H^2(\mathbb{D}^N)$. As for an expression of $\beta_N^\pm(C_\Phi)$ in terms of ‘‘capacity’’, only partial results are known so far ([13] and [14]) and the application to a result like (1.1) fails in general. We gave an example of such a phenomenon in [13, Theorem 5.12]. In the present paper we give a shaper result.

2 Background and notation

Let \mathbb{D} be the open unit disk, $H^2(\mathbb{D}^N)$ the Hardy space of the polydisk \mathbb{D}^N , and $\Phi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ an analytic map. When $N = 1$, it is well-known (see [4] or [17]) that Φ induces a composition operator $C_\Phi: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ by the formula:

$$C_\Phi(f) = f \circ \Phi,$$

and the connection between the ‘‘symbol’’ Φ and the properties of the operator C_Φ , in particular its compactness, can be further studied (see [4] or [17]). When $N > 1$, C_Φ is not bounded in general (see [4]).

Let \mathbb{T} be the unit circle, and m the normalized Haar measure on \mathbb{T}^N . A positive Borel measure μ on \mathbb{D}^N is called a Carleson measure (for the space $H^2(\mathbb{D}^N)$) if the canonical injection $J: H^2(\mathbb{D}^N) \rightarrow L^2(\mu)$ is bounded. When $\Phi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ is analytic and induces a bounded composition operator on $H^2(\mathbb{D}^N)$, the pullback measure $m_\Phi = \Phi^*(m)$, defined, for any test function u , by:

$$\int_{\mathbb{D}^N} u(w) dm_\Phi(w) = \int_{\mathbb{T}^N} u[\Phi^*(\xi)] dm(\xi),$$

is a Carleson measure. Here Φ^* is the radial limit function, defined for m -almost every $\xi \in \mathbb{T}^N$, by $\Phi^*(\xi) = \lim_{r \rightarrow 1^-} \Phi(r\xi)$.

For $\xi \in \mathbb{T} = \partial\mathbb{D}$ and $h > 0$, the Carleson window $S(\xi, h)$ is defined as:

$$(2.1) \quad S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\}.$$

If $f \in \text{Hol}(\mathbb{D}^2)$, $D_j^k f$ denotes the k -th derivative of f with respect to the j -th variable ($j = 1, 2$).

We denote by $A(\mathbb{D})$ the disk algebra, i.e. the space of functions holomorphic in \mathbb{D} and continuous on $\overline{\mathbb{D}}$. We similarly define the bidisk algebra $A(\mathbb{D}^2)$.

Let H_1 and H_2 be Hilbert spaces, and $T: H_1 \rightarrow H_2$ an operator. The n -th approximation number $a_n(T)$ of T , $n = 1, 2, \dots$, is defined (see [2]) as the distance (for the operator-norm) of T to operators of rank $< n$:

$$(2.2) \quad a_n(T) = \inf_{\text{rank } R < n} \|T - R\|.$$

The approximation numbers have the ideal property:

$$a_n(ATB) \leq \|A\| a_n(T) \|B\|.$$

The n -th Gelfand number $c_n(T)$ of T is defined by:

$$(2.3) \quad c_n(T) = \inf_{\text{codim } E < n} \|T|_E\|.$$

As an easy consequence of the Schmidt decomposition, we have for any compact operator between Hilbert spaces:

$$(2.4) \quad c_n(T) = a_n(T).$$

If $T, T_1, T_2: H \rightarrow H'$ are operators between Hilbert spaces H and H' , we write $T = T_1 \oplus T_2$ if $T = T_1 + T_2$ and:

$$\|Tx\|^2 = \|T_1x\|^2 + \|T_2x\|^2, \quad \text{for all } x \in H.$$

The subadditivity of approximation numbers is then expressed by:

$$(2.5) \quad a_{j+k}(T_1 \oplus T_2) \leq a_j(T_1) + a_k(T_2).$$

We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of non-negative integers, and by $[x]$ the integral part of the real number x .

We write $X \lesssim Y$ to indicate that $X \leq cY$ for some constant $c > 0$, and $X \approx Y$ to indicate that $X \lesssim Y$ and $Y \lesssim X$.

3 Purpose of the paper

Let us recall that the Hardy space of the polydisk is the space:

$$H^2(\mathbb{D}^N) = \left\{ f: \mathbb{D}^N \rightarrow \mathbb{C}; f(z) = \sum_{\alpha \in \mathbb{N}^N} a_\alpha z^\alpha \text{ and } \|f\|_2^2 := \sum |a_\alpha|^2 < \infty \right\}.$$

If $\Phi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ is an analytic map, the associated composition operator C_Φ (which is not always bounded on $H^2(\mathbb{D}^N)$) is defined by:

$$C_\Phi(f) = f \circ \Phi.$$

We will mainly here be interested in the case $N = 2$.

The reproducing kernel K_a of $H^2(\mathbb{D}^2)$ is, with $a = (a_1, a_2)$ and $z = (z_1, z_2)$:

$$(3.1) \quad K_a(z) = \frac{1}{(1 - \overline{a_1}z_1)(1 - \overline{a_2}z_2)}.$$

As a consequence:

$$(3.2) \quad |f(a)| = |\langle f, K_a \rangle| \leq \frac{\|f\|_2}{\sqrt{(1 - |a_1|^2)(1 - |a_2|^2)}}.$$

In particular, the functions in the unit ball of $H^2(\mathbb{D}^2)$ are uniformly bounded on compact subsets of \mathbb{D}^2 .

In [13, Theorem 5.12], we gave an example of a holomorphic self-map $\Phi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$, continuous on the closure $\overline{\mathbb{D}^2}$, such that $\|\Phi\|_\infty = 1$, that is:

$$(3.3) \quad \Phi(\mathbb{T}^2) \cap \partial\mathbb{D}^2 \neq \emptyset,$$

and yet:

$$(3.4) \quad \beta_2^+(C_\Phi) < 1,$$

in contrast with the one-dimensional case ([10, Theorem 3.4]).

Understanding where the difference really lies when we pass to the multidimensional case is a big challenge: it does not seem to be a matter of regularity of the boundary, and a similar example probably holds for the Hardy space of the ball. It might be a matter of boundary: the Shilov boundary of the ball is its usual boundary, but that of the polydisk is its distinguished boundary:

$$\partial_e \mathbb{D}^N = \{z = (z_j); |z_j| = 1 \text{ for all } j = 1, \dots, N\} = \mathbb{T}^N$$

(indeed, the distinguished maximum principle tells that, for f analytic in \mathbb{D}^N and continuous on $\overline{\mathbb{D}^N}$, it holds $\max_{z \in \mathbb{D}^N} |f(z)| = \max_{z \in \partial_e \mathbb{D}^N} |f(z)|$). The aim of this paper is to show that this is not the case and, improving on ([13, Theorem 5.12]) and (3.3), to build an analytic self-map $\Phi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$, continuous on $\overline{\mathbb{D}^2}$, non-degenerate and such that:

$$(3.5) \quad \Phi(\mathbb{T}^2) \cap \partial_e \mathbb{D}^2 \neq \emptyset, \quad \text{but} \quad \beta_2^+(C_\Phi) < 1.$$

The paper is organized as follows. In Section 4, we recall with some detail the definition and main properties of a so-called *cuspidal map* $\chi \in A(\mathbb{D})$, to be of essential use in our counterexample. In Section 5, we prove several lemmas which constitute the core of the proof. In Section 6, we state and prove our main theorem.

4 The cuspidal map

The *cuspidal map* $\chi: \mathbb{D} \rightarrow \mathbb{D}$ is analytic in \mathbb{D} and extends continuously on $\overline{\mathbb{D}}$. The boundary of its image is formed by three circular arcs of respective centers $\frac{1}{2}$, $1 + \frac{i}{2}$, $1 - \frac{i}{2}$, and of radius $\frac{1}{2}$ (see Figure 1). However, the parametrization $t \mapsto \chi(e^{it})$ involves logarithms.

It was often used by the authors ([11], [8]) as an extremal example.

We first recall the definition of χ .

Let $\mathbb{D}^+ = \{z \in \mathbb{D}; \Re z > 0\}$ be the right half-disk. Let now \mathbb{H} be the upper half-plane, and $T: \mathbb{D} \rightarrow \mathbb{H}$ defined by:

$$T(u) = i \frac{1+u}{1-u}, \quad \text{with} \quad T^{-1}(s) = \frac{s-i}{s+i}.$$

Taking the square root of T , we map \mathbb{D} onto the first quadrant defined by $Q_1 = \{z \in \mathbb{C}; \Re z > 0\}$; we go back to the half-disk $\{z \in \mathbb{D}; \Im z < 0\}$ by T^{-1} . Finally, make a rotation by i to go onto \mathbb{D}^+ . We get:

$$\chi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1}.$$

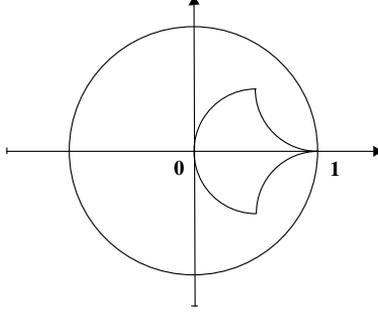


Figure 1: *Cusp map domain*

One has $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, and $\chi_0(-i) = i$. The half-circle $\{z \in \mathbb{T}; \Re z \geq 0\}$ is mapped by χ_0 onto the segment $[-i, i]$ and the segment $[-1, 1]$ onto the segment $[0, 1]$.

Set now, successively:

$$(4.1) \quad \chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{1}{\chi_2(z)}$$

and finally:

$$(4.2) \quad \chi(z) = 1 - \chi_3(z).$$

We now summarize the properties of the cusp map χ in the following proposition.

Proposition 4.1. *The cusp map satisfies:*

- 1) $1 - |\chi(z)| \lesssim \frac{1}{\log(2/|1-z|)}$;
- 2) $|1 - \chi(z)| \leq K(1 - |\chi(z)|)$ for all $z \in \mathbb{D}$, where K is a positive constant;
- 3) $\chi(\mathbb{D})$ is the intersection of the open disk $D(\frac{1}{2}, \frac{1}{2})$ with the exterior of the two open disks $D(1 + \frac{i}{2}, \frac{1}{2})$ and $D(1 - \frac{i}{2}, \frac{1}{2})$;
- 4) $\chi(1) = 1$, $\chi(\bar{z}) = \overline{\chi(z)}$ and $|\chi(z) - 1| \leq 1$ for all $z \in \mathbb{D}$;
- 5) for $0 < |t| \leq \pi/4$, we have $1 - \Re \chi(e^{it}) \approx 1/(\log 1/|t|)$;
- 6) $\chi(\overline{\mathbb{D}}) \subseteq \{z = x + iy; 0 \leq x \leq 1 \text{ and } |y| \leq 2(1-x)^2\}$.

Proof. Items 1) to 5) are proved in [11, Lemma 4.2]. To prove 6), write $\chi(z) = (1 - h) + iy$. Since $\chi(\bar{z}) = \overline{\chi(z)}$, we can assume $y \geq 0$. Since $\chi(\mathbb{D}) \cap D(1 + \frac{i}{2}, \frac{1}{2}) = \emptyset$, we have $|\chi(z) - (1 + \frac{i}{2})| \geq \frac{1}{2}$; hence:

$$h^2 + \left(y - \frac{1}{2}\right)^2 = \left|\chi(z) - \left(1 + \frac{i}{2}\right)\right|^2 \geq \frac{1}{4},$$

so that $y \leq y^2 + h^2$. But $y \leq 1/2$, since $\chi(z) \in D(\frac{1}{2}, \frac{1}{2})$; therefore $y^2 \leq y/2$, so we get $y \leq 2h^2$. \square

5 Preliminary lemmas

In this section, we collect some lemmas, which will reveal essential in the proof of our counterexample.

We consider the map $\varphi = \varphi_\theta$, $0 < \theta < 1$, defined, for $z \in \overline{\mathbb{D}} \setminus \{1\}$, by:

$$(5.1) \quad \varphi(z) = \exp\left(- (1 - z)^{-\theta}\right).$$

We observe, since $\Re(1 - z) \geq 0$ for $z \in \overline{\mathbb{D}}$, that:

$$(5.2) \quad |\varphi(z)| \leq \exp\left(-\delta |1 - z|^{-\theta}\right),$$

where $\delta = \cos \pi\theta/2 > 0$. Moreover, (5.2) shows that $\varphi \in A(\mathbb{D})$, since:

$$\lim_{z \rightarrow 1, z \in \overline{\mathbb{D}}} \varphi(z) = 0 =: \varphi(1).$$

Our first lemma will allow us to define our symbol Φ .

Lemma 5.1. *One can adjust $0 < c < 1$ so as to get:*

$$(5.3) \quad |\chi(z)| + 2c|\varphi \circ \chi(z)| < 1 \quad \text{for all } z \in \mathbb{D}.$$

Hence, if we set, for any $g \in A(\mathbb{D})$ with $\|g\|_\infty \leq 1$:

$$(5.4) \quad \Phi(z_1, z_2) = \left(\chi(z_1), \chi(z_1) + c(\varphi \circ \chi)(z_1)g(z_2)\right),$$

we have $\Phi(\mathbb{D}^2) \subseteq \mathbb{D}^2$.

Remark. The factor 2 in (5.3) is needed in order to get the following inequalities, to be used later, for $z \in \mathbb{D}$ and $w = \chi(z) + c(\varphi \circ \chi)(z)u$, with $|u| \leq 1$:

$$(5.5) \quad |w| \leq \frac{1 + |\chi(z)|}{2},$$

or, equivalently:

$$(5.6) \quad 1 - |w| \geq \frac{1 - |\chi(z)|}{2}.$$

Indeed:

$$|w| \leq |\chi(z)| + c|\varphi \circ \chi(z)| \leq |\chi(z)| + \frac{1 - |\chi(z)|}{2} = \frac{1 + |\chi(z)|}{2}.$$

Proof of Lemma 5.1. Set $X = |1 - \chi(z)|$, so that, with K the constant of Proposition 4.1, 2):

$$(5.7) \quad |\chi(z)| \leq 1 - \frac{|1 - \chi(z)|}{K} = 1 - \frac{X}{K}.$$

For $z \in \mathbb{D}$ and X close enough to zero, say $X < \eta$, we have $2 \exp(-\delta X^{-\theta}) < \frac{X}{K}$. If we adjust $0 < c < 1$ so as to have $c < \frac{\eta}{2K}$, it follows from (5.2) and (5.7) that, for $X < \eta$:

$$|\chi(z)| + 2c|\varphi \circ \chi(z)| \leq 1 - \frac{X}{K} + 2 \exp(-\delta X^{-\theta}) < 1.$$

However, for $X \geq \eta$, (5.7) says that $|\chi(z)| \leq 1 - \frac{\eta}{K}$, so:

$$|\chi(z)| + 2c|\varphi \circ \chi(z)| \leq 1 - \frac{\eta}{K} + 2c < 1,$$

as well and this ends the proof of Lemma 5.1. \square

Our second lemma estimates some integrals and ensures that Φ induces a compact composition operator on $H^2(\mathbb{D}^2)$.

Lemma 5.2. *For $0 < h \leq 1$, the following estimate holds:*

$$(5.8) \quad I_0(h) := \int_{|\chi(e^{it}) - 1| \leq h} \frac{1}{(1 - |\chi(e^{it})|)^2} dt \lesssim e^{-\tau/h},$$

Proof. By Proposition 4.1, 5), there exist two constants c_1, c_2 such that:

$$\frac{c_1}{\log 1/|t|} \leq |\chi(e^{it}) - 1| \leq \frac{c_2}{\log 1/|t|}, \quad |t| \leq \pi;$$

hence:

$$I_0(h) \lesssim \int_{|t| \leq e^{-c_1/h}} [\log(1/|t|)]^2 dt = 2 \int_{c_1/h}^{\infty} x^2 e^{-x} dx \lesssim h^{-2} e^{-c_1/h}. \quad \square$$

Corollary 5.3. For $g \in A(\mathbb{D})$ with $0 < \|g\|_\infty \leq 1$, set:

$$I(h) := \int_{|\chi(e^{it_1})-1| \leq h} \frac{dt_1 dt_2}{(1 - |\chi(e^{it_1})|)(1 - |\chi(e^{it_1}) + c(\varphi \circ \chi)(e^{it_1})g(e^{it_2})|)}.$$

Then:

$$I(h) \lesssim e^{-\tau/h}.$$

Consequently, the composition operator C_Φ defined in (5.4) is bounded from $H^2(\mathbb{D}^2)$ to $H^2(\mathbb{D}^2)$ and is compact.

Proof. Using (5.6), we have, thanks to (5.8):

$$I(h) \leq \int_{|\chi(e^{it_1})-1| \leq h} \frac{2}{(1 - |\chi(e^{it_1})|)^2} dt_1 dt_2 \lesssim e^{-\tau/h}.$$

In particular, $I(1) < \infty$, showing that C_Φ is Hilbert-Schmidt and hence bounded. \square

For the rest of the paper, we fix a number σ in $(0, 1)$, that for convenience we take as:

$$(5.9) \quad \sigma = \frac{7}{8},$$

a positive integer j_0 such that:

$$(5.10) \quad 2\sigma^{j_0} \leq 1/8$$

(i.e. $j_0 \geq 21$), and we set:

$$(5.11) \quad a_j = 1 - \sigma^j$$

and:

$$(5.12) \quad \rho_j = \frac{\sigma^j}{4} = \frac{1}{4}(1 - a_j).$$

We also define, for $n \geq 1$ and θ being the parameter used in (5.1):

$$(5.13) \quad N_n = \left\lceil \frac{\log 2n}{\theta \log 1/\sigma} \right\rceil + 1 > \frac{\log 2n}{\log 1/\sigma}.$$

The next lemma gives a cutting off for $\chi(\mathbb{D})$.

Lemma 5.4. *For every $n \geq 1$, the image $\chi(\mathbb{D})$ of the cusp map, deprived of the closed Euclidean disk $\overline{D}(0, 1 - \sigma^{j_0}/K)$ and of $\chi(\mathbb{D}) \cap S(1, 1/n)$, can be covered by the open Euclidean disks $D(a_j, \rho_j)$, with $j_0 \leq j \leq N_n$.*

Proof. Let $z \in \mathbb{D}$ such that $|\chi(z)| > 1 - \sigma^{j_0}/K$ and $|\chi(z) - 1| > 1/n$. We write $\chi(z) = x + iy =: 1 - h + iy$.

Let j with $a_j \leq x < a_{j+1}$, i.e. $\sigma^{j+1} < h \leq \sigma^j$. We have $j \geq j_0$, since $h < \sigma^{j_0}$.

Now, since $0 \leq x - a_j < a_{j+1} - a_j = \sigma^{j+1} - \sigma^j$, that $y^2 \leq 4h^4$ (Proposition 4.1, 6)), and $h \leq \sigma^j$, we have:

$$|\chi(z) - a_j|^2 < (\sigma^j - \sigma^{j+1})^2 + y^2 \leq (1 - \sigma)^2 \sigma^{2j} + 4\sigma^{4j};$$

hence:

$$|\chi(z) - a_j| < \sigma^j(1 - \sigma) + 2\sigma^{2j} = \sigma^j(1 - \sigma + 2\sigma^j).$$

Subsequently, since $1 - \sigma = 1/8$, $j \geq j_0$, and $2\sigma^{j_0} \leq 1/8$:

$$|\chi(z) - a_j| < \sigma^j(1 - \sigma + 2\sigma^{j_0}) \leq \frac{\sigma^j}{4} = \rho_j,$$

showing that $\chi(z) \in D(a_j, \rho_j)$.

Moreover, we have $j \leq N_n$. Indeed, if $j > N_n$, we would have:

$$|\chi(z) - 1| \leq |\chi(z) - a_j| + (1 - a_j) \leq \rho_j + \sigma^j = \frac{5}{4}\sigma^j \leq \frac{5}{4}\sigma^{N_n+1} \leq 2\sigma^{N_n} \leq 1/n,$$

contradicting the fact that $\chi(z) \notin S(1, 1/n)$. \square

Our next two lemmas give estimates on derivatives for the functions belonging to $H^2(\mathbb{D}^2)$.

Lemma 5.5. *Let $f \in H^2(\mathbb{D}^2)$, k a non-negative integer, $b \in \mathbb{D}$, and let $h_k(z) = (D_2^k f)(z, z)$. Then:*

$$|h_k(b)| \leq \frac{k! 2^{k+1}}{(1 - |b|)^{k+1}} \|f\|_2.$$

Proof. The Cauchy inequalities give for $0 < s < 1 - |b|$ and $\alpha \in \mathbb{N}^2$:

$$|D^\alpha f(b, b)| \leq \frac{\alpha!}{s^{|\alpha|}} \sup_{|w_1 - b| = s, |w_2 - b| = s} |f(w_1, w_2)|.$$

The choice $s = \frac{1-|b|}{2}$ gives $1 - |w_j| \geq \frac{1-|b|}{2}$ for $|w_j - b| = s$, $j = 1, 2$; hence, thanks to the estimate (3.2):

$$|f(w_1, w_2)| \leq \frac{\|f\|_2}{\sqrt{(1-|w_1|)(1-|w_2|)}} \leq \frac{2}{1-|b|} \|f\|_2.$$

Specializing to $\alpha = (0, k)$ now gives the result. \square

Lemma 5.6. *With the notations of Lemma 5.5, assume that $h_k^{(l)}(a) = 0$ for some $a \in \mathbb{D}$ and for $0 \leq l < n$. Then, for $0 < \rho < 1$ and $|b - a| \leq \frac{\rho}{2}(1 - |a|)$, it holds:*

$$|h_k(b)| \leq \rho^n \frac{k! 4^{k+1}}{(1 - |a|)^{k+1}} \|f\|_2.$$

Proof. We may assume $\|f\|_2 \leq 1$. Consider the function defined, for $w \in \mathbb{D}$, by:

$$H_k(w) = h_k\left(a + w \frac{1 - |a|}{2}\right).$$

It is a bounded and holomorphic function in \mathbb{D} .

For $w \in \mathbb{D}$, let $\beta = a + w \frac{1-|a|}{2}$, which satisfies $1 - |\beta| \geq \frac{1-|a|}{2}$. Lemma 5.5 gives:

$$|H_k(w)| = |h_k(\beta)| \leq \frac{k! 4^{k+1}}{(1 - |a|)^{k+1}}.$$

Now, $H_k^{(l)}(0) = h_k^{(l)}(a) = 0$ for $0 \leq l < n$; hence the Schwarz lemma says that H_k satisfies $|H_k(w)| \leq |w|^n \|H_k\|_\infty$ for all $w \in \mathbb{D}$. Take $w = \frac{2(b-a)}{1-|a|}$, which satisfies $|w| \leq \rho$, to get:

$$|h_k(b)| = |H_k(w)| \leq |w|^n \|H_k\|_\infty \leq \rho^n \frac{k! 4^{k+1}}{(1 - |a|)^{k+1}}. \quad \square$$

6 The main result

Recall that χ is the cusp map and that φ is defined in (5.1). The map g appearing in the formula below plays an inert role, and is just designed to ensure that Φ is non-degenerate; we can take, for example $g(z_2) = z_2$. This seems to mean that non-degeneracy is not the only issue in the question of estimating $\beta_2^+(C_\Phi)$.

Our example appears as a perturbation of the diagonal map defined by $\Delta(z_1, z_2) = (\chi(z_1), \chi(z_1))$ for which we already know ([15, Theorem 2.4])

that $\Delta(1, 1) = (1, 1)$ and $\beta_2^+(C_\Delta) < 1$. This map is degenerate, but the perturbation clearly gives a non degenerate one since its Jacobian is $J_\Phi(z_1, z_2) = c(\varphi \circ \chi)(z_1) \chi'(z_1) g'(z_2)$.

Theorem 6.1. *Let:*

$$\Phi(z_1, z_2) = (\chi(z_1), \chi(z_1) + c(\varphi \circ \chi)(z_1) g(z_2))$$

be the function defined in (5.4).

Then:

- 1) $\Phi(\mathbb{D}^2) \subseteq \mathbb{D}^2$ and $C_\Phi: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$ is compact;
- 2) Φ is non degenerate, and its components belong to the bidisk algebra;
- 3) $\Phi(\mathbb{T}^2) \cap \mathbb{T}^2 = \Phi(\mathbb{T}^2) \cap \partial_e \mathbb{D}^2 \neq \emptyset$;
- 4) $a_{n^2}(C_\Phi) \lesssim \exp(-\tau n)$, for some $\tau > 0$, implying $\beta_2^+(C_\Phi) < 1$.

Proof. That Φ maps \mathbb{D}^2 to itself is proved in Lemma 5.1 and that the composition operator $C_\Phi: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$ is compact, in Corollary 5.3. Item 2) is due to the presence of g , as explained above. The fact that $\Phi(\mathbb{T}^2) \cap \mathbb{T}^2 \neq \emptyset$ is clear since $\Phi(1, 1) = (1, 1)$. It remains to prove 4).

Once more, the proof will be conveniently divided into several steps. We begin by a lemma which is in fact obvious, but explains well what is going on.

Lemma 6.2. *Let $\lambda = 1 - \frac{\sigma^{j_0}}{2K}$, where σ , K and j_0 are as in (5.9), Proposition 4.1, 2), and (5.10). Let $r_n = 1 - \frac{1}{n}$, and let μ_1, μ_2, μ_3 the respective restrictions of m_Φ to the disk $\overline{\lambda \mathbb{D}^2}$, the annulus $r_n \mathbb{D}^2 \setminus \overline{\lambda \mathbb{D}^2}$, and the annulus $\mathbb{D}^2 \setminus r_n \mathbb{D}^2$. We then have:*

$$C_\Phi = T_1 \oplus T_2 \oplus T_3,$$

where T_j is the canonical injection of $H^2(\mathbb{D}^2)$ into $L^2(\mu_j)$.

This is indeed obvious since:

$$\|C_\Phi f\|^2 = \int_{\mathbb{D}^2} |f|^2 dm_\Phi,$$

and by splitting the integral into three parts.

We now majorize separately the numbers $a_p(T_j)$, for $j = 1, 2, 3$. In the sequel, the positive constant τ may vary from one formula to another.

Step 1. It holds:

$$(6.1) \quad a_{n^2}(T_1) \lesssim e^{-\tau n}.$$

Proof. Let $V = z_1^n H^2(\mathbb{D}^2) + z_2^n H^2(\mathbb{D}^2)$; this is a subspace of $H^2(\mathbb{D}^2)$ of codimension $\leq n^2$, since:

$$V = \{f \in H^2(\mathbb{D}^2); D_1^j D_2^k f(0,0) = 0 \text{ for } 0 \leq j, k < n\}.$$

If $f(z) = \sum_{\max(j,k) \geq n} a_{j,k} z_1^j z_2^k \in V$ and $\|f\|_2 = 1$, one can write:

$$f(z) = z_1^n q_1(z_1, z_2) + z_2^n q_2(z_1, z_2),$$

with:

$$q_1(z) = \sum_{j \geq n, k \geq 0} a_{j,k} z_1^{j-n} z_2^k \quad \text{and} \quad q_2(z) = \sum_{j < n, k \geq n} a_{j,k} z_1^j z_2^{k-n},$$

which satisfy $\|q_j\|_2 \leq \|f\|_2 = 1$, $j = 1, 2$.

An easy estimate now gives (since $\max(|z_1|^n, |z_2^n|) \leq \lambda^n$ on $\overline{\lambda\mathbb{D}^2}$):

$$\begin{aligned} \|T_1 f\|^2 &\leq 2 \left(\int_{\lambda\mathbb{D}^2} (|z_1^n|^2 |q_1(z_1, z_2)|^2 + |z_2^n|^2 |q_2(z_1, z_2)|^2) dm_\Phi \right) \\ &\lesssim \lambda^{2n} \int_{\lambda\mathbb{D}^2} (|q_1|^2 + |q_2|^2) dm_\Phi \lesssim \lambda^{2n} (\|q_1\|_2^2 + \|q_2\|_2^2) \lesssim \lambda^{2n}, \end{aligned}$$

since we know by Corollary 5.3 that C_Φ is bounded on $H^2(\mathbb{D}^2)$ and hence that m_Φ is a Carleson measure for $H^2(\mathbb{D}^2)$. Alternatively, we could majorize $|q_j(z_1, z_2)|$ uniformly on the support of μ_1 . We hence obtain:

$$(6.2) \quad a_{n^2+1}(T_1) = c_{n^2+1}(T_1) \lesssim e^{-\tau n}. \quad \square$$

Step 2. It holds:

$$(6.3) \quad a_{n^2}(T_3) \lesssim e^{-\tau n}.$$

Proof. In one variable, we could use the Carleson embedding theorem; but this theorem for the bidisk and the Hardy space $H^2(\mathbb{D}^2)$ notably has a more complicated statement ([3]; see also [5]), and cannot be used efficiently here. Our strategy will be to replace it by a sharp estimation of a Hilbert-Schmidt norm.

We set $h_n = 1 - r_n = 1/n$.

Clearly, denoting by S_2 the Hilbert-Schmidt class:

$$\|T_3\|^2 \leq \|T_3\|_{S_2}^2 = \int \frac{d\mu_3(w)}{(1 - |w_1|^2)(1 - |w_2|^2)} \leq \int \frac{d\mu_3(w)}{(1 - |w_1|)(1 - |w_2|)}.$$

Now, if $w = (w_1, w_2) = (\chi(z_1), \chi(z_1) + c(\varphi \circ \chi)(z_1)g(z_2))$ belongs to the support of μ_3 , we have $\max(|w_1|, |w_2|) \geq r_n = 1 - h_n$, and, recalling (5.5):

$$(6.4) \quad |w_1| \geq 2|w_2| - 1,$$

we have in either case $|w_1| \geq 1 - 2h_n$. By Proposition 4.1, 2), this implies that:

$$|1 - w_1| \leq 2Kh_n.$$

Corollary 5.3 gives:

$$\begin{aligned} \|T_3\|^2 &\lesssim \int_{|\chi(e^{it_1}) - 1| \leq 2Kh_n} \frac{dt_1 dt_2}{(1 - |\chi(e^{it_1})|)(1 - |\chi(e^{it_1}) + c(\varphi \circ \chi)(e^{it_1})g(e^{it_2})|)} \\ &= I(2Kh_n) \lesssim e^{-\tau/h_n}. \end{aligned}$$

But $h_n = 1/n$, so that:

$$(6.5) \quad a_{n^2}(T_3) \leq \|T_3\| \lesssim e^{-\tau n}. \quad \square$$

Step 3. It holds:

$$(6.6) \quad a_{n^2}(T_2) \lesssim e^{-\tau n}.$$

This estimate follows from the following key auxiliary lemma. In fact, this lemma will give, for the Gelfand numbers, $c_{n^2}(T_2) \lesssim e^{-\tau n}$, and we know that they are equal to the approximation numbers.

Let $M: H^2(\mathbb{D}^2) \rightarrow \text{Hol}(\mathbb{D})$ be the linear map defined by:

$$Mf(z) = f(z, z),$$

Recall that $a_j = 1 - \sigma^j$ and $N_n = \left\lceil \frac{\log 2n}{\theta \log 1/\sigma} \right\rceil + 1$.

Lemma 6.3. *Let E be the closed subspace of $H^2(\mathbb{D}^2)$ defined by:*

$$\begin{aligned} E = \left\{ f \in H^2(\mathbb{D}^2); [M(D_2^k f)]^{(l)}(a_j) = 0 \right. \\ \left. \text{for } 0 \leq l < n, 0 \leq k \leq m_j, 1 \leq j \leq N_n \right\}. \end{aligned}$$

Then, we can adjust the numbers m_j so as to guarantee that, for some positive constant τ :

$$\text{codim } E \lesssim n^2$$

and, for all $f \in E$ with $\|f\|_2 \leq 1$:

$$\|T_2(f)\|_2 \lesssim \exp(-\tau n).$$

Proof. This is the most delicate part.

Recall that:

$$h_n = 1/n, \quad r_n = 1 - h_n, \quad \lambda = 1 - \frac{\sigma^{j_0}}{2K}.$$

We need a uniform estimate of $|f(w)|$ for $f \in E$ with $\|f\|_2 \leq 1$ and for:

$$w = (w_1, w_2) \in \text{supp } m_\Phi \cap (r_n \mathbb{D}^2 \setminus \overline{\lambda \mathbb{D}^2}).$$

This estimate will be given by Lemma 5.4, Lemma 5.5 and Lemma 5.6. Note that we have:

$$\chi(z_1) \in \mathbb{D} \setminus [S(1, 1/n) \cup \overline{D}(0, 2\lambda - 1)].$$

Indeed, if $(w_1, w_2) = \Phi(z_1, z_2) \notin \lambda \mathbb{D}^2$, we have $\max(|w_1|, |w_2|) > \lambda$; so either $|w_1| > \lambda \geq 2\lambda - 1$, or $|w_2| > \lambda$ and again $|w_1| > 2\lambda - 1$ since $|w_1| \geq 2|w_2| - 1$, by (5.5). Hence $w_1 \notin \overline{D}(0, 2\lambda - 1)$. Moreover, we have $|1 - w_1| \geq 1 - |w_1| > 1/n$, so $w_1 \notin S(1, 1/n)$.

Using Lemma 5.4, select $j_0 \leq j \leq N_n$ such that $|\chi(z_1) - a_j| \leq \frac{1}{4}(1 - a_j)$. Now set:

$$A = (\chi(z_1), \chi(z_1)) \quad \text{and} \quad \Delta = (0, (\varphi \circ \chi)(z_1) g(z_2)).$$

Our strategy will be the following. We write:

$$\begin{aligned} f[\Phi(z_1, z_2)] &= f(A + \Delta) = \sum_{k=0}^{\infty} \frac{D_2^k f(A)}{k!} \Delta^k \\ &= \sum_{k=0}^{\infty} \frac{M(D_2^k f)[\chi(z_1)]}{k!} \Delta^k = \sum_{k=0}^{\infty} \frac{h_k[\chi(z_1)]}{k!} \Delta^k, \end{aligned}$$

with $h_k = M(D_2^k f)$, and we put:

$$S_j = \sum_{k=0}^{m_j} \frac{h_k[\chi(z_1)]}{k!} \Delta^k$$

and

$$R_j = \sum_{k > m_j} \frac{h_k[\chi(z_1)]}{k!} \Delta^k.$$

We will estimate separately S_j and R_j .

a) *Estimation of R_j .*

Recall that j is such that $j_0 \leq j \leq N_n$ and $|\chi(z_1) - a_j| \leq \frac{1}{4}(1 - a_j)$. We saw in the proof of this Lemma 5.4 that $1 - |\chi(z_1)| \leq |1 - \chi(z_1)| \leq \frac{5}{4}\sigma^j$. Hence:

$$|\Delta| \leq |(\varphi \circ \chi)(z_1)| \leq \exp\left(-\frac{\delta}{|1 - \chi(z_1)|^\theta}\right) \lesssim \exp(-\tau\sigma^{-j\theta}).$$

Now, use Lemma 5.5 and (5.2) to get:

$$\begin{aligned} |R_j| &\leq \sum_{k>m_j} \frac{2^{k+1}}{(1 - |\chi(z_1)|)^{k+1}} |\Delta|^k \lesssim \sum_{k>m_j} 2^k \sigma^{-jk} \exp(-\tau k \sigma^{-j\theta}) \\ &\lesssim 2^{m_j} \sigma^{-jm_j} \exp(-\tau m_j \sigma^{-j\theta}) \lesssim \exp(-\tau m_j \sigma^{-j\theta}) \end{aligned}$$

for some absolute constant $\tau > 0$, that is:

$$(6.7) \quad |R_j| \lesssim \exp(-\tau n)$$

if we take:

$$(6.8) \quad m_j = [n\sigma^{j\theta}] + 1.$$

b) *Estimation of S_j .*

We saw in the estimation of R_j that $1 - |\chi(z_1)| \gtrsim \sigma^j$. Now, remember that $h_k^{(l)}(a_j) = 0$ for $l < n$, since $f \in E$, we then use Lemma 5.6 to get, when we take the values:

$$a = a_j, \quad 1 - a_j = \sigma^j, \quad b = \chi(z_1), \quad \rho = \frac{1}{2},$$

a good upper bound for $\frac{h_k[\chi(z_1)]}{k!}$ when $k \leq m_j$, namely:

$$\left| \frac{h_k[\chi(z_1)]}{k!} \right| \lesssim \frac{4^{k+1}}{\sigma^{j(k+1)}} \rho^n.$$

We then obtain an estimate of the form:

$$\begin{aligned} |S_j| &\lesssim \sum_{k=0}^{m_j} \rho^n \frac{4^{k+1}}{\sigma^{j(k+1)}} \lesssim \rho^n \frac{4^{m_j}}{\sigma^{jm_j}} = \exp\left(-n \log 2 + m_j \log 4 - jm_j \log \frac{7}{8}\right) \\ &\lesssim \exp(-4\tau n + Bj m_j) \end{aligned}$$

with $\tau = \frac{1}{4} \log 2$ and $B \leq \log 4 + \log(8/7) \leq 2$; or else, using (6.8):

$$|S_j| \lesssim \exp(-4\tau n + Bjn \sigma^{j\theta} + Bj).$$

But since $\sigma = 7/8 < 1$, the implied exponent, for $j_0 \leq j \leq N_n$:

$$-4\tau n + Bjn \sigma^{j\theta} + Bj = n(-4\tau + Bj \sigma^{j\theta}) + Bj,$$

is $\leq -2\tau n + B' \log n$, provided that we choose j_0 large enough, namely such that $j_0 (\frac{7}{8})^{j_0 \theta} \leq 1/4$. This implies an inequality of the form:

$$(6.9) \quad |S_j| \lesssim e^{-2\tau n} n^{B'} \lesssim e^{-\tau n}.$$

Putting the estimates (6.7) and (6.9) on R_j and S_j together, we obtain, for every $f \in E$ with $\|f\|_2 \leq 1$:

$$(6.10) \quad \|T_2 f\| \lesssim e^{-\tau n}.$$

It remains to bound from above the codimension of E . Since $N_n = \lceil \frac{\log 2n}{\theta \log 1/\sigma} \rceil + 1$ with $\sigma = 7/8$ and $m_j = \lceil n \sigma^{j\theta} \rceil + 1$, we see that:

$$\text{codim } E \leq \sum_{l=0}^{n-1} \sum_{j=1}^{N_n} m_j \leq \sum_{l=0}^{n-1} \sum_{j=1}^{N_n} (n \sigma^{j\theta} + 1) \lesssim n^2 \sum_{j=1}^{\infty} \sigma^{j\theta} + n \log n < q n^2.$$

Therefore (6.10) can be read as well, remembering the equality of approximation numbers and Gelfand numbers:

$$(6.11) \quad a_{qn^2}(T_2) = c_{qn^2}(T_2) \lesssim e^{-\tau n}.$$

Putting the estimates (6.2), (6.5), and (6.11) together ends the proof of Lemma 6.3. \square

Finally, Lemma 6.2 and (2.5) give:

$$a_{3n^2}(C_\Phi) = a_{3n^2}(T_1 \oplus T_2 \oplus T_3) \leq a_{n^2}(T_1) + a_{n^2}(T_2) + a_{n^2}(T_3) \lesssim e^{-\tau n},$$

thereby finishing the proof of Theorem 6.1. \square

Acknowledgments. This work was initiated during a stay of the first and second-named authors in the University of Sevilla in February 2018. They wish to thank their Spanish colleagues for their kindness, and for the excellent working conditions which they provided.

L. Rodríguez-Piazza is partially supported by the project MTM2015-63699-P (Spanish MINECO and FEDER funds).

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