An extremal composition operator on the Hardy space of the bidisk with small approximation numbers

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Abstract. We construct an analytic self-map $\Phi$ of the bidisk $\mathbb{D}^2$ whose image touches the distinguished boundary, but whose approximation numbers of the associated composition operator on $H^2(\mathbb{D}^2)$ are small in the sense that

$$\limsup_{n \to \infty} \left[ a_n(C_{\Phi}) \right]^{1/n} < 1.$$ 

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1 Introduction

For composition operators $C_{\Phi} : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ on the Hardy space of the unit disk, the decay of their approximation numbers $a_n(C_{\Phi})$ cannot be arbitrarily fast, and actually cannot supersede a geometric speed ([16]; see also [10, Theorem 3.1]): there exists a positive constant $c$ such that:

$$a_n(C_{\Phi}) \gtrsim e^{-cn}, \quad n = 1, 2, \ldots$$

It is easy to see that this speed occurs when $\|\Phi\|_\infty < 1$, and we proved in [10, Theorem 3.4] that a geometrical speed only takes place in this case; in other words:

$$\|\Phi\|_\infty = 1 \iff \lim_{n \to \infty} \left[ a_n(C_{\Phi}) \right]^{1/n} = 1.$$ 

This leads to the introduction, for an operator $T$ between Banach spaces, of the parameters:

$$\beta^{-}(T) = \liminf_{n \to \infty} \left[ a_n(T) \right]^{1/n} \quad \text{and} \quad \beta^{+}(T) = \limsup_{n \to \infty} \left[ a_n(T) \right]^{1/n},$$
where \(a_n(T)\) is the \(n\)-th approximation number of \(T\). When \([a_n(T)]^{1/n}\) actually has a limit, i.e. when \(\beta^-(T) = \beta^+(T)\), we write it \(\beta(T)\).

What is proved in [10, Theorem 3.4] is that \(\beta(C_\Phi) = 1\) if and only if \(\|\Phi\|_\infty = 1\). Later, in [12], we gave, when \(\|\Phi\|_\infty < 1\), a formula for this parameter in terms of the Green capacity of \(\Phi(D)\), which allowed us to recover (1.1).

More generally, for \(N \geq 1\), we introduce:

\[
\beta_N^-(T) = \liminf_{n \to \infty} [a_{nN}(T)]^{1/n} \quad \text{and} \quad \beta_N^+(T) = \limsup_{n \to \infty} [a_{nN}(T)]^{1/n},
\]

and:

\[
\beta_N(T) = \lim_{n \to \infty} [a_{nN}(T)]^{1/n}
\]

when the limit exists. It is clear that \(0 \leq \beta_N^\pm(T) \leq 1\), and it is interesting to know when the extreme cases \(\beta_N^\pm(T) = 0\) or \(\beta_N^\pm(T) = 1\) occur. For example:

\[
\beta_N^-(T) > 0 \iff a_{nN}(T) \gtrsim e^{-\tau n}, \quad \text{with } \tau > 0
\]
\[
\beta_N^+(T) = 1 \iff a_{nN}(T) \gtrsim e^{-n\varepsilon_n}, \quad \text{with } \varepsilon_n \to 0.
\]

It is coined in [1] (see also [13] and [14]) that \(\beta_N^\pm(C_\Phi)\) are the suitable parameters for the composition operators on \(H^2(D^N)\), and it is proved, for any \(N \geq 1\), that \(\beta_N^-(C_\Phi) > 0\), as soon as \(\Phi\) is non degenerate (i.e. the Jacobian \(J_\Phi\) is not identically 0) and the operator \(C_\Phi\) is bounded on \(H^2(D^N)\). As for an expression of \(\beta_N^\pm(C_\Phi)\) in terms of “capacity”, only partial results are known so far ([13] and [14]) and the application to a result like (1.1) fails in general. We gave an example of such a phenomenon in [13, Theorem 5.12]. In the present paper we give a shaper result.

## 2 Background and notation

Let \(\mathbb{D}\) be the open unit disk, \(H^2(\mathbb{D}^N)\) the Hardy space of the polydisk \(\mathbb{D}^N\), and \(\Phi: \mathbb{D}^N \to \mathbb{D}^N\) an analytic map. When \(N = 1\), it is well-known (see [4] or [17]) that \(\Phi\) induces a composition operator \(C_\Phi: H^2(\mathbb{D}) \to H^2(\mathbb{D})\) by the formula:

\[
C_\Phi(f) = f \circ \Phi,
\]

and the connection between the “symbol” \(\Phi\) and the properties of the operator \(C_\Phi\), in particular its compactness, can be further studied (see [4] or [17]). When \(N > 1\), \(C_\Phi\) is not bounded in general (see [4]).
Let $\mathbb{T}$ be the unit circle, and $m$ the normalized Haar measure on $\mathbb{T}^N$. A positive Borel measure $\mu$ on $DN$ is called a Carleson measure (for the space $H^2(DN)$) if the canonical injection $J: H^2(DN) \to L^2(\mu)$ is bounded. When $\Phi: DN \to DN$ is analytic and induces a bounded composition operator on $H^2(DN)$, the pullback measure $m_\Phi = \Phi^*(m)$, defined, for any test function $u$, by:

$$\int_{DN} u(w) dm_\Phi(w) = \int_{\mathbb{T}^N} u[\Phi^*(\xi)] dm(\xi),$$

is a Carleson measure. Here $\Phi^*$ is the radial limit function, defined for $m$-almost every $\xi \in \mathbb{T}^N$, by $\Phi^*(\xi) = \lim_{r \to 1^-} \Phi(r\xi)$.

For $\xi \in \mathbb{T} = \partial \mathbb{D}$ and $h > 0$, the Carleson window $S(\xi, h)$ is defined as:

$$S(\xi, h) = \{z \in \mathbb{D} : |z - \xi| \leq h\}.$$

If $f \in \text{Hol}(\mathbb{D})^2$, $D^k f$ denotes the $k$-th derivative of $f$ with respect to the $j$-th variable ($j = 1, 2$).

We denote by $A(\mathbb{D})$ the disk algebra, i.e. the space of functions holomorphic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. We similarly define the bidisk algebra $A(\mathbb{D}^2)$.

Let $H_1$ and $H_2$ be Hilbert spaces, and $T: H_1 \to H_2$ an operator. The $n$-th approximation number $a_n(T)$ of $T$, $n = 1, 2, \ldots$, is defined (see [2]) as the distance (for the operator-norm) of $T$ to operators of rank $< n$:

$$a_n(T) = \inf_{\text{rank } R < n} \|T - R\|.$$

The approximation numbers have the ideal property:

$$a_n(ATB) \leq \|A\| a_n(T) \|B\|.$$

The $n$-th Gelfand number $c_n(T)$ of $T$ is defined by:

$$c_n(T) = \inf_{\text{codim } E < n} \|T|_E\|.$$

As an easy consequence of the Schmidt decomposition, we have for any compact operator between Hilbert spaces:

$$c_n(T) = a_n(T).$$

If $T, T_1, T_2: H \to H'$ are operators between Hilbert spaces $H$ and $H'$, we write $T = T_1 \oplus T_2$ if $T = T_1 + T_2$ and:

$$\|Tx\|^2 = \|T_1x\|^2 + \|T_2x\|^2,$$

for all $x \in H$. 

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The subadditivity of approximation numbers is then expressed by:

$$a_{j+k}(T_1 \oplus T_2) \leq a_j(T_1) + a_k(T_2).$$

We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers, and by $[x]$ the integral part of the real number $x$.

We write $X \lesssim Y$ to indicate that $X \leq cY$ for some constant $c > 0$, and $X \approx Y$ to indicate that $X \lesssim Y$ and $Y \lesssim X$.

3 Purpose of the paper

Let us recall that the Hardy space of the polydisk is the space:

$$H^2(\mathbb{D}^N) = \left\{ f : \mathbb{D}^N \to \mathbb{C} ; f(z) = \sum_{\alpha \in \mathbb{N}^N} a_\alpha z^\alpha \text{ and } \|f\|_2^2 := \sum |a_\alpha|^2 < \infty \right\}.$$  

If $\Phi : \mathbb{D}^N \to \mathbb{D}^N$ is an analytic map, the associated composition operator $C_\Phi$ (which is not always bounded on $H^2(\mathbb{D}^N)$) is defined by:

$$C_\Phi(f) = f \circ \Phi.$$  

We will mainly here be interested in the case $N = 2$.

The reproducing kernel $K_a$ of $H^2(\mathbb{D}^2)$ is, with $a = (a_1, a_2)$ and $z = (z_1, z_2)$:

$$\begin{align*}
K_a(z) &= \frac{1}{(1 - \overline{a_1}z_1)(1 - \overline{a_2}z_2)}.
\end{align*}$$

As a consequence:

$$\begin{align*}
|f(a)| &= |\langle f, K_a \rangle| \leq \frac{\|f\|_2}{\sqrt{(1 - |a_1|^2)(1 - |a_2|^2)}}.
\end{align*}$$

In particular, the functions in the unit ball of $H^2(\mathbb{D}^2)$ are uniformly bounded on compact subsets of $\mathbb{D}^2$.

In [13, Theorem 5.12], we gave an example of a holomorphic self-map $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$, continuous on the closure $\overline{\mathbb{D}^2}$, such that $\|\Phi\|_\infty = 1$, that is:

$$\Phi(\mathbb{T}^2) \cap \partial \mathbb{D}^2 \neq \emptyset,$$

and yet:

$$\beta_2^+(C_\Phi) < 1,$$

in contrast with the one-dimensional case ([10, Theorem 3.4]).
Understanding where the difference really lies when we pass to the multidimensional case is a big challenge: it does not seem to be a matter of regularity of the boundary, and a similar example probably holds for the Hardy space of the ball. It might be a matter of boundary: the Shilov boundary of the ball is its usual boundary, but that of the polydisk is its distinguished boundary:

$$\partial_e \mathbb{D}^N = \{ z = (z_j); |z_j| = 1 \text{ for all } j = 1, \ldots, N \} = \mathbb{T}^N$$

(Indeed, the distinguished maximum principle tells that, for $f$ analytic in $\mathbb{D}^N$ and continuous on $\overline{\mathbb{D}^N}$, it holds $\max_{z \in \mathbb{D}^N} |f(z)| = \max_{z \in \partial_e \mathbb{D}^N} |f(z)|$).

The aim of this paper is to show that this is not the case and, improving on ([13, Theorem 5.12]) and (3.3), to build an analytic self-map $\Phi: \mathbb{D}^2 \to \mathbb{D}^2$, continuous on $\overline{\mathbb{D}^2}$, non-degenerate and such that:

$$\Phi(\mathbb{T}^2) \cap \partial_e \mathbb{D}^2 \neq \emptyset, \quad \text{but} \quad \beta^+_2(C_\Phi) < 1.$$ 

The paper is organized as follows. In Section 4, we recall with some detail the definition and main properties of a so-called cusp map $\chi \in A(\mathbb{D})$, to be of essential use in our counterexample. In Section 5, we prove several lemmas which constitute the core or the proof. In Section 6, we state and prove our main theorem.

4 The cusp map

The cusp map $\chi: \mathbb{D} \to \mathbb{D}$ is analytic in $\mathbb{D}$ and extends continuously on $\overline{\mathbb{D}}$. The boundary of its image is formed by three circular arcs of respective centers $\frac{1}{2}, 1 + \frac{i}{2}, 1 - \frac{i}{2}$, and of radius $\frac{1}{2}$ (see Figure 1). However, the parametrization $t \mapsto \chi(e^{it})$ involves logarithms.

It was often used by the authors ([11], [8]) as an extremal example.

We first recall the definition of $\chi$.

Let $\mathbb{D}^+ = \{ z \in \mathbb{D}; \Re z > 0 \}$ be the right half-disk. Let now $\mathbb{H}$ be the upper half-plane, and $T: \mathbb{D} \to \mathbb{H}$ defined by:

$$T(u) = i \frac{1 + u}{1 - u}, \quad \text{with} \quad T^{-1}(s) = \frac{s - i}{s + i}.$$ 

Taking the square root of $T$, we map $\mathbb{D}$ onto the first quadrant defined by $Q_1 = \{ z \in \mathbb{C}; \Re z > 0 \}$; we go back to the half-disk $\{ z \in \mathbb{D}; \Im z < 0 \}$ by $T^{-1}$. Finally, make a rotation by $i$ to go onto $\mathbb{D}^+$. We get:

$$\chi_0(z) = \frac{(z - i \sqrt{iz - 1} - 1)^{1/2} - i}{-i(z - i \sqrt{iz - 1} + 1)^{1/2} + 1}.$$
One has $\chi_0(1) = 0$, $\chi_0(-1) = 1$, $\chi_0(i) = -i$, and $\chi_0(-i) = i$. The half-circle $\{z \in \mathbb{T}; \Re z \geq 0\}$ is mapped by $\chi_0$ onto the segment $[-i, i]$ and the segment $[-1, 1]$ onto the segment $[0, 1]$.

Set now, successively:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{1}{\chi_2(z)}$$

and finally:

$$\chi(z) = 1 - \chi_3(z).$$

We now summarize the properties of the cusp map $\chi$ in the following proposition.

**Proposition 4.1.** The cusp map satisfies:

1) $1 - |\chi(z)| \lesssim \frac{1}{\log(2/|1-z|)}$;

2) $|1 - \chi(z)| \leq K(1 - |\chi(z)|)$ for all $z \in \mathbb{D}$, where $K$ is a positive constant;

3) $\chi(\mathbb{D})$ is the intersection of the open disk $D\left(\frac{1}{2}, \frac{1}{2}\right)$ with the exterior of the two open disks $D\left(1 + \frac{i}{2}, \frac{1}{2}\right)$ and $D\left(1 - \frac{i}{2}, \frac{1}{2}\right)$;

4) $\chi(1) = 1$, $\chi(\overline{z}) = \overline{\chi(z)}$ and $|\chi(z) - 1| \leq 1$ for all $z \in \mathbb{D}$;

5) for $0 < |t| \leq \pi/4$, we have $1 - \Re \chi(e^{it}) \approx 1/(\log 1/|t|)$;

6) $\chi(\overline{\mathbb{D}}) \subseteq \{z = x + iy; \ 0 \leq x \leq 1 \text{ and } |y| \leq 2(1 - x)^2\}$. 
Proof. Items 1) to 5) are proved in [11, Lemma 4.2]. To prove 6), write \(\chi(z) = (1 - h) + iy\). Since \(\chi(\overline{z}) = \overline{\chi(z)}\), we can assume \(y \geq 0\). Since \(\chi(D) \cap D(1 + \frac{i}{2}, \frac{1}{2}) = \emptyset\), we have \(\left|\chi(z) - (1 + \frac{i}{2})\right| \geq \frac{1}{4}\); hence:

\[
h^2 + \left(y - \frac{1}{2}\right)^2 = \left|\chi(z) - (1 + \frac{i}{2})\right|^2 \geq \frac{1}{4},
\]

so that \(y \leq y^2 + h^2\). But \(y \leq 1/2\), since \(\chi(z) \in D(\frac{1}{2}, \frac{1}{2})\); therefore \(y^2 \leq y/2\), so we get \(y \leq 2h^2\). \(\square\)

5 Preliminary lemmas

In this section, we collect some lemmas, which will reveal essential in the proof of our counterexample.

We consider the map \(\varphi = \varphi_\theta\), \(0 < \theta < 1\), defined, for \(z \in D(1) \setminus \{1\}\), by:

\[
\varphi(z) = \exp \left( - (1 - z)^{-\theta} \right).
\]

We observe, since \(\Re(1 - z) \geq 0\) for \(z \in D\), that:

\[
|\varphi(z)| \leq \exp \left( - \delta |1 - z|^{-\theta} \right),
\]

where \(\delta = \cos \pi \theta / 2 > 0\). Moreover, (5.2) shows that \(\varphi \in A(D)\), since:

\[
\lim_{z \to 1, z \in D^*} \varphi(z) = 0 =: \varphi(1).
\]

Our first lemma will allow us to define our symbol \(\Phi\).

**Lemma 5.1.** One can adjust \(0 < c < 1\) so as to get:

\[
|\chi(z)| + 2c|\varphi \circ \chi(z)| < 1 \quad \text{for all } z \in D.
\]

Hence, if we set, for any \(g \in A(D)\) with \(\|g\|_\infty \leq 1\):

\[
\Phi(z_1, z_2) = \left(\chi(z_1), \chi(z_1) + c(\varphi \circ \chi)(z_1)g(z_2)\right),
\]

we have \(\Phi(D^2) \subseteq D^2\).

**Remark.** The factor 2 in (5.3) is needed in order to get the following inequalities, to be used later, for \(z \in D\) and \(w = \chi(z) + c(\varphi \circ \chi)(z)u\), with \(|u| \leq 1\):

\[
|w| \leq \frac{1 + |\chi(z)|}{2},
\]
or, equivalently:

\begin{equation}
1 - |w| \geq \frac{1 - |\chi(z)|}{2}.
\end{equation}

Indeed:

\[ |w| \leq |\chi(z)| + c|\varphi \circ \chi(z)| \leq |\chi(z)| + \frac{1 - |\chi(z)|}{2} = \frac{1 + |\chi(z)|}{2}. \]

**Proof of Lemma 5.1.** Set \( X = |1 - \chi(z)| \), so that, with \( K \) the constant of Proposition 4.1, 2):

\begin{equation}
|\chi(z)| \leq 1 - \frac{|1 - \chi(z)|}{K} = 1 - \frac{X}{K}.
\end{equation}

For \( z \in \mathbb{D} \) and \( X \) close enough to zero, say \( X < \eta \), we have \( 2 \exp(-\delta X^{-\theta}) < \frac{\eta}{X} \). If we adjust \( 0 < c < 1 \) so as to have \( c < \frac{\eta}{2K} \), it follows from (5.2) and (5.7) that, for \( X < \eta \):

\[ |\chi(z)| + 2c|\varphi \circ \chi(z)| \leq 1 - \frac{X}{K} + 2 \exp(-\delta X^{-\theta}) < 1. \]

However, for \( X \geq \eta \), (5.7) says that \( |\chi(z)| \leq 1 - \frac{\eta}{K} \), so:

\[ |\chi(z)| + 2c|\varphi \circ \chi(z)| \leq 1 - \frac{\eta}{K} + 2c < 1, \]

as well and this ends the proof of Lemma 5.1. \( \square \)

Our second lemma estimates some integrals and ensures that \( \Phi \) induces a compact composition operator on \( H^2(\mathbb{D}^2) \).

**Lemma 5.2.** For \( 0 < h \leq 1 \), the following estimate holds:

\begin{equation}
I_0(h) := \int_{|\chi(e^{it}) - 1| \leq h} \frac{1}{(1 - |\chi(e^{it})|)^2} dt \lesssim e^{-\tau/h},
\end{equation}

**Proof.** By Proposition 4.1, 5), there exist two constants \( c_1, c_2 \) such that:

\[ \frac{c_1}{\log 1/|t|} \leq |\chi(e^{it}) - 1| \leq \frac{c_2}{\log 1/|t|}, \quad |t| \leq \pi; \]

hence:

\[ I_0(h) \lesssim \int_{|t| \leq e^{-c_1/h}} [\log(1/|t|)]^2 dt = 2 \int_{c_1/h}^{\infty} x^2 e^{-x} dx \lesssim h^{-2} e^{-c_1/h}. \] \( \square \)
Corollary 5.3. For $g \in A(\mathbb{D})$ with $0 < \|g\|_{\infty} \leq 1$, set:

$$I(h) := \int_{|\chi(e^{it_1}) - 1| \leq h} \frac{dt_1 dt_2}{(1 - |\chi(e^{it_1})|)(1 - |\chi(e^{it_1}) + c \varphi \chi(e^{it_1}) g(e^{it_2}))}.$$ 

Then:

$$I(h) \lesssim e^{-\tau/h}.$$ 

Consequently, the composition operator $C_{\Phi}$ defined in (5.4) is bounded from $H^2(\mathbb{D}^2)$ to $H^2(\mathbb{D}^2)$ and is compact.

Proof. Using (5.6), we have, thanks to (5.8):

$$I(h) \leq \int_{|\chi(e^{it_1}) - 1| \leq h} \frac{2}{(1 - |\chi(e^{it_1})|)^2} dt_1 dt_2 \lesssim e^{-\tau/h}.$$ 

In particular, $I(1) < \infty$, showing that $C_{\Phi}$ is Hilbert-Schmidt and hence bounded.

For the rest of the paper, we fix a number $\sigma$ in $(0, 1)$, that for convenience we take as:

$$\sigma = \frac{7}{8},$$

a positive integer $j_0$ such that:

$$2\sigma^{j_0} \leq 1/8$$

(i.e. $j_0 \geq 21$), and we set:

$$a_j = 1 - \sigma^j$$

and:

$$\rho_j = \frac{\sigma^j}{4} = \frac{1}{4}(1 - a_j).$$

We also define, for $n \geq 1$ and $\theta$ being the parameter used in (5.1):

$$N_n = \left\lfloor \frac{\log 2n}{\theta \log 1/\sigma} \right\rfloor + 1 > \frac{\log 2n}{\log 1/\sigma}.$$

The next lemma gives a cutting off for $\chi(\mathbb{D})$. 

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Lemma 5.4. For every $n \geq 1$, the image $\chi(\mathbb{D})$ of the cusp map, deprived of the closed Euclidean disk $\overline{D}(0, 1 - \sigma^{j_0}/K)$ and of $\chi(\mathbb{D}) \cap S(1,1/n)$, can be covered by the open Euclidean disks $D(a_j, \rho_j)$, with $j_0 \leq j \leq N_n$.

Proof. Let $z \in \mathbb{D}$ such that $|\chi(z)| > 1 - \sigma^{j_0}/K$ and $|\chi(z) - 1| > 1/n$. We write $\chi(z) = x + iy =: 1 - h + iy$.

Let $j$ with $a_j \leq x < a_{j+1}$, i.e. $\sigma^{j+1} < h \leq \sigma^j$. We have $j \geq j_0$, since $h < \sigma^{j_0}$.

Now, since $0 \leq x - a_j < a_{j+1} - a_j = \sigma^{j+1} - \sigma^j$, that $y^2 \leq 4h^4$ (Proposition 4.1, 6)), and $h \leq \sigma^j$, we have:

$$|\chi(z) - a_j|^2 < (\sigma^j - \sigma^{j+1})^2 + y^2 \leq (1 - \sigma)^2 \sigma^{2j} + 4 \sigma^{4j};$$

hence:

$$|\chi(z) - a_j| < \sigma^j (1 - \sigma) + 2 \sigma^{2j} = \sigma^j (1 - \sigma + 2 \sigma^j).$$

Subsequently, since $1 - \sigma = 1/8$, $j \geq j_0$, and $2 \sigma^{j_0} \leq 1/8$:

$$|\chi(z) - a_j| < \sigma^j (1 - \sigma + 2 \sigma^{j_0}) \leq \frac{\sigma^j}{4} = \rho_j,$$

showing that $\chi(z) \in D(a_j, \rho_j)$.

Moreover, we have $j \leq N_n$. Indeed, if $j > N_n$, we would have:

$$|\chi(z) - 1| \leq |\chi(z) - a_j| + (1 - a_j) \leq \rho_j + \sigma^j = \frac{5}{4} \sigma^j \leq \frac{5}{4} \sigma^{N_n+1} \leq 2 \sigma^{N_n} \leq 1/n,$$

contradicting the fact that $\chi(z) \notin S(1,1/n)$. 

Our next two lemmas give estimates on derivatives for the functions belonging to $H^2(\mathbb{D}^2)$.

Lemma 5.5. Let $f \in H^2(\mathbb{D}^2)$, $k$ a non-negative integer, $b \in \mathbb{D}$, and let $h_k(z) = (D_k f)(z, z)$. Then:

$$|h_k(b)| \leq \frac{k! 2^{k+1}}{(1 - |b|)^{k+1}} \|f\|_2.$$

Proof. The Cauchy inequalities give for $0 < s < 1 - |b|$ and $\alpha \in \mathbb{N}^2$:

$$|D^\alpha f(b, b)| \leq \sup_{s^{|\alpha|}} \frac{\alpha!}{s^{|\alpha|}} \sup_{|w_1 - b| = s, |w_2 - b| = s} |f(w_1, w_2)|.$$
The choice $s = \frac{1 - |b|}{2}$ gives $1 - |w_j| \geq \frac{1 - |b|}{2}$ for $|w_j - b| = s$, $j = 1, 2$; hence, thanks to the estimate (3.2):

$$|f(w_1, w_2)| \leq \frac{\|f\|_2}{\sqrt{(1 - |w_1|)(1 - |w_2|)}} \leq \frac{2}{1 - |b|} \|f\|_2.$$ 

Specializing to $\alpha = (0, k)$ now gives the result. □

**Lemma 5.6.** With the notations of Lemma 5.5, assume that $h_k^{(l)}(a) = 0$ for some $a \in \mathbb{D}$ and for $0 \leq l < n$. Then, for $0 < \rho < 1$ and $|b - a| \leq \frac{\rho}{2} (1 - |a|)$, it holds:

$$|h_k(b)| \leq \rho^n \frac{k! 4^{k+1}}{(1 - |a|)^{k+1}} \|f\|_2.$$

**Proof.** We may assume $\|f\|_2 \leq 1$. Consider the function defined, for $w \in \mathbb{D}$, by:

$$H_k(w) = h_k\left(a + w \frac{1 - |a|}{2}\right).$$

It is a bounded and holomorphic function in $\mathbb{D}$.

For $w \in \mathbb{D}$, let $\beta = a + w \frac{1 - |a|}{2}$, which satisfies $1 - |\beta| \geq \frac{1 - |a|}{2}$. Lemma 5.5 gives:

$$|H_k(w)| = |h_k(\beta)| \leq \frac{k! 4^{k+1}}{(1 - |a|)^{k+1}}.$$

Now, $H_k^{(l)}(0) = h_k^{(l)}(a) = 0$ for $0 \leq l < n$; hence the Schwarz lemma says that $H_k$ satisfies $|H_k(w)| \leq |w|^n \|H_k\|_{\infty}$ for all $w \in \mathbb{D}$. Take $w = \frac{2(b - a)}{1 - |a|}$, which satisfies $|w| \leq \rho$, to get:

$$|h_k(b)| = |H_k(w)| \leq |w|^n \|H_k\|_{\infty} \leq \rho^n \frac{k! 4^{k+1}}{(1 - |a|)^{k+1}}.$$ □

6 The main result

Recall that $\chi$ is the cusp map and that $\varphi$ is defined in (5.1). The map $g$ appearing in the formula below plays an inert role, and is just designed to ensure that $\Phi$ is non-degenerate; we can take, for example $g(z_2) = z_2$. This seems to mean that non-degeneracy is not the only issue in the question of estimating $\beta_\varphi^+(C_\varphi)$.

Our example appears as a perturbation of the diagonal map defined by $\Delta(z_1, z_2) = (\chi(z_1), \chi(z_1))$ for which we already know ([15, Theorem 2.4])
that $\Delta(1,1) = (1,1)$ and $\beta_2^+(C_\Delta) < 1$. This map is degenerate, but the perturbation clearly gives a non degenerate one since its Jacobian is $J_\Phi(z_1, z_2) = c(\varphi \circ \chi)(z_1) \chi'(z_1) g'(z_2)$.

**Theorem 6.1.** Let:

$$\Phi(z_1, z_2) = (\chi(z_1), \chi(z_1) + c(\varphi \circ \chi)(z_1) g(z_2))$$

be the function defined in (5.4).

Then:

1) $\Phi(\mathbb{D}^2) \subseteq \mathbb{D}^2$ and $C_\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is compact;

2) $\Phi$ is non degenerate, and its components belong to the bidisk algebra;

3) $\Phi(T^2) \cap T^2 = \Phi(T^2) \cap \partial_e \mathbb{D}^2 \neq \emptyset$;

4) $a_{n^2}(C_\Phi) \lesssim \exp(-\tau n)$, for some $\tau > 0$, implying $\beta_2^+(C_\Phi) < 1$.

**Proof.** That $\Phi$ maps $\mathbb{D}^2$ to itself is proved in Lemma 5.1 and that the composition operator $C_\Phi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is compact, in Corollary 5.3. Item 2) is due to the presence of $g$, as explained above. The fact that $\Phi(T^2) \cap T^2 \neq \emptyset$ is clear since $\Phi(1,1) = (1,1)$. It remains to prove 4).

Once more, the proof will be conveniently divided into several steps. We begin by a lemma which is in fact obvious, but explains well what is going on.

**Lemma 6.2.** Let $\lambda = 1 - \frac{\sigma j_0}{2K}$, where $\sigma$, $K$ and $j_0$ are as in (5.9), Proposition 4.1, 2), and (5.10). Let $r_n = 1 - \frac{1}{n}$, and let $\mu_1, \mu_2, \mu_3$ the respective restrictions of $m_\Phi$ to the disk $\lambda \mathbb{D}^2$, the annulus $r_n \mathbb{D}^2 \setminus \lambda \mathbb{D}^2$, and the annulus $\mathbb{D}^2 \setminus r_n \mathbb{D}^2$. We then have:

$$C_\Phi = T_1 \oplus T_2 \oplus T_3,$$

where $T_j$ is the canonical injection of $H^2(\mathbb{D}^2)$ into $L^2(\mu_j)$.

This is indeed obvious since:

$$\|C_\Phi f\|^2 = \int_{\mathbb{D}^2} |f|^2 dm_\Phi,$$

and by splitting the integral into three parts.

We now majorize separately the numbers $a_p(T_j)$, for $j = 1, 2, 3$. In the sequel, the positive constant $\tau$ may vary from one formula to another.

**Step 1.** It holds:

$$a_{n^2}(T_1) \lesssim e^{-\tau n}.$$
Proof. Let $V = z_1^n H^2(\mathbb{D}^2) + z_2^n H^2(\mathbb{D}^2)$; this is a subspace of $H^2(\mathbb{D}^2)$ of codimension $\leq n^2$, since:

$$V = \{ f \in H^2(\mathbb{D}^2) ; \ D_j D_k f(0,0) = 0 \text{ for } 0 \leq j, k < n \}.$$ 

If $f(z) = \sum_{\max(j,k) \geq n} a_{j,k} z_1^j z_2^k \in V$ and $\|f\|_2 = 1$, one can write:

$$f(z) = z_1^n q_1(z_1, z_2) + z_2^n q_2(z_1, z_2),$$

with:

$$q_1(z) = \sum_{j \geq n, k \geq 0} a_{j,k} z_1^{j-n} z_2^k \text{ and } q_2(z) = \sum_{j < n, k \geq n} a_{j,k} z_1^j z_2^{k-n},$$

which satisfy $\|q_j\|_2 \leq \|f\|_2 = 1$, $j = 1, 2$.

An easy estimate now gives (since $\max(|z_1|^n, |z_2|^n) \leq \lambda^n$ on $\lambda \mathbb{D}^2$):

$$\|T_1 f\|^2 \leq 2 \left( \int_{\mathbb{D}^2} (|z_1|^2 |q_1(z_1, z_2)|^2 + |z_2|^2 |q_2(z_1, z_2)|^2) \, dm_\Phi \right) \lesssim \lambda^{2n} \int_{\mathbb{D}^2} (||q_1||_2^2 + ||q_2||_2^2) \, dm_\Phi \lesssim \lambda^{2n} \lesssim \lambda^{2n},$$

since we know by Corollary 5.3 that $C_\Phi$ is bounded on $H^2(\mathbb{D}^2)$ and hence that $m_\Phi$ is a Carleson measure for $H^2(\mathbb{D}^2)$. Alternatively, we could majorize $|q_j(z_1, z_2)|$ uniformly on the support of $\mu_1$. We hence obtain:

\begin{equation}
(6.2) \quad a_{n^2+1}(T_1) = c_{n^2+1}(T_1) \lesssim e^{-\tau n}.
\end{equation}

Step 2. It holds:

\begin{equation}
(6.3) \quad a_{n^2}(T_3) \lesssim e^{-\tau n}.
\end{equation}

Proof. In one variable, we could use the Carleson embedding theorem; but this theorem for the bidisk and the Hardy space $H^2(\mathbb{D}^2)$ notably has a more complicated statement ([3]; see also [5]), and cannot be used efficiently here. Our strategy will be to replace it by a sharp estimation of a Hilbert-Schmidt norm.

We set $h_n = 1 - r_n = 1/n$.

Clearly, denoting by $S_2$ the Hilbert-Schmidt class:

$$\|T_3\|^2 \leq \|T_3\|_{S_2}^2 = \int \frac{d\mu_3(w)}{(1 - |w_1|^2)(1 - |w_2|^2)} \leq \int \frac{d\mu_3(w)}{(1 - |w_1|)(1 - |w_2|)}.$$
Now, if \( w = (w_1, w_2) = (\chi(z_1), \chi(z_1) + c(\varphi \circ \chi)(z_1) g(z_2)) \) belongs to the support of \( \mu_3 \), we have \( \max(|w_1|, |w_2|) \geq r_n = 1 - h_n \), and, recalling (5.5):

\[
|w_1| \geq 2|w_2| - 1,
\]
we have in either case \( |w_1| \geq 1 - 2h_n \). By Proposition 4.1, 2), this implies that:

\[
|1 - w_1| \leq 2Kh_n.
\]

Corollary 5.3 gives:

\[
\|T_3\|^2 \lesssim \int_{|\chi(e^{it_1}) - 1| \leq 2Kh_n} \frac{dt_1 dt_2}{(1 - |\chi(e^{it_1})|)(1 - |\chi(e^{it_1}) + c(\varphi \circ \chi)(e^{it_1}) g(e^{it_2})|)} = I(2Kh_n) \lesssim e^{-\tau/h_n}.
\]

But \( h_n = 1/n \), so that:

\[
(6.5) \quad a_n^2(T_3) \leq \|T_3\| \lesssim e^{-\tau n}.
\]

Step 3. It holds:

\[
(6.6) \quad a_n^2(T_2) \lesssim e^{-\tau n}.
\]

This estimate follows from the following key auxiliary lemma. In fact, this lemma will give, for the Gelfand numbers, \( c_n^2(T_2) \lesssim e^{-\tau n} \), and we know that they are equal to the approximation numbers.

Let \( M : H^2(\mathbb{D}^2) \to \text{Hol}(\mathbb{D}) \) be the linear map defined by:

\[
Mf(z) = f(z, z),
\]

Recall that \( a_j = 1 - \sigma^j \) and \( N_n = \left\lfloor \frac{\log 2n}{\log 1/\sigma} \right\rfloor + 1 \).

**Lemma 6.3.** Let \( E \) be the closed subspace of \( H^2(\mathbb{D}^2) \) defined by:

\[
E = \left\{ f \in H^2(\mathbb{D}^2) : \left[ M(D^k_z f) \right]^{(l)}(a_j) = 0 \quad \text{for} \quad 0 \leq l < n, \ 0 \leq k \leq m_j, \ 1 \leq j \leq N_n \right\}.
\]

Then, we can adjust the numbers \( m_j \) so as to guarantee that, for some positive constant \( \tau \):

\[
\text{codim } E \lesssim n^2
\]
and, for all \( f \in E \) with \( \|f\|_2 \leq 1 \):

\[
\|T_2(f)\|_2 \lesssim \exp(-\tau n).
\]
Proof. This is the most delicate part.

Recall that:

\[ h_n = 1/n, \quad r_n = 1 - h_n, \quad \lambda = 1 - \frac{\sigma h}{2K}. \]

We need a uniform estimate of \(|f(w)|\) for \(f \in E\) with \(\|f\|_2 \leq 1\) and for:

\[ w = (w_1, w_2) \in \text{supp} m_\Phi \cap (r_n \mathbb{D}^2 \setminus \lambda \mathbb{D}^2). \]

This estimate will be given by Lemma 5.4, Lemma 5.5 and Lemma 5.6.

Note that we have:

\[ \chi(z_1) \in \mathbb{D} \setminus [S(1, 1/n) \cup \overline{D}(0, 2\lambda - 1)]. \]

Indeed, if \((w_1, w_2) = \Phi(z_1, z_2) \notin \lambda \mathbb{D}^2\), we have \(\max(|w_1|, |w_2|) > \lambda\); so either \(|w_1| > \lambda \geq 2\lambda - 1\), or \(|w_2| > \lambda\) and again \(|w_1| > 2\lambda - 1\) since \(|w_1| \geq 2|w_2| - 1\), by (5.5). Hence \(w_1 \notin \overline{D}(0, 2\lambda - 1)\). Moreover, we have \(|1 - w_1| > 1 - |w_1| > 1/n\), so \(w_1 \notin S(1, 1/n)\).

Using Lemma 5.4, select \(j_0 \leq j \leq N_n\) such that \(|\chi(z_1) - a_j| \leq \frac{1}{4}(1 - a_j)\).

Now set:

\[ A = (\chi(z_1), \chi(z_1)) \quad \text{and} \quad \Delta = (0, (\varphi \circ \chi)(z_1) g(z_2)). \]

Our strategy will be the following. We write:

\[
f[\Phi(z_1, z_2)] = f(A + \Delta) = \sum_{k=0}^{\infty} \frac{D_k f(A)}{k!} \Delta^k = \sum_{k=0}^{\infty} \frac{M(D_k^2 f)[\chi(z_1)]}{k!} \Delta^k = \sum_{k=0}^{\infty} h_k[\chi(z_1)] \Delta^k,
\]

with \(h_k = M(D_k^2 f)\), and we put:

\[
S_j = \sum_{k=0}^{m_j} \frac{h_k[\chi(z_1)]}{k!} \Delta^k
\]

and

\[
R_j = \sum_{k=m_j}^{\infty} \frac{h_k[\chi(z_1)]}{k!} \Delta^k.
\]

We will estimate separately \(S_j\) and \(R_j\).
a) Estimation of $R_j$.

Recall that $j$ is such that $0 \leq j \leq N_n$ and $|\chi(z_1) - a_j| \leq \frac{1}{4} (1 - a_j)$. We saw in the proof of this Lemma 5.4 that $1 - |\chi(z_1)| \leq |1 - \chi(z_1)| \leq \frac{5}{4} \sigma^j$. Hence:

$$|\Delta| \leq |(\varphi \circ \chi)(z_1)| \leq \exp \left( - \frac{\delta}{|1 - \chi(z_1)|^\theta} \right) \lesssim \exp(-\tau \sigma^{-j\theta}).$$

Now, use Lemma 5.5 and (5.2) to get:

$$|R_j| \leq \sum_{k\geq m_j} \frac{2^{k+1}}{(1 - |\chi(z_1)|)^{k+1}} |\Delta|^k \lesssim \sum_{k\geq m_j} 2^k \sigma^{-j^k} \exp(-\tau k \sigma^{-j^k})$$

$$\lesssim 2^{m_j} \sigma^{-m_j} \exp(-\tau m_j \sigma^{-j^k}) \lesssim \exp(-\tau m_j \sigma^{-j^k})$$

for some absolute constant $\tau > 0$, that is:

(6.7) $|R_j| \lesssim \exp(-\tau n)$

if we take:

(6.8) $m_j = \lceil n \sigma^{j\theta} \rceil + 1$.

b) Estimation of $S_j$.

We saw in the estimation of $R_j$ that $1 - |\chi(z_1)| \gtrsim \sigma^j$. Now, remember that $h_l^{(l)}(a_j) = 0$ for $l < n$, since $f \in E$, we then use Lemma 5.6 to get, when we take the values:

$$a = a_j, \; 1 - a_j = \sigma^j, \; b = \chi(z_1), \; \rho = \frac{1}{2},$$

a good upper bound for $\frac{h_k [\chi(z_1)]}{k!}$ when $k \leq m_j$, namely:

$$\left| \frac{h_k [\chi(z_1)]}{k!} \right| \lesssim \frac{4^{k+1}}{\sigma^{j(k+1)}} \rho^n.$$

We then obtain an estimate of the form:

$$|S_j| \lesssim \sum_{k=0}^{m_j} \rho^n \frac{4^{k+1}}{\sigma^{j(k+1)}} \lesssim \rho^n \frac{4^{m_j}}{\sigma^{jm_j}} = \exp \left( - n \log 2 + m_j \log 4 - jm_j \log \frac{7}{8} \right)$$

$$\lesssim \exp(-4\tau n + B jm_j)$$
with \( \tau = \frac{1}{4} \log 2 \) and \( B \leq \log 4 + \log(8/7) \leq 2 \); or else, using (6.8):

\[
|S_j| \lesssim \exp(-4\tau n + B_j n \sigma^{j\theta} + B_j).
\]

But since \( \sigma = 7/8 < 1 \), the implied exponent, for \( j_0 \leq j \leq N_n \):

\[
-4\tau n + B_j n \sigma^{j\theta} + B_j = n(-4\tau + B_j \sigma^{j\theta}) + B_j,
\]

is \( \leq -2\tau n + B' \log n \), provided that we choose \( j_0 \) large enough, namely such that \( j_0 \left(\frac{7}{8}\right)^{j_0} \theta \leq 1/4 \). This implies an inequality of the form:

\[
(6.9) \quad |S_j| \lesssim e^{-2\tau n} n^{B'} \lesssim e^{-\tau n}.
\]

Putting the estimates (6.7) and (6.9) on \( R_j \) and \( S_j \) together, we obtain, for every \( f \in E \) with \( \|f\|_2 \leq 1 \):

\[
\|T_2 f\| \lesssim e^{-\tau n}.
\]

It remains to bound from above the codimension of \( E \). Since \( N_n = \left\lfloor \frac{\log 2n}{\theta \log 1/\sigma} \right\rfloor + 1 \) with \( \sigma = 7/8 \) and \( m_j = \left\lfloor n \sigma^{j\theta} \right\rfloor + 1 \), we see that:

\[
\text{codim } E \leq \sum_{i=0}^{n-1} \sum_{j=1}^{N_n} m_j \leq \sum_{i=0}^{n-1} \sum_{j=1}^{N_n} (n \sigma^{j\theta} + 1) \lesssim n^2 \sum_{j=1}^{\infty} \sigma^{j\theta} + n \log n < q n^2.
\]

Therefore (6.10) can be read as well, remembering the equality of approximation numbers and Gelfand numbers:

\[
(6.11) \quad a_{q n^2}(T_2) = c_{q n^2}(T_2) \lesssim e^{-\tau n}.
\]

Putting the estimates (6.2), (6.5), and (6.11) together ends the proof of Lemma 6.3.

Finally, Lemma 6.2 and (2.5) give:

\[
a_{3n^2}(C_\Phi) = a_{3n^2}(T_1 \oplus T_2 \oplus T_3) \leq a_{n^2}(T_1) + a_{n^2}(T_2) + a_{n^2}(T_3) \lesssim e^{-\tau n},
\]

thereby finishing the proof of Theorem 6.1.

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References


[15] D. Li, H. Queffélec, L. Rodríguez-Piazza, Composition operators with surjective symbol and small approximation numbers, preprint


