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# Entropy numbers and spectral radius type formula for composition operators on the polydisk

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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**Abstract.** We give estimates of the entropy numbers of composition operators on the Hardy space of the disk and of the polydisk.

MSC 2010 Primary 47B33 Secondary 47B06

**Key-words** composition operator; entropy numbers; polydisk; plurica-pacity

## 1 Introduction

This short paper was motivated by a question of J. Wengenroth ([19]) about entropy numbers of composition operators on Hardy spaces  $H^2$ , which stand a little apart in the jungle of "s-numbers", even though they seem the most natural for the study of compactness, since their membership in  $c_0$  characterizes compactness, even in the general framework of arbitrary Banach spaces. Indeed, in various papers (see [1, 10, 11, 12, 13]), we studied in detail the approximation numbers of composition operators, and here we will essentially transfer those results to entropy numbers thanks to the polar (Schmidt) decomposition and a general result on entropy numbers of diagonal operators on  $\ell^2$ .

So, the proofs are easy, but the statements feature a very different behavior of those entropy numbers. In particular, we will investigate a few properties related with a so-called "spectral radius type formula" which we obtained, in dimension one through a result of Widom ([12]), and, partially in dimension N ([13, 14]), through a result of Nivoche ([16]) and Zakharyuta ([22]).

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# 2 Entropy numbers

We begin by recalling some facts on s-numbers.

Given an operator  $T: X \to Y$  between Banach spaces, recall ([4]) that we can attach to this operator five non-increasing sequences  $(a_n)$ ,  $(b_n)$ ,  $(c_n)$ ,  $(d_n)$ ,  $(e_n)$  of non-negative numbers (depending on T), respectively the sequences of approximation, Bernstein, Gelfand, Kolmogorov, and entropy numbers of T. The latter are defined ([4, Chapter 1], or [17, Chapter 5]), for  $n \ge 1$  by:

(2.1) 
$$e_n(T) = \inf\{\varepsilon > 0; \ N(T(B_X), \varepsilon B_Y) \le 2^{n-1}\},$$

where  $B_X$  and  $B_Y$  are the respective closed unit balls of X and Y, and where, for  $A, B \subseteq Y$ , N(A, B) denotes the smallest number of translates of B needed to cover A.

All those sequences  $(a_n)$ ,  $(b_n)$ ,  $(c_n)$ ,  $(d_n)$ ,  $(e_n)$ , say  $(u_n)$ , share the ideal property:

$$u_n(ATB) \le ||A|| u_n(T) ||B||.$$

For Hilbert spaces, it turns out that  $a_n = b_n = c_n = d_n = s_n$ , where  $(s_n)$  designates the sequence of singular numbers, but entropy numbers stay a little apart.

For general Banach spaces X and Y and  $T: X \to Y$ , we have, in general ([3, Theorem 1], see also [17, Theorem 5.2]), for  $\alpha > 0$ :

$$\sup_{1 \le k \le n} k^{\alpha} e_k(T) \le C_{\alpha} \sup_{1 \le k \le n} k^{\alpha} a_k(T),$$

and, if X and  $Y^*$  are of type 2:

$$a_n(T) \le K e_n(T)$$
, for all  $n \ge 1$ 

([7, Corollary 1.6]), where  $K = \kappa [T_2(X)T_2(Y^*)]^2$ ; in particular, if T acts between Hilbert spaces (see [17, Theorem 5.3]):

$$a_n(T) \le 4 e_n(T)$$
, for all  $n \ge 1$ .

Those inequalities indicate that entropy numbers are always bigger than singular numbers, up to a constant, and that, as far as the scale of powers  $n^{\alpha}$ 

is implied, they are dominated by approximation numbers in a weak sense. But it turns out that, individually, they can be much bigger than the latter for composition operators, as we shall see.

We will rely on the following estimate ([4, p. 17]), in which  $\ell^2$  denotes the space of square-summable sequences  $x = (x_k)_{k \ge 1}$  of complex numbers. This estimate is given for the sequence  $(\varepsilon_n)$  of covering numbers and with the scale of powers of 2, but  $e_n = \varepsilon_{2^{n-1}}$ , by definition, and the change of 2 to e only affects constants.

**Theorem 2.1.** (see [4, p. 17]) There exist absolute constants 0 < a < b such that, for any diagonal compact operator  $\Delta \colon \ell^2 \to \ell^2$  with positive and non-increasing eigenvalues  $(\sigma_k)_{k\geq 1}$ , namely  $\Delta((x_k)_k) = (\sigma_k x_k)_k$ , we have, for all  $n \geq 1$ :

$$(2.2) \quad a \sup_{k \ge 1} \left[ e^{-n/k} \left( \prod_{j=1}^k \sigma_j \right)^{1/k} \right] \le e_n(\Delta) \le b \sup_{k \ge 1} \left[ e^{-n/k} \left( \prod_{j=1}^k \sigma_j \right)^{1/k} \right].$$

A useful corollary of Theorem 2.1 is the following.

**Theorem 2.2.** Let  $T: H_1 \to H_2$  be a compact operator between the Hilbert spaces  $H_1$  and  $H_2$ , and let  $(a_n)_{n\geq 1}$  be its sequence of approximation numbers. Then, for all  $n\geq 1$ :

(2.3) 
$$\alpha \sup_{k \ge 1} \left[ e^{-n/k} \left( \prod_{j=1}^k a_j \right)^{1/k} \right] \le e_n(T) \le \beta \sup_{k \ge 1} \left[ e^{-n/k} \left( \prod_{j=1}^k a_j \right)^{1/k} \right],$$

where  $\alpha$  and  $\beta$  are positive numerical constants.

Proof. Let  $Tx = \sum_{n=1}^{\infty} s_n(x \mid u_n)v_n$  the Schmidt decomposition of T, where  $(u_n)_n$  and  $(v_n)_n$  are orthonormal sequences of  $H_1$  and  $H_2$ , respectively, and  $(s_n)_n$  is the sequence of singular numbers of T. Let  $\Delta \colon \ell_2 \to \ell_2$  the diagonal operator with diagonal values  $s_n$ ,  $n \ge 1$ . Then  $T = V_1 \Delta U_1$  and  $\Delta = V_2 T U_2$ , with  $U_1 x = \left( (x \mid u_n) \right)_n$ ,  $V_1 \left( (t_n)_n \right) = \sum_n t_n v_n$ ,  $U_2 \left( (t_n)_n \right) = \sum_n t_n u_n$  and  $V_2 x = \left( (x \mid v_n) \right)_n$ . We have  $\|U_1\|, \|V_1\|, \|U_2\|, \|V_2\| \le 1$ ; hence the result follows from Theorem 2.1 and the ideal property.

This theorem might be thought useless, because we don't know better the  $a_n$ 's than the  $e_n$ 's! In our situation, this is not the case, since we made a more or less systematic study of the  $a_n$ 's for composition operators in [1, 11, 10, 12] for example.

We now pass to applications to composition operators  $C_{\varphi}$ , defined as  $C_{\varphi}(f) = f \circ \varphi$  when they act on the Hardy space  $H^2(\mathbb{D}^N)$  (which is always

the case if N=1). Here,  $\varphi$  denotes an analytic and non-degenerate self-map of  $\mathbb{D}^N$ . For clarity, we separate the cases of dimension N=1 and of dimension  $N\geq 2$ .

# 3 Applications in dimension 1

# 3.1 General results

In [12], we had coined the parameter:

(3.1) 
$$\beta_1(T) = \lim_{n \to \infty} \left[ a_n(T) \right]^{1/n}$$

and its versions  $\beta_1^+(T)$ ,  $\beta_1^-(T)$  with a upper limit and a lower limit respectively. The following result ([12]) shows in particular that no lower or upper limit is needed for  $\beta = \beta_1$ , and provides a simpler proof of the second item in Theorem 3.1 than in our initial proof of [10].

For the definition of the Green capacity  $\operatorname{Cap}(A)$  of a Borel subset A of  $\mathbb{D}$ ,  $0 \leq \operatorname{Cap}(A) \leq \infty$ , we refer to [12].

**Theorem 3.1.** Let  $\Omega = \varphi(\mathbb{D})$ , with  $\varphi \colon \mathbb{D} \to \mathbb{D}$  a non-constant analytic map. Then:

1) One always has  $\beta_1^-(C_\varphi) = \beta_1^+(C_\varphi) =: \beta_1(C_\varphi)$  and:

(3.2) 
$$\beta_1(C_{\varphi}) = \exp[-1/\operatorname{Cap}(\Omega)] > 0.$$

2) In particular, one has the equivalence:

(3.3) 
$$\beta_1(C_{\varphi}) = 1 \quad \Longleftrightarrow \quad \|\varphi\|_{\infty} = 1.$$

Here, another parameter emerges.

(3.4) 
$$\gamma_1(T) = \lim_{n \to \infty} \left[ e_n(T) \right]^{1/\sqrt{n}}$$

and its  $\gamma_1^+(T), \gamma_1^-(T)$  versions.

**Theorem 3.2.** Let  $\varphi \colon \mathbb{D} \to \mathbb{D}$  be a symbol and  $\Omega = \varphi(\mathbb{D})$ . Then:

1) 
$$\gamma_1^-(C_{\varphi}) = \gamma_1^+(C_{\varphi}) =: \gamma_1(C_{\varphi}) \text{ and:}$$

(3.5) 
$$\gamma_1(C_{\varphi}) = \exp\left[-\sqrt{2/\operatorname{Cap}(\Omega)}\right] > 0.$$

2) In particular, one has the equivalence:

(3.6) 
$$\gamma_1(C_{\varphi}) = 1 \iff \|\varphi\|_{\infty} = 1.$$

*Proof.* Set  $\rho = 1/\text{Cap}(\Omega)$  for simplicity of notations. Let  $\varepsilon > 0$ , and  $C_{\varepsilon}$  a positive constant which depends only on  $\varepsilon$  and can vary from a formula to another. Theorem 3.1 implies  $a_k \leq C_{\varepsilon} e^{\varepsilon k} e^{-k\rho}$ , whence:

$$(a_1 \cdots a_k)^{1/k} \le C_{\varepsilon} e^{\varepsilon k/2} e^{-\rho k/2}$$

Theorem 2.2 now gives:

$$e_n(C_{\varphi}) \le C_{\varepsilon} \sup_{k > 1} \left[ e^{\varepsilon k/2} e^{-(n/k + \rho k/2)} \right].$$

This supremum is essentially attained for  $k = \left[\sqrt{2n/\rho}\right]$  where [.] stands for the integer part, and gives:

$$e_n(C_{\varphi}) \le C_{\varepsilon} e^{\varepsilon \sqrt{n/(2\rho)}} e^{-\sqrt{2n\rho}}.$$

This implies  $\gamma_1^+(C_\varphi) \leq e^{\varepsilon \sqrt{1/(2\rho)}} e^{-\sqrt{2\rho}}$ , and finally:

$$\gamma_1^+(C_\varphi) \le e^{-\sqrt{2\rho}}$$
.

The lower bound  $\gamma_1^-(C_\varphi) \ge e^{-\sqrt{2\rho}}$  is proved similarly.

This clearly ends the proof, since we know from [12] that  $\operatorname{Cap}(\Omega) = \infty$  if ond only if  $\|\varphi\|_{\infty} = 1$ .

#### 3.2 Specific results

For  $0 < \theta < 1$ , the lens map  $\lambda_{\theta}$  of parameter  $\theta$  is defined by:

(3.7) 
$$\lambda_{\theta}(z) = \frac{(1+z)^{\theta} - (1-z)^{\theta}}{(1+z)^{\theta} + (1-z)^{\theta}}$$

(see [18] or [10]).

**Theorem 3.3.** Let  $\lambda_{\theta}$  be the lens map with parameter  $\theta$ . Then, with positive constants a, b, a', b' depending only on  $\theta$ :

(3.8) 
$$a' e^{-b'n^{1/3}} \le e_n(C_{\lambda_\theta}) \le a e^{-bn^{1/3}}.$$

*Proof.* We proved in [8, Theorem 2.1] (see also [10, Proposition 6.3] that  $a_k = a_k(C_{\lambda_\theta}) \le a e^{-b\sqrt{k}}$ . It follows, using Theorem 2.2, that  $(a_1 \cdots a_k)^{1/k} \le a e^{-b\sqrt{k}}$  and that, for some positive constant C:

$$e_n(C_{\lambda_\theta}) \le C \exp\left[-\left((n/k) + bk^{1/2}\right)\right].$$

Taking  $k = [n^{2/3}]$  gives the claimed upper bound. The lower bound is proved similarly, using the left inequality in Theorem 2.2, since we know ([12]) that  $a_k \ge a' e^{-b'\sqrt{k}}$ .

We refer to [11, Section 4.1] for the definition of the cusp map  $\chi$ . We have:

**Theorem 3.4.** Let  $\chi$  be the cusp map. Then, with positive constants a, b, a', b':

(3.9) 
$$a' e^{-b'\sqrt{n/\log n}} \le e_n(C_{\chi}) \le a e^{-b\sqrt{n/\log n}}.$$

*Proof.* We proved in [11] that:

(3.10) 
$$a' e^{-b'k/\log k} \le a_k(C_x) \le a e^{-bk/\log k}$$
.

The proof then follows the same lines as in Theorem 3.3, with the choice  $k = \lceil \sqrt{n \log n} \rceil$ .

# 4 The multidimensional case

#### 4.1 General results

Let  $\varphi \colon \mathbb{D}^N \to \mathbb{D}^N$  be an analytic map. We will say that  $\varphi$  is non-degenerate if  $\varphi(\mathbb{D}^N)$  has non-empty interior, equivalently if  $\det \varphi'(z) \neq 0$  for at least one point  $z \in \mathbb{D}^N$ .

Let now  $\varphi \colon \mathbb{D}^N \to \mathbb{D}^N$  be a non-degenerate analytic map inducing a bounded composition operator  $C_{\varphi} \colon H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$  (this is not always the case as soon as N > 1, even if  $\varphi$  is injective and hence non-degenerate, see for example [5, p. 246], when the polydisk is replaced by the ball; but similar examples exist for the polydisk). Assume moreover that  $C_{\varphi}$  is a compact operator.

**Theorem 4.1.** Let  $C_{\varphi} \colon H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$  be a compact composition operator, with  $\varphi$  non-degenerate. We have:

- 1)  $e_n(C_{\varphi}) \ge c \exp\left(-C n^{\frac{1}{N+1}}\right)$ , for some constants C > c > 0, depending on  $\varphi$ ;
- 2) if  $\|\varphi\|_{\infty} < 1$ , then  $e_n(C_{\varphi}) \leq C \exp\left(-c n^{\frac{1}{N+1}}\right)$ , with C > c > 0 depending on  $\varphi$ .

*Proof.* 1) It is proved in [1, Theorem 3.1] that, for a non-degenerate map  $\varphi$ , it holds:

$$a_k(C_{\varphi}) \ge a' e^{-b'k^{1/N}}$$
.

As in the previous section, it follows from Theorem 2.2, that  $(a_1 \cdots a_k)^{1/k} \ge e^{-b''k^{1/N}}$ , and then, taking  $k = [n^{N/(N+1)}]$ , that:

$$e_n(C_{\varphi}) \ge c e^{-Cn^{1/(N+1)}}.$$

2) Similarly, for  $\|\varphi\|_{\infty} < 1$ , it is proved in [1, Theorem 5.2] that:

$$a_k(C_{\varphi}) \le C e^{-ck^{1/N}};$$

and we get the result from Theorem 2.2.

Those estimates motivate the introduction of the parameter:

(4.1) 
$$\gamma_N(C_{\varphi}) = \lim_{n \to \infty} \left[ e_n(C_{\varphi}) \right]^{\frac{1}{n^{1/(N+1)}}}.$$

We define similarly  $\gamma_N^{\pm}(C_{\varphi})$ , and will say more on it in next section.

## 4.2 Specific results

# 4.2.1 Multi-lens maps

Let  $\lambda_{\theta}$  be lens maps with parameter  $\theta$ . We define the multi-lens map  $\Lambda_{\theta}$  of parameter  $\theta$  on the polydisk  $\mathbb{D}^{N}$  as:

(4.2) 
$$\Lambda_{\theta}(z_1, \dots, z_N) = (\lambda_{\theta}(z_1), \lambda_{\theta}(z_2), \dots, \lambda_{\theta}(z_N)),$$

for 
$$(z_1,\ldots,z_N)\in\mathbb{D}^N$$
.

The following result is proved in [1, Theorem 6.1].

**Theorem 4.2.** Let  $\Lambda_{\theta}$  be the multi-lens map with parameter  $\theta$ . Then, for positive constants a, b, a', b' depending only on  $\theta$  and N, one has:

(4.3) 
$$a' e^{-b'n^{1/(2N)}} \le a_n(C_{\Lambda_\theta}) \le a e^{-b n^{1/(2N)}}.$$

The version of Theorem 4.2 for entropy numbers, stated without proof, is:

**Theorem 4.3.** Let  $\Lambda_{\theta}$  be the multi-lens map with parameter  $\theta$ . Then:

$$(4.4) a' \exp\left(-b'n^{1/(2N+1)}\right) \le e_n(C_{\Lambda_\theta}) \le a \exp\left(-b n^{1/(2N+1)}\right).$$

### 4.2.2 Multi-cusp maps

Let  $\chi \colon \mathbb{D} \to \mathbb{D}$  be the cusp map and  $\varphi \colon \mathbb{D}^N \to \mathbb{D}^N$  be the multi-cusp map defined by:

$$(4.5) \Xi(z_1,\ldots,z_N) = (\chi(z_1),\chi(z_2),\ldots,\chi(z_N)).$$

It is proved in [1, Theorem 6.2]:

**Theorem 4.4.** Let  $\chi \colon \mathbb{D} \to \mathbb{D}$  be the cusp map and  $\Xi \colon \mathbb{D}^N \to \mathbb{D}^N$  be the multi-cusp map. Then:

(4.6) 
$$a' e^{-b' n^{1/N}/\log n} \le a_n(C_{\Xi}) \le a e^{-b n^{1/N}/\log n},$$

where a, b, a', b' are positive constants depending only on N.

The version of Theorem 4.4 for entropy numbers, stated without proof, is:

**Theorem 4.5.** let  $\chi \colon \mathbb{D} \to \mathbb{D}$  be the cusp map and  $\Xi \colon \mathbb{D}^N \to \mathbb{D}^N$  be the multi-cusp map. Then:

(4.7) 
$$a' \exp \left[ -b' n^{1/(N+1)} (\log n)^{-N/(N+1)} \right] \leq e_n(C_\Xi) \leq a \exp \left[ -b n^{1/(N+1)} (\log n)^{-N/(N+1)} \right].$$

# 5 Connections with pluricapacity and Zakharyuta's results

Here, in dimension  $N \geq 2$ , the situation is satisfactory for upper bounds (see [13]); for lower bounds, see [14]. The notion involved is now that of pluricapacity, or Monge-Ampère capacity, coined by Bedford and Taylor in [2]. More precisely, if A is a Borel subset of  $\mathbb{D}^N$ , we refer to [13] or [14] for the definition of its pluricapacity  $\operatorname{Cap}_N(A)$ , belonging to  $[0, +\infty]$ , and set:

(5.1) 
$$\tau_N(A) = \frac{1}{(2\pi)^N} \operatorname{Cap}_N(A)$$

(5.2) 
$$\Gamma_N(A) = \exp\left[-\left(\frac{N!}{\tau_N(A)}\right)^{1/N}\right]$$

(5.3) 
$$\beta_N^+(T) = \limsup_{n \to \infty} \left[ a_n(T) \right]^{1/n^{1/N}}.$$

We temporarily assume that  $\|\varphi\|_{\infty} < 1$  so that  $K = \overline{\varphi(\mathbb{D}^N)}$  is a compact subset of  $\mathbb{D}^N$ . We proved in [13, Theorem 6.4], relying on positive results of Nivoche ([16]) and Zaharyuta ([22, Proposition 6.1]) on the so-called Kolmogorov conjecture, that:

Theorem 5.1. It holds:

$$\beta_N^+(C_\varphi) \le \Gamma_N(K) \,.$$

We have the following result, which extends the previous result in dimension 1.

**Theorem 5.2.** The following upper bound holds:

(5.5) 
$$\gamma_N^+(C_\varphi) \le \exp\left(-\beta_N \rho^{N/(N+1)}\right),\,$$

where:

(5.6) 
$$\rho = \left(\frac{N!}{\tau_N(K)}\right)^{1/N} = 2\pi \left(\frac{N!}{\operatorname{Cap}_N(K)}\right)^{1/N},$$

and

(5.7) 
$$\beta_N = \left(\frac{N}{N+1}\right)^{N/(N+1)} \left(N^{-N/(N+1)} + N^{1/(N+1)}\right) > e^{-1/(N+1)} N^{1/(N+1)}.$$

*Proof.* Abbreviate  $a_n(C_{\varphi})$  and  $e_n(C_{\varphi})$  to  $a_n$  and  $e_n$ , and set  $\alpha = N/(N+1)$ . Let  $\varepsilon > 0$ . Theorem 5.1 implies:

$$a_k \le C_{\varepsilon} e^{\varepsilon k^{1/N}} e^{-\rho k^{1/N}},$$

so:

$$(a_1 \cdots a_k)^{1/k} \le C_{\varepsilon} e^{\varepsilon k^{1/N}} e^{-\rho \alpha k^{1/N}}.$$

Apply once more Theorem 2.2 to obtain:

$$e_n \le C_{\varepsilon} \sup_{k>1} e^{\varepsilon k^{1/N}} \exp\left[-(n/k + \rho \alpha k^{1/N})\right].$$

The supremum is essentially attained for k the integral part of  $(N/\rho\alpha)^{\alpha}n^{\alpha}$  and then, in view of (5.7) and  $\alpha/N = 1 - \alpha$ , up to a negligible term:

$$\frac{n}{k} + \rho \alpha k^{1/N} = n^{1-\alpha} \left(\frac{\rho \alpha}{N}\right)^{\alpha} + \rho \alpha n^{1-\alpha} \left(\frac{N}{\rho \alpha}\right)^{1-\alpha}$$
$$= n^{1-\alpha} (\rho \alpha)^{\alpha} (N^{-\alpha} + N^{1-\alpha}).$$

Finally,

$$e_n \le C_{\varepsilon} e^{\varepsilon n^{1-\alpha}} \exp\left(-\beta_N \rho^{\alpha} n^{1-\alpha}\right) = C_{\varepsilon} e^{\varepsilon n^{1/(N+1)}} \exp\left(-\beta_N \rho^{\alpha} n^{1/(N+1)}\right).$$

This clearly ends the proof of Theorem 5.2.

**Remark.** We have so far no sharp lower bound for entropy numbers, at least when  $\|\varphi\|_{\infty} = 1$ , since we already fail to have one in general for approximation numbers (see however [14]).

Besides, let  $J: H^{\infty}(\mathbb{D}^N) \to \mathcal{C}(K)$  be the canonical embedding, when  $K \subseteq \mathbb{D}^N$  is a "condenser", namely a compact subset of  $\mathbb{D}^N$  such that any bounded analytic function on  $\mathbb{D}^N$  which vanishes on K vanishes identically, which is moreover "regular". The positive solution to the Kolmogorov conjecture can be expressed in terms of the Kolmogorov numbers  $d_n(J)$  of J or equivalently, in terms of the entropy numbers  $e_n(J)$  of J ([21, Theorem 5], generalizing Erokhin's result in dimension 1 appearing in his posthumous paper [6] and methods due to Mityagin [15] and Levin and Tikhomirov [9]; see also [22, Lemma 2.2]). The result is that, taking  $K = \varphi(\mathbb{D}^N)$ , one has, with sharp constants  $c_K$ ,  $c'_K$  depending on the pluricapacity of K in  $\mathbb{D}^N$ :

(5.8) 
$$d_n(J) \approx e^{-c_K n^{1/N}}$$
 and  $e_n(J) \approx e^{-c_K' n^{1/(N+1)}}$ .

This jump from the exponent 1/N to the exponent 1/(N+1) is reflected in our Theorem 5.2, through the new parameter  $\gamma_N^+$ .

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