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Entropy numbers and spectral radius type formula for composition operators on the polydisk

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

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Abstract. We give estimates of the entropy numbers of composition operators on the Hardy space of the disk and of the polydisk.

MSC 2010 Primary 47B33 Secondary 47B06

Key-words composition operator ; entropy numbers ; polydisk ; pluricapacity

1 Introduction

This short paper was motivated by a question of J. Wengenroth ([19]) about entropy numbers of composition operators on Hardy spaces H^2 , which stand a little apart in the jungle of “ s -numbers”, even though they seem the most natural for the study of compactness, since their membership in c_0 characterizes compactness, even in the general framework of arbitrary Banach spaces. Indeed, in various papers (see [1, 10, 11, 12, 13]), we studied in detail the approximation numbers of composition operators, and here we will essentially transfer those results to entropy numbers thanks to the polar (Schmidt) decomposition and a general result on entropy numbers of diagonal operators on ℓ^2 .

So, the proofs are easy, but the statements feature a very different behavior of those entropy numbers. In particular, we will investigate a few properties related with a so-called “spectral radius type formula” which we obtained, in dimension one through a result of Widom ([12]), and, partially in dimension N ([13, 14]), through a result of Nivoche ([16]) and Zakharyuta ([22]).

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2 Entropy numbers

We begin by recalling some facts on s -numbers.

Given an operator $T: X \rightarrow Y$ between Banach spaces, recall ([4]) that we can attach to this operator five non-increasing sequences (a_n) , (b_n) , (c_n) , (d_n) , (e_n) of non-negative numbers (depending on T), respectively the sequences of *approximation*, *Bernstein*, *Gelfand*, *Kolmogorov*, and *entropy* numbers of T . The latter are defined ([4, Chapter 1], or [17, Chapter 5]), for $n \geq 1$ by:

$$(2.1) \quad e_n(T) = \inf\{\varepsilon > 0; N(T(B_X), \varepsilon B_Y) \leq 2^{n-1}\},$$

where B_X and B_Y are the respective closed unit balls of X and Y , and where, for $A, B \subseteq Y$, $N(A, B)$ denotes the smallest number of translates of B needed to cover A .

All those sequences (a_n) , (b_n) , (c_n) , (d_n) , (e_n) , say (u_n) , share the ideal property:

$$u_n(ATB) \leq \|A\| u_n(T) \|B\|.$$

For Hilbert spaces, it turns out that $a_n = b_n = c_n = d_n = s_n$, where (s_n) designates the sequence of singular numbers, but entropy numbers stay a little apart.

For general Banach spaces X and Y and $T: X \rightarrow Y$, we have, in general ([3, Theorem 1], see also [17, Theorem 5.2]), for $\alpha > 0$:

$$\sup_{1 \leq k \leq n} k^\alpha e_k(T) \leq C_\alpha \sup_{1 \leq k \leq n} k^\alpha a_k(T),$$

and, if X and Y^* are of type 2:

$$a_n(T) \leq K e_n(T), \quad \text{for all } n \geq 1$$

([7, Corollary 1.6]), where $K = \kappa [T_2(X)T_2(Y^*)]^2$; in particular, if T acts between Hilbert spaces (see [17, Theorem 5.3]):

$$a_n(T) \leq 4 e_n(T), \quad \text{for all } n \geq 1.$$

Those inequalities indicate that entropy numbers are always bigger than singular numbers, up to a constant, and that, as far as the scale of powers n^α

is implied, they are dominated by approximation numbers in a weak sense. But it turns out that, individually, they can be much bigger than the latter for composition operators, as we shall see.

We will rely on the following estimate ([4, p. 17]), in which ℓ^2 denotes the space of square-summable sequences $x = (x_k)_{k \geq 1}$ of complex numbers. This estimate is given for the sequence (ε_n) of covering numbers and with the scale of powers of 2, but $e_n = \varepsilon_{2^{n-1}}$, by definition, and the change of 2 to e only affects constants.

Theorem 2.1. (see [4, p. 17]) *There exist absolute constants $0 < a < b$ such that, for any diagonal compact operator $\Delta: \ell^2 \rightarrow \ell^2$ with positive and non-increasing eigenvalues $(\sigma_k)_{k \geq 1}$, namely $\Delta((x_k)_k) = (\sigma_k x_k)_k$, we have, for all $n \geq 1$:*

$$(2.2) \quad a \sup_{k \geq 1} \left[e^{-n/k} \left(\prod_{j=1}^k \sigma_j \right)^{1/k} \right] \leq e_n(\Delta) \leq b \sup_{k \geq 1} \left[e^{-n/k} \left(\prod_{j=1}^k \sigma_j \right)^{1/k} \right].$$

A useful corollary of Theorem 2.1 is the following.

Theorem 2.2. *Let $T: H_1 \rightarrow H_2$ be a compact operator between the Hilbert spaces H_1 and H_2 , and let $(a_n)_{n \geq 1}$ be its sequence of approximation numbers. Then, for all $n \geq 1$:*

$$(2.3) \quad \alpha \sup_{k \geq 1} \left[e^{-n/k} \left(\prod_{j=1}^k a_j \right)^{1/k} \right] \leq e_n(T) \leq \beta \sup_{k \geq 1} \left[e^{-n/k} \left(\prod_{j=1}^k a_j \right)^{1/k} \right],$$

where α and β are positive numerical constants.

Proof. Let $Tx = \sum_{n=1}^{\infty} s_n(x | u_n)v_n$ the Schmidt decomposition of T , where $(u_n)_n$ and $(v_n)_n$ are orthonormal sequences of H_1 and H_2 , respectively, and $(s_n)_n$ is the sequence of singular numbers of T . Let $\Delta: \ell_2 \rightarrow \ell_2$ the diagonal operator with diagonal values s_n , $n \geq 1$. Then $T = V_1 \Delta U_1$ and $\Delta = V_2 T U_2$, with $U_1 x = ((x | u_n))_n$, $V_1((t_n)_n) = \sum_n t_n v_n$, $U_2((t_n)_n) = \sum_n t_n u_n$ and $V_2 x = ((x | v_n))_n$. We have $\|U_1\|, \|V_1\|, \|U_2\|, \|V_2\| \leq 1$; hence the result follows from Theorem 2.1 and the ideal property. \square

This theorem might be thought useless, because we don't know better the a_n 's than the e_n 's! In our situation, this is not the case, since we made a more or less systematic study of the a_n 's for composition operators in [1, 11, 10, 12] for example.

We now pass to applications to composition operators C_φ , defined as $C_\varphi(f) = f \circ \varphi$ when they act on the Hardy space $H^2(\mathbb{D}^N)$ (which is always

the case if $N = 1$). Here, φ denotes an analytic and non-degenerate self-map of \mathbb{D}^N . For clarity, we separate the cases of dimension $N = 1$ and of dimension $N \geq 2$.

3 Applications in dimension 1

3.1 General results

In [12], we had coined the parameter:

$$(3.1) \quad \beta_1(T) = \lim_{n \rightarrow \infty} [a_n(T)]^{1/n}$$

and its versions $\beta_1^+(T), \beta_1^-(T)$ with a upper limit and a lower limit respectively. The following result ([12]) shows in particular that no lower or upper limit is needed for $\beta = \beta_1$, and provides a simpler proof of the second item in Theorem 3.1 than in our initial proof of [10].

For the definition of the Green capacity $\text{Cap}(A)$ of a Borel subset A of \mathbb{D} , $0 \leq \text{Cap}(A) \leq \infty$, we refer to [12].

Theorem 3.1. *Let $\Omega = \varphi(\mathbb{D})$, with $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ a non-constant analytic map. Then:*

1) *One always has $\beta_1^-(C_\varphi) = \beta_1^+(C_\varphi) =: \beta_1(C_\varphi)$ and:*

$$(3.2) \quad \beta_1(C_\varphi) = \exp[-1/\text{Cap}(\Omega)] > 0.$$

2) *In particular, one has the equivalence:*

$$(3.3) \quad \beta_1(C_\varphi) = 1 \iff \|\varphi\|_\infty = 1.$$

Here, another parameter emerges.

$$(3.4) \quad \gamma_1(T) = \lim_{n \rightarrow \infty} [e_n(T)]^{1/\sqrt{n}}$$

and its $\gamma_1^+(T), \gamma_1^-(T)$ versions.

Theorem 3.2. *Let $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ be a symbol and $\Omega = \varphi(\mathbb{D})$. Then:*

1) *$\gamma_1^-(C_\varphi) = \gamma_1^+(C_\varphi) =: \gamma_1(C_\varphi)$ and:*

$$(3.5) \quad \gamma_1(C_\varphi) = \exp[-\sqrt{2/\text{Cap}(\Omega)}] > 0.$$

2) *In particular, one has the equivalence:*

$$(3.6) \quad \gamma_1(C_\varphi) = 1 \iff \|\varphi\|_\infty = 1.$$

Proof. Set $\rho = 1/\text{Cap}(\Omega)$ for simplicity of notations. Let $\varepsilon > 0$, and C_ε a positive constant which depends only on ε and can vary from a formula to another. Theorem 3.1 implies $a_k \leq C_\varepsilon e^{\varepsilon k} e^{-k\rho}$, whence:

$$(a_1 \cdots a_k)^{1/k} \leq C_\varepsilon e^{\varepsilon k/2} e^{-\rho k/2}.$$

Theorem 2.2 now gives:

$$e_n(C_\varphi) \leq C_\varepsilon \sup_{k \geq 1} [e^{\varepsilon k/2} e^{-(n/k + \rho k/2)}].$$

This supremum is essentially attained for $k = \lceil \sqrt{2n/\rho} \rceil$ where $\lceil \cdot \rceil$ stands for the integer part, and gives:

$$e_n(C_\varphi) \leq C_\varepsilon e^{\varepsilon \sqrt{n/(2\rho)}} e^{-\sqrt{2n\rho}}.$$

This implies $\gamma_1^+(C_\varphi) \leq e^{\varepsilon \sqrt{1/(2\rho)}} e^{-\sqrt{2\rho}}$, and finally:

$$\gamma_1^+(C_\varphi) \leq e^{-\sqrt{2\rho}}.$$

The lower bound $\gamma_1^-(C_\varphi) \geq e^{-\sqrt{2\rho}}$ is proved similarly.

This clearly ends the proof, since we know from [12] that $\text{Cap}(\Omega) = \infty$ if and only if $\|\varphi\|_\infty = 1$. \square

3.2 Specific results

For $0 < \theta < 1$, the lens map λ_θ of parameter θ is defined by:

$$(3.7) \quad \lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}$$

(see [18] or [10]).

Theorem 3.3. *Let λ_θ be the lens map with parameter θ . Then, with positive constants a, b, a', b' depending only on θ :*

$$(3.8) \quad a' e^{-b'n^{1/3}} \leq e_n(C_{\lambda_\theta}) \leq a e^{-bn^{1/3}}.$$

Proof. We proved in [8, Theorem 2.1] (see also [10, Proposition 6.3] that $a_k = a_k(C_{\lambda_\theta}) \leq a e^{-b\sqrt{k}}$. It follows, using Theorem 2.2, that $(a_1 \cdots a_k)^{1/k} \leq a e^{-b\sqrt{k}}$ and that, for some positive constant C :

$$e_n(C_{\lambda_\theta}) \leq C \exp[-((n/k) + bk^{1/2})].$$

Taking $k = \lceil n^{2/3} \rceil$ gives the claimed upper bound. The lower bound is proved similarly, using the left inequality in Theorem 2.2, since we know ([12]) that $a_k \geq a' e^{-b'\sqrt{k}}$. \square

We refer to [11, Section 4.1] for the definition of the cusp map χ . We have:

Theorem 3.4. *Let χ be the cusp map. Then, with positive constants a, b, a', b' :*

$$(3.9) \quad a' e^{-b' \sqrt{n/\log n}} \leq e_n(C_\chi) \leq a e^{-b \sqrt{n/\log n}}.$$

Proof. We proved in [11] that:

$$(3.10) \quad a' e^{-b' k/\log k} \leq a_k(C_\chi) \leq a e^{-b k/\log k}.$$

The proof then follows the same lines as in Theorem 3.3, with the choice $k = \lceil \sqrt{n \log n} \rceil$. \square

4 The multidimensional case

4.1 General results

Let $\varphi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ be an analytic map. We will say that φ is non-degenerate if $\varphi(\mathbb{D}^N)$ has non-empty interior, equivalently if $\det \varphi'(z) \neq 0$ for at least one point $z \in \mathbb{D}^N$.

Let now $\varphi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ be a non-degenerate analytic map inducing a bounded composition operator $C_\varphi: H^2(\mathbb{D}^N) \rightarrow H^2(\mathbb{D}^N)$ (this is not always the case as soon as $N > 1$, even if φ is injective and hence non-degenerate, see for example [5, p. 246], when the polydisk is replaced by the ball; but similar examples exist for the polydisk). Assume moreover that C_φ is a compact operator.

Theorem 4.1. *Let $C_\varphi: H^2(\mathbb{D}^N) \rightarrow H^2(\mathbb{D}^N)$ be a compact composition operator, with φ non-degenerate. We have:*

1) $e_n(C_\varphi) \geq c \exp(-C n^{\frac{1}{N+1}})$, for some constants $C > c > 0$, depending on φ ;

2) if $\|\varphi\|_\infty < 1$, then $e_n(C_\varphi) \leq C \exp(-c n^{\frac{1}{N+1}})$, with $C > c > 0$ depending on φ .

Proof. 1) It is proved in [1, Theorem 3.1] that, for a non-degenerate map φ , it holds:

$$a_k(C_\varphi) \geq a' e^{-b' k^{1/N}}.$$

As in the previous section, it follows from Theorem 2.2, that $(a_1 \cdots a_k)^{1/k} \geq e^{-b''k^{1/N}}$, and then, taking $k = \lceil n^{N/(N+1)} \rceil$, that:

$$e_n(C_\varphi) \geq c e^{-Cn^{1/(N+1)}}.$$

2) Similarly, for $\|\varphi\|_\infty < 1$, it is proved in [1, Theorem 5.2] that:

$$a_k(C_\varphi) \leq C e^{-ck^{1/N}};$$

and we get the result from Theorem 2.2. \square

Those estimates motivate the introduction of the parameter:

$$(4.1) \quad \gamma_N(C_\varphi) = \lim_{n \rightarrow \infty} [e_n(C_\varphi)]^{\frac{1}{n^{1/(N+1)}}}.$$

We define similarly $\gamma_N^\pm(C_\varphi)$, and will say more on it in next section.

4.2 Specific results

4.2.1 Multi-lens maps

Let λ_θ be lens maps with parameter θ . We define the multi-lens map Λ_θ of parameter θ on the polydisk \mathbb{D}^N as:

$$(4.2) \quad \Lambda_\theta(z_1, \dots, z_N) = (\lambda_\theta(z_1), \lambda_\theta(z_2), \dots, \lambda_\theta(z_N)),$$

for $(z_1, \dots, z_N) \in \mathbb{D}^N$.

The following result is proved in [1, Theorem 6.1].

Theorem 4.2. *Let Λ_θ be the multi-lens map with parameter θ . Then, for positive constants a, b, a', b' depending only on θ and N , one has:*

$$(4.3) \quad a' e^{-b'n^{1/(2N)}} \leq a_n(C_{\Lambda_\theta}) \leq a e^{-bn^{1/(2N)}}.$$

The version of Theorem 4.2 for entropy numbers, stated without proof, is:

Theorem 4.3. *Let Λ_θ be the multi-lens map with parameter θ . Then:*

$$(4.4) \quad a' \exp(-b'n^{1/(2N+1)}) \leq e_n(C_{\Lambda_\theta}) \leq a \exp(-bn^{1/(2N+1)}).$$

4.2.2 Multi-cusp maps

Let $\chi: \mathbb{D} \rightarrow \mathbb{D}$ be the cusp map and $\varphi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ be the multi-cusp map defined by:

$$(4.5) \quad \Xi(z_1, \dots, z_N) = (\chi(z_1), \chi(z_2), \dots, \chi(z_N)).$$

It is proved in [1, Theorem 6.2]:

Theorem 4.4. *Let $\chi: \mathbb{D} \rightarrow \mathbb{D}$ be the cusp map and $\Xi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ be the multi-cusp map. Then:*

$$(4.6) \quad a' e^{-b' n^{1/N}/\log n} \leq a_n(C_\Xi) \leq a e^{-bn^{1/N}/\log n},$$

where a, b, a', b' are positive constants depending only on N .

The version of Theorem 4.4 for entropy numbers, stated without proof, is:

Theorem 4.5. *let $\chi: \mathbb{D} \rightarrow \mathbb{D}$ be the cusp map and $\Xi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ be the multi-cusp map. Then:*

$$(4.7) \quad \begin{aligned} a' \exp \left[-b' n^{1/(N+1)} (\log n)^{-N/(N+1)} \right] \\ \leq e_n(C_\Xi) \leq a \exp \left[-bn^{1/(N+1)} (\log n)^{-N/(N+1)} \right]. \end{aligned}$$

5 Connections with pluricapacity and Zakharyuta's results

Here, in dimension $N \geq 2$, the situation is satisfactory for upper bounds (see [13]); for lower bounds, see [14]. The notion involved is now that of pluricapacity, or Monge-Ampère capacity, coined by Bedford and Taylor in [2]. More precisely, if A is a Borel subset of \mathbb{D}^N , we refer to [13] or [14] for the definition of its pluricapacity $\text{Cap}_N(A)$, belonging to $[0, +\infty]$, and set:

$$(5.1) \quad \tau_N(A) = \frac{1}{(2\pi)^N} \text{Cap}_N(A)$$

$$(5.2) \quad \Gamma_N(A) = \exp \left[- \left(\frac{N!}{\tau_N(A)} \right)^{1/N} \right]$$

$$(5.3) \quad \beta_N^+(T) = \limsup_{n \rightarrow \infty} [a_n(T)]^{1/n^{1/N}}.$$

We temporarily assume that $\|\varphi\|_\infty < 1$ so that $K = \overline{\varphi(\mathbb{D}^N)}$ is a compact subset of \mathbb{D}^N . We proved in [13, Theorem 6.4], relying on positive results of Nivoche ([16]) and Zaharyuta ([22, Proposition 6.1]) on the so-called Kolmogorov conjecture, that:

Theorem 5.1. *It holds:*

$$(5.4) \quad \beta_N^+(C_\varphi) \leq \Gamma_N(K).$$

We have the following result, which extends the previous result in dimension 1.

Theorem 5.2. *The following upper bound holds:*

$$(5.5) \quad \gamma_N^+(C_\varphi) \leq \exp(-\beta_N \rho^{N/(N+1)}),$$

where:

$$(5.6) \quad \rho = \left(\frac{N!}{\tau_N(K)} \right)^{1/N} = 2\pi \left(\frac{N!}{\text{Cap}_N(K)} \right)^{1/N},$$

and

$$(5.7) \quad \beta_N = \left(\frac{N}{N+1} \right)^{N/(N+1)} (N^{-N/(N+1)} + N^{1/(N+1)}) \geq e^{-1/(N+1)} N^{1/(N+1)}.$$

Proof. Abbreviate $a_n(C_\varphi)$ and $e_n(C_\varphi)$ to a_n and e_n , and set $\alpha = N/(N+1)$. Let $\varepsilon > 0$. Theorem 5.1 implies:

$$a_k \leq C_\varepsilon e^{\varepsilon k^{1/N}} e^{-\rho k^{1/N}},$$

so:

$$(a_1 \cdots a_k)^{1/k} \leq C_\varepsilon e^{\varepsilon k^{1/N}} e^{-\rho \alpha k^{1/N}}.$$

Apply once more Theorem 2.2 to obtain:

$$e_n \leq C_\varepsilon \sup_{k \geq 1} e^{\varepsilon k^{1/N}} \exp[-(n/k + \rho \alpha k^{1/N})].$$

The supremum is essentially attained for k the integral part of $(N/\rho \alpha)^\alpha n^\alpha$ and then, in view of (5.7) and $\alpha/N = 1 - \alpha$, up to a negligible term:

$$\begin{aligned} \frac{n}{k} + \rho \alpha k^{1/N} &= n^{1-\alpha} \left(\frac{\rho \alpha}{N} \right)^\alpha + \rho \alpha n^{1-\alpha} \left(\frac{N}{\rho \alpha} \right)^{1-\alpha} \\ &= n^{1-\alpha} (\rho \alpha)^\alpha (N^{-\alpha} + N^{1-\alpha}). \end{aligned}$$

Finally,

$$e_n \leq C_\varepsilon e^{\varepsilon n^{1-\alpha}} \exp(-\beta_N \rho^\alpha n^{1-\alpha}) = C_\varepsilon e^{\varepsilon n^{1/(N+1)}} \exp(-\beta_N \rho^\alpha n^{1/(N+1)}).$$

This clearly ends the proof of Theorem 5.2. \square

Remark. We have so far no sharp lower bound for entropy numbers, at least when $\|\varphi\|_\infty = 1$, since we already fail to have one in general for approximation numbers (see however [14]).

Besides, let $J: H^\infty(\mathbb{D}^N) \rightarrow \mathcal{C}(K)$ be the canonical embedding, when $K \subseteq \mathbb{D}^N$ is a “condenser”, namely a compact subset of \mathbb{D}^N such that any bounded analytic function on \mathbb{D}^N which vanishes on K vanishes identically, which is moreover “regular”. The positive solution to the Kolmogorov conjecture can be expressed in terms of the Kolmogorov numbers $d_n(J)$ of J or equivalently, in terms of the entropy numbers $e_n(J)$ of J ([21, Theorem 5], generalizing Erokhin’s result in dimension 1 appearing in his posthumous paper [6] and methods due to Mityagin [15] and Levin and Tikhomirov [9]; see also [22, Lemma 2.2]). The result is that, taking $K = \overline{\varphi(\mathbb{D}^N)}$, one has, with sharp constants c_K, c'_K depending on the pluricapacity of K in \mathbb{D}^N :

$$(5.8) \quad d_n(J) \approx e^{-c_K n^{1/N}} \quad \text{and} \quad e_n(J) \approx e^{-c'_K n^{1/(N+1)}}.$$

This jump from the exponent $1/N$ to the exponent $1/(N+1)$ is reflected in our Theorem 5.2, through the new parameter γ_N^+ .

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