



# Composition operators with surjective symbol and small approximation numbers

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

## ► To cite this version:

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza. Composition operators with surjective symbol and small approximation numbers. 2018. hal-01919077

**HAL Id: hal-01919077**

**<https://univ-artois.hal.science/hal-01919077>**

Preprint submitted on 12 Nov 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Composition operators with surjective symbol and small approximation numbers

*Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza*

November 12, 2018

**Abstract.** We give a new proof of the existence of a surjective symbol whose associated composition operator on  $H^2(\mathbb{D})$  is in all Schatten classes, with the improvement that its approximation numbers can be, in some sense, arbitrarily small. We show, as an application, that, contrary to the 1-dimensional case, for  $N \geq 2$ , the behavior of the approximation numbers  $a_n = a_n(C_\varphi)$ , or rather of  $\beta_N^- = \liminf_{n \rightarrow \infty} [a_n]^{1/n^{1/N}}$  or  $\beta_N^+ = \limsup_{n \rightarrow \infty} [a_n]^{1/n^{1/N}}$ , of composition operators on  $H^2(\mathbb{D}^N)$  cannot be determined by the image of the symbol.

**MSC 2010** Primary: 47B33 Secondary: 32A35 ; 46B28

**Key-words** approximation numbers ; cusp map ; composition operator ; Hardy space ; lens map ; polydisk

## 1 Introduction

We start by recalling some notations and facts.

Let  $\mathbb{D}$  be the open unit disk,  $H^2$  the Hardy space on  $\mathbb{D}$ , and  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  a non-constant analytic self-map. It is well-known ([14]) that  $\varphi$  induces a composition operator  $C_\varphi: H^2 \rightarrow H^2$  by the formula:

$$C_\varphi(f) = f \circ \varphi,$$

and the connection between the “symbol”  $\varphi$  and the properties of the operator  $C_\varphi: H^2 \rightarrow H^2$ , in particular its compactness, can be further studied ([14]).

We also recall that the  $n$ th approximation number  $a_n(T)$ ,  $n = 1, 2, \dots$ , of an operator  $T: H_1 \rightarrow H_2$ , between Hilbert spaces  $H_1$  and  $H_2$ , is defined as the distance of  $T$  to operators of rank  $< n$ , for the operator-norm:

$$(1.1) \quad a_n(T) = \inf_{\text{rank } R < n} \|T - R\|.$$

The  $p$ -Schatten class  $S_p(H_1, H_2)$ ,  $p > 0$  consists of all  $T: H_1 \rightarrow H_2$  such that  $(a_n(T))_n \in \ell^p$ . The approximation numbers have the ideal property:

$$a_n(ATB) \leq \|A\| a_n(T) \|B\|.$$

Let now, for  $\xi \in \mathbb{T} = \partial\mathbb{D}$  and  $h > 0$ , the Carleson window  $S(\xi, h)$  be defined as:

$$(1.2) \quad S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\}.$$

For a symbol  $\varphi$ , we define  $m_\varphi = \varphi^*(m)$  where  $m$  is the Haar measure of  $\mathbb{T}$  and  $\varphi^*: \mathbb{T} \rightarrow \overline{\mathbb{D}}$  the (almost everywhere defined) radial limit function associated with  $\varphi$ , namely:

$$\varphi^*(\xi) = \lim_{r \rightarrow 1^-} \varphi(r\xi).$$

Finally, we set for  $h > 0$ :

$$(1.3) \quad \rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m_\varphi[S(\xi, h)].$$

It is known ([14]) that  $\rho_\varphi(h) = O(h)$  and ([12]) that  $C_\varphi$  is compact if and only if  $\rho_\varphi(h) = o(h)$  as  $h \rightarrow 0$ . Simpler criteria ([14]) exist when  $\varphi$  is injective, or even  $p$ -valent, meaning that for any  $w \in \mathbb{D}$ , the equation  $\varphi(z) = w$  has at most  $p$  solutions.

A measure  $\mu$  on  $\mathbb{D}$  is called  $\alpha$ -Carleson,  $\alpha \geq 1$ , if  $\sup_{|\xi|=1} \mu[S(\xi, h)] = O(h^\alpha)$ .

B. MacCluer and J. Shapiro showed in [13, Example 3.12] the following result, paradoxical at first glance.

**Theorem 1.1** (MacCluer-Shapiro). *There exists a surjective and four-valent symbol  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  such that the composition operator  $C_\varphi: H^2 \rightarrow H^2$  is compact.*

Observe that such a symbol  $\varphi$  cannot be one-valent (injective), because it would be an automorphism of  $\mathbb{D}$ , and  $C_\varphi$  would be invertible and therefore not compact. In [6, Theorem 4.1], we gave the following improved statement.

**Theorem 1.2.** *For every non-decreasing function  $\delta: (0, 1) \rightarrow (0, 1)$ , there exists a two-valent symbol and nearly surjective (i.e.  $\varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$ ) symbol  $\varphi$ , and  $0 < h_0 < 1$ , such that:*

$$(1.4) \quad m(\{z \in \mathbb{T}; |\varphi^*(z)| \geq 1 - h\}) \leq \delta(h) \quad \text{for } 0 < h \leq h_0.$$

*As a consequence, there exists a surjective and four-valent symbol  $\psi: \mathbb{D} \rightarrow \mathbb{D}$  such that the composition operator  $C_\psi: H^2 \rightarrow H^2$  is in every Schatten class  $S_p(H^2)$ ,  $p > 0$ .*

Our proof was rather technical and complicated, and based on arguments of barriers and harmonic measures.

The goal of this paper is to give a more precise statement of Theorem 1.2 in terms of approximation numbers  $a_n(C_\varphi)$ , and not only in terms of Schatten classes, and with a simpler proof. We then apply this result to show that for the polydisk  $\mathbb{D}^N$ ,  $N \geq 2$ , the nature (boundedness, compactness, asymptotic behavior of approximation numbers) of the composition operator cannot be determined by the geometry of the image  $\varphi(\mathbb{D}^N)$  of its symbol  $\varphi$ . For certain asymptotic behavior of approximation numbers, this is contrary to the 1-dimensional case (see [10, Theorem 3.1 and Theorem 3.14]).

The notation  $A \lesssim B$  means that  $A \leq CB$  for some positive constant  $C$ , and  $A \approx B$  that  $A \lesssim B$  and  $B \lesssim A$ .

## 2 Background and preliminary results

We initiated the study of approximation numbers of composition operators on  $H^2$  in [8], and proved the following basic results:

**Theorem 2.1.** *If  $\varphi$  is any symbol, then, for some  $\delta > 0$  and  $r > 0$ , or  $a > 0$ :*

$$a_n(C_\varphi) \geq \delta r^n = \delta e^{-an}.$$

Moreover, as soon as  $\|\varphi\|_\infty = 1$ , there exists some sequence  $\varepsilon_n$  tending to 0 such that:

$$a_n(C_\varphi) \geq \delta e^{-n\varepsilon_n}.$$

We also proved in [8, Theorem 5.1] that:

**Proposition 2.2.** *For any symbol  $\varphi$ , we have:*

$$a_n(C_\varphi) \lesssim \inf_{0 < h < 1} \left[ e^{-nh} + \sqrt{\frac{\rho_\varphi(h)}{h}} \right].$$

We also recall (see [8]) that, for  $\gamma > -1$ , the weighted Bergman space  $\mathcal{B}_\gamma$  is the space of functions  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  such that:

$$(2.1) \quad \|f\|_\gamma^2 := \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{\gamma+1}} < \infty.$$

Equivalently,  $\mathcal{B}_\gamma$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that:

$$(2.2) \quad \int_{\mathbb{D}} |f(z)|^2 (\gamma+1)(1-|z|^2)^\gamma dA(z) < \infty,$$

where  $dA$  is the normalized area measure on  $\mathbb{D}$ , and then:

$$(2.3) \quad \int_{\mathbb{D}} |f(z)|^2 (\gamma + 1)(1 - |z|^2)^\gamma dA(z) \approx \|f\|_\gamma^2.$$

The case  $\gamma = 0$  corresponds to the usual Bergman space  $\mathcal{B}^2$ , and the limiting case  $\gamma = -1$  to the Hardy space  $H^2$ . We wish to note in passing (we will make use of that elsewhere) that the proof of Theorem 5.1 in [8] easily gives the following result.

**Proposition 2.3.** *Let  $\gamma > -1$  and  $\varphi$  a symbol inducing a bounded composition operator  $C_\varphi: \mathcal{B}_\gamma \rightarrow H^2$ . Then:*

$$a_n(C_\varphi: \mathcal{B}_\gamma \rightarrow H^2) \lesssim \inf_{0 < h < 1} \left( (n+1)^{(\gamma+1)/2} e^{-nh} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_\varphi(t)}{t^{2+\gamma}}} \right).$$

*Proof.* Take  $E = z^n \mathcal{B}_\gamma$ ; this is a subspace of  $\mathcal{B}_\gamma$  of codimension  $\leq n$ . Let  $f \in E$  with  $\|f\|_\gamma = 1$ . Writing  $f = z^n g$  with  $\|g\|_\gamma^2 \leq (n+1)^{\gamma+1}$  and splitting the integral into two parts, we have, for  $0 < h < 1$ :

$$\|C_\varphi f\|_{H^2}^2 = \int_{\mathbb{D}} |f|^2 dm_\varphi \leq (1-h)^{2n} \int_{(1-h)\mathbb{D}} |g|^2 dm_\varphi + \int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 dm_\varphi.$$

For the first integral, we have:

$$(2.4) \quad \int_{(1-h)\mathbb{D}} |g|^2 dm_\varphi \leq \int_{\mathbb{D}} |g|^2 dm_\varphi = \|C_\varphi g\|_{H^2}^2 \leq \|C_\varphi\|_{\mathcal{B}_\gamma \rightarrow H^2}^2 \|g\|_\gamma^2.$$

For the second integral, we have:

$$\int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 dm_\varphi \leq \|J: \mathcal{B}_\gamma \rightarrow L^2(\mu_h)\|^2,$$

where  $\mu_h$  is the restriction of  $m_\varphi$  to the annulus  $\{z \in \mathbb{D}; 1-h < |z| < 1\}$  and  $J$  the canonical injection of  $\mathcal{B}_\gamma$  into  $L^2(\mu_h)$ . Hence Stegenga's version of the Carleson embedding theorem for  $\mathcal{B}_\gamma$  ([16, Theorem 1.2]; see [4] for the unweighted case; see also [3, p. 62] or [17, p. 167]) gives us:

$$(2.5) \quad \int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 dm_\varphi \lesssim \sup_{0 < t \leq h} \frac{\rho_\varphi(t)}{t^{2+\gamma}}.$$

Putting (2.4) and (2.5) together, that gives:

$$\|C_\varphi f\|_{H^2} \lesssim e^{-nh} (n+1)^{(\gamma+1)/2} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_\varphi(t)}{t^{2+\gamma}}}.$$

In other terms, using the Gelfand numbers  $c_k$ :

$$c_{n+1}(C_\varphi: \mathcal{B}_\gamma \rightarrow H^2) \lesssim (n+1)^{(\gamma+1)/2} e^{-nh} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_\varphi(t)}{t^{2+\gamma}}}.$$

As  $a_{n+1} = c_{n+1}$  and as we can ignore the difference between  $a_n$  and  $a_{n+1}$ , that finishes the proof.  $\square$

As an application, we mention the following result. We refer to [9, Section 4.1] for the definition of the cusp map, denoted  $\chi$ .

**Theorem 2.4.** *Let  $\chi: \mathbb{D} \rightarrow \mathbb{D}$  be the cusp map and  $\Phi: \mathbb{D}^N \rightarrow \mathbb{D}^N$  the diagonal map defined by:*

$$(2.6) \quad \Phi(z_1, z_2, \dots, z_N) = (\chi(z_1), \chi(z_1), \dots, \chi(z_1)).$$

*Then, the composition operator  $C_\Phi$  maps  $H^2(\mathbb{D}^N)$  to itself and:*

$$(2.7) \quad a_n(C_\Phi) \lesssim e^{-d\sqrt{n}}$$

*where  $d$  is a positive constant depending only on  $N$ .*

**Remark.** We have to compare with [1, Theorem 6.2] where, for:

$$\Psi(z_1, \dots, z_N) = (\chi(z_1), \dots, \chi(z_N)),$$

it is shown that, for constants  $b \geq a > 0$  depending only on  $N$ :

$$e^{-b(n^{1/N}/\log n)} \lesssim a_n(C_\Psi) \lesssim e^{-a(n^{1/N}/\log n)}.$$

Note also that for  $N = 1$ , the estimate of Theorem 2.4 is very crude.

*Proof of Theorem 2.4.* Take  $\gamma = N - 2$ . As in [11, Section 4], we have thanks to the Cauchy-Schwarz inequality, and the fact that  $\sum_{|\alpha|=n} 1 \approx (n+1)^{N-1}$ , a factorization:

$$C_\Phi = JC_\chi M,$$

where  $M: H^2(\mathbb{D}^N) \rightarrow \mathcal{B}_\gamma$  is defined by  $Mf = g$  with:

$$(2.8) \quad g(z) = f(z, z, \dots, z) = \sum_{n=0}^{\infty} \left( \sum_{|\alpha|=n} a_\alpha \right) z^n, \quad z \in \mathbb{D},$$

for

$$f(z_1, z_2, \dots, z_N) = \sum_{\alpha} a_\alpha z_1^{\alpha_1} \cdots z_N^{\alpha_N},$$

and where  $J: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^N)$  is the canonical injection given by:

$$(2.9) \quad (Jh)(z_1, z_2, \dots, z_N) = h(z_1).$$

This corresponds to a diagram:

$$(2.10) \quad H^2(\mathbb{D}^N) \xrightarrow{M} \mathcal{B}_\gamma \xrightarrow{C_\chi} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^N),$$

where  $C_\chi: \mathcal{B}_\gamma = \mathcal{B}_{N-2} \rightarrow H^2(\mathbb{D})$  is a bounded operator. Indeed, we have the behavior ([9, Lemma 4.2]):

$$|1 - \chi^*(e^{i\theta})| \approx \frac{1}{\log(1/|\theta|)},$$

and this implies, with  $c$  an absolute constant:

$$(2.11) \quad \begin{aligned} m_\chi[S(\xi, h)] &\lesssim m_\chi[S(1, h)] = m(\{|\chi^*(e^{i\theta}) - 1| < h\}) \\ &\lesssim m(\{c/\log(1/|\theta|) < h\}) \leq e^{-c/h}; \end{aligned}$$

in particular  $\rho_\chi(h) \leq e^{-c/h} = O(h^N)$ , so  $m_\chi$  is an  $N$ -Carleson measure and the Stengenga-Carleson theorem ([16, Theorem 1.2]) says that the operator  $C_\chi: \mathcal{B}_{N-2} \rightarrow H^2(\mathbb{D})$  is bounded.

Now Proposition 2.3 with (2.11) give:

$$a_n(C_\chi: \mathcal{B}_\gamma \rightarrow H^2) \lesssim \inf_{0 < h < 1} [(n+1)^{(N-1)/2} e^{-nh} + e^{-c/h} h^{-N/2}].$$

Adjusting  $h = 1/\sqrt{n}$ , we get  $a_n(C_\chi: \mathcal{B}_\gamma \rightarrow H^2) \lesssim e^{-d\sqrt{n}}$  for some positive constant  $d$ . Finally, the factorization  $C_\Phi = JC_\chi M$  and the ideal property of approximation numbers give the result.  $\square$

In the case of lens maps, Proposition 2.3 gives very poor estimates. We avoid using this theorem in [11, Section 4], when  $N = 2$ , using the semi-group property of those lens maps. The same proof gives for arbitrary  $N \geq 2$  the following result.

**Theorem 2.5.** *Let  $\lambda_\theta$  the lens map with parameter  $\theta$ ,  $0 < \theta < 1$ , and let  $\Phi: \mathbb{D}^N \rightarrow \mathbb{D}^N$  be the diagonal map defined by:*

$$(2.12) \quad \Phi(z_1, z_2, \dots, z_N) = (\lambda_\theta(z_1), \lambda_\theta(z_1), \dots, \lambda_\theta(z_1)).$$

*Then:*

- 1) *if  $\theta > 1/N$ ,  $C_\Phi$  is unbounded on  $H^2(\mathbb{D}^N)$ ;*

- 2) if  $\theta = 1/N$ ,  $C_\Phi$  is bounded and not compact on  $H^2(\mathbb{D}^N)$ ;  
3) if  $\theta < 1/N$ ,  $C_\Phi$  is compact on  $H^2(\mathbb{D}^N)$  and moreover:

$$(2.13) \quad a_n(C_\Phi) \lesssim e^{-d\sqrt{n}}$$

for a constant  $d > 0$  depending only on  $\theta$  and  $N$ .

**Remark.** In [1, Theorem 6.1], it is shown that, for:

$$\Psi(z_1, \dots, z_N) = (\lambda_\theta(z_1), \dots, \lambda_\theta(z_N)),$$

we have, for constants  $b \geq a > 0$ , depending only on  $\theta$  and  $N$ :

$$e^{-bn^{1/(2N)}} \lesssim a_n(C_\Psi) \lesssim e^{-an^{1/(2N)}}.$$

*Proof of Theorem 2.5.* That had been proved, for  $N = 2$  in [11, Theorem 4.2 and Theorem 4.4]. For convenience of the reader, we sketch the proof.

Assume first  $\theta \leq 1/N$ , and write  $\lambda_\theta = \lambda_{N\theta} \circ \lambda_{1/N}$ , where we set, for convenience,  $\lambda_1(z) = z$ , so  $C_{\lambda_1} = \text{Id}$ . As in the proof of Theorem 2.4 (see [11, Section 4]), we have a factorization:

$$C_\Phi = JC_{\lambda_{N\theta}}C_{\lambda_{1/N}}M,$$

where  $M$  and  $J$  are defined in (2.8) and (2.9).

This corresponds to a diagram (recall that  $\gamma = N - 2$ ):

$$H^2(\mathbb{D}^N) \xrightarrow{M} \mathcal{B}_\gamma \xrightarrow{C_{\lambda_{1/N}}} H^2(\mathbb{D}) \xrightarrow{C_{\lambda_{N\theta}}} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^N).$$

The second arrow is bounded, since we know ([7, Lemma 3.3]) that the pullback measure  $m_{\lambda_{1/N}}$  is  $N$ -Carleson, so that  $C_{\lambda_{1/N}}$  maps  $\mathcal{B}_{N-2}$  to  $H^2(\mathbb{D})$  by the Stegenga-Carleson embedding theorem ([16, Theorem 1.2]).

For  $\theta < 1/N$ , we have  $N\theta < 1$  and  $C_{\lambda_{N\theta}}$  is compact and, for some constant  $b = b(\theta)$ , we have  $a_n(C_{\lambda_{N\theta}}) \lesssim e^{-b\sqrt{n}}$  ([7, Theorem 2.1]). Hence  $C_\Phi$  is compact and  $a_n(C_\Phi) \lesssim e^{-b\sqrt{n}}$ .

Now, for  $\theta \geq 1/N$ , we consider the reproducing kernels:

$$K_{a_1, \dots, a_N}(z_1, \dots, z_N) = \prod_{j=1}^N \frac{1}{1 - \bar{a}_j z_j}.$$

We have:

$$\|K_{a_1, \dots, a_N}\|^2 = \prod_{j=1}^N \frac{1}{1 - |a_j|^2}$$



and:

$$C_{\Phi}^*(K_{a_1, \dots, a_N}) = K_{\lambda_{\theta}(a_1), \dots, \lambda_{\theta}(a_N)},$$

so:

$$\|C_{\Phi}^*(K_{a_1, \dots, a_N})\|^2 = \left( \frac{1}{1 - |\lambda_{\theta}(a_1)|^2} \right)^N.$$

Since:

$$1 - |\lambda_{\theta}(a_1)|^2 \approx 1 - |\lambda_{\theta}(a_1)| \approx (1 - |a_1|)^{\theta},$$

we see that  $\|C_{\Phi}^*(K_{a_1, \dots, a_N})\|/\|K_{a_1, \dots, a_N}\|$  is not bounded for  $\theta > 1/N$ , so  $C_{\varphi}$  is then not bounded; and it does not converge to 0 for  $\theta = 1/N$ , so  $C_{\Phi}$  is then not compact.  $\square$

### 3 Surjectivity

Let us come back to our surjectivity issues.

Let us first remark that Theorem 1.2 gives the following result.

**Theorem 3.1.** *For every non-decreasing function  $\delta: (0, 1) \rightarrow (0, 1)$ , there exists a surjective and four-valent symbol  $\psi$ , and  $0 < h_0 < 1$ , such that, for  $0 < h \leq h_0$ :*

$$(3.1) \quad m(\{z \in \mathbb{T}; |\varphi^*(z)| \geq 1 - h\}) \leq \delta(h).$$

*Proof.* Just observe that the passage from “ $\varphi$  two-valent and nearly surjective” to “ $\psi$  four-valent and surjective” is harmless: for this, consider the Blaschke product:

$$B(z) = \left( \frac{z - a}{1 - az} \right)^2,$$

where  $0 < a < 1$ , and take  $\psi = B \circ \varphi$ ; we observe that  $B(\mathbb{D} \setminus \{0\}) = \mathbb{D}$  since  $a^2 = B(\frac{2a}{1+a^2})$ , and, for  $z \in \mathbb{D}$ :

$$\frac{1 - |B(z)|}{1 - |z|} \geq \frac{1 - \left| \frac{z - a}{1 - az} \right|^2}{1 - |z|^2} = \frac{1 - a^2}{|1 - az|^2} \geq \frac{1 - a^2}{4},$$

so that:

$$m(|\psi^*| > 1 - h) = m(1 - |B \circ \varphi^*| < h) \leq m(1 - |\varphi^*| \leq \kappa_a h),$$

with  $\kappa_a = 4/(1 - a^2)$ . Hence, this map  $\psi$  is surjective, four-valent, and satisfies (3.1), as well, up to a change of  $\delta(h)$  to  $\delta(h/\kappa_a)$  for  $\varphi$  at the beginning.  $\square$

### 3.1 A more precise statement

Our new statement is as follows.

**Theorem 3.2.** *For every positive sequence  $(\varepsilon_n)_n$  with limit 0, there exists a surjective and four-valent symbol  $\varphi$  such that:*

$$a_n(C_\varphi) \lesssim e^{-n\varepsilon_n}.$$

*Consequently, there exists a surjective and four-valent symbol  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  such that the composition operator  $C_\varphi: H^2 \rightarrow H^2$  is in every Schatten class  $S_p(H^2)$ ,  $p > 0$ .*

*Proof.* Observe first that  $\|\varphi\|_\infty = 1$  when  $\varphi$  is surjective, so that, in view of Theorem 2.1, we cannot dispense with the numbers  $\varepsilon_n$ , even if they can tend to 0 arbitrarily slowly.

Now, we can choose  $\delta: (0, 1) \rightarrow (0, 1)$  non-decreasing such that  $\delta(\varepsilon_n) \leq e^{-n\varepsilon_n}$  for all  $n$ , and then, using Theorem 3.1, we get a surjective and four-valent symbol  $\varphi$ , satisfying for all  $h$  small enough:

$$\rho_\varphi(h) \leq h \delta^2(h).$$

Proposition 2.2 gives:

$$a_n(C_\varphi) \lesssim \inf_{0 < h < 1} [e^{-nh} + \delta(h)].$$

Adjusting  $h = \varepsilon_n$ , we get  $a_n(C_\varphi) \lesssim e^{-n\varepsilon_n}$ .

To get the second part of the theorem, just take  $\varepsilon_n = n^{-1/2}$ . □

### 3.2 A simplified proof of Theorem 1.2

We give here the announced simplified proof of Theorem 1.2. This proof is based on the following key lemma, in which  $\mathcal{H}(\mathbb{D})$  denotes the set of holomorphic functions on  $\mathbb{D}$ .

**Lemma 3.3.** *There exists a numerical constant  $C$  such that, if  $f \in \mathcal{H}(\mathbb{D})$  satisfies, for some  $\alpha \in \mathbb{R}$ :*

$$\begin{cases} \Im[f(0)] < \alpha \\ f(\mathbb{D}) \subseteq \{z \in \mathbb{C}; 0 < \Re z < \pi\} \cup \{z \in \mathbb{C}; \Im z < \alpha\}, \end{cases}$$

*then:*

$$m(\{\Im f^* > y\}) \leq C e^{\alpha-y}, \quad \text{for } y \geq \alpha.$$

We first show how this lemma allows us to conclude.

*Proof of Theorem 1.2.* Let  $g: (0, \infty) \rightarrow (0, \infty)$  be a continuous decreasing function such that:

$$\lim_{t \rightarrow 0^+} g(t) = +\infty, \quad g(\pi) = \pi, \quad \lim_{t \rightarrow +\infty} g(t) = 0.$$

Then let  $\Omega$  be the simply connected region defined by:

$$\Omega = \{x + iy; x > 0, \quad g(x) < y < g(x) + 4\pi\},$$

and  $f: \mathbb{D} \rightarrow \Omega$  be a Riemann map such that  $f(0) = \pi + 3i\pi$ . Observe that we can apply Lemma 3.3 to  $f$  with  $\alpha = 5\pi$  since  $\Im f(0) = 3\pi$  and if  $f(z) = x + iy$  with  $x \geq \pi$ ; hence:

$$\Im f(z) = y < g(x) + 4\pi \leq g(\pi) + 4\pi = 5\pi.$$

Finally, consider the symbol  $\varphi = e^{-f}$ . It is nearly surjective:  $\varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$ , and two-valent, as easily checked.

For  $0 < h \leq 1/2$ , we have for  $\xi \in \mathbb{T}$  and  $|\varphi^*(\xi)| > 1 - h$ :

$$e^{-2h} \leq 1 - h < |\varphi^*(\xi)| = \exp(-\Re f^*(\xi));$$

hence  $\Re f^*(\xi) < 2h$ .

But if  $2h > x = \Re f^*(\xi)$ , we have  $g(x) > g(2h)$ . As  $f^*(\xi) = x + iy \in \overline{\Omega}$ , we get  $\Im f^*(\xi) = y \geq g(x) > g(2h)$ . Lemma 3.3 now gives:

$$(3.2) \quad m(\{\xi; |\varphi^*(\xi)| > 1 - h\}) \leq m(\{\xi; \Im f^*(\xi) > g(2h)\}) \leq C e^{5\pi - g(2h)}.$$

It is now enough to adjust  $g$  so as to have  $e^{g(t)} \geq C e^{5\pi} / \delta(t/2)$  for  $t$  small enough to get (1.4) from (3.2).  $\square$

*Proof of Lemma 3.3.* We now prove Lemma 3.3. If  $e^{y-\alpha} < 2$ , there is nothing to prove, since then:

$$m(\Im f^* > y) \leq 1 \leq 2 e^{\alpha-y}.$$

We can hence assume that  $e^{y-\alpha} \geq 2$ . First, we make a comment. If the Riemann mapping theorem is very general and flexible, it gives very few informations on the parametrization  $t \mapsto f^*(e^{it})$  when  $f: \mathbb{D} \rightarrow \Omega$  is a conformal map, except in some specific cases (lens maps, cusps, etc.: see [9]). Here, the Kolmogorov weak type inequality provides a substitute. Write:

$$f = u + iv$$

and set:

$$f_1 = -if + i\frac{\pi}{2} - \alpha = v - \alpha + i\left(\frac{\pi}{2} - u\right)$$

and:

$$F_1 = 1 + e^{f_1} = (1 + e^{v-\alpha} \sin u) + ie^{v-\alpha} \cos u.$$

If  $v < \alpha$ , then  $\Re F_1 > 1 - |\sin u| \geq 0$ . If  $v \geq \alpha$ , then  $0 < u < \pi$  and  $\Re F_1 \geq 1$ . Hence  $F_1$  maps  $\mathbb{D}$  to the right half-plane  $\mathbb{C}_0 = \{z; \Re z > 0\}$ . Finally, let  $F = U + iV: \mathbb{D} \rightarrow \mathbb{C}_0$  be defined by:

$$F = F_1 - i\Im F_1(0),$$

so that  $V(0) = 0$ . By the Kolmogorov inequality for the conjugation map  $U \mapsto V$ , and the harmonicity of  $U$ , we have, for all  $\lambda > 0$  ( $a$  designating an absolute constant):

$$(3.3) \quad m(|F^*| > \lambda) \leq \frac{a}{\lambda} \|U^*\|_1 = \frac{a}{\lambda} \int_{\mathbb{T}} U^* dm = \frac{a}{\lambda} U(0).$$

Next, we claim that:

$$(3.4) \quad |\Im F_1(0)| < 1 \quad \text{and} \quad U(0) < 2.$$

Indeed,  $v(0) < \alpha$  by hypothesis, so that  $|\Im F_1(0)| = e^{v(0)-\alpha} |\cos u(0)| < 1$ , and  $U(0) = 1 + e^{v(0)-\alpha} \sin u(0) < 2$ . Suppose now that, for some  $y > \alpha$  and  $z \in \mathbb{D}$ , we have  $v(z) > y$ . Then,  $0 < u(z) < \pi$  by our second assumption, and this implies  $\Re e^{f_1(z)} = e^{v(z)-\alpha} \sin u(z) > 0$ , so that, using  $|1 + w| \geq |w|$  if  $\Re w > 0$  and (3.4), and remembering that  $e^{y-\alpha} \geq 2$ :

$$\begin{aligned} |F(z)| &= |1 + e^{f_1(z)} - i\Im F_1(0)| \geq |1 + e^{f_1(z)}| - 1 \\ &\geq |e^{f_1(z)}| - 1 = e^{v(z)-\alpha} - 1 > e^{y-\alpha} - 1 \geq \frac{1}{2} e^{y-\alpha}. \end{aligned}$$

Taking radial limits and using (3.3) and (3.4), we get:

$$m(\Im f^* > y) \leq m(|F^*| > e^{y-\alpha}/2) \leq 4a e^{\alpha-y}.$$

This ends the proof of Lemma 3.3 with  $C = \max(2, 4a)$ . □

## 4 Application to the multidimensional case

In this section, we apply Theorem 3.1 and Theorem 3.2 to show that, for  $N \geq 2$ , the image of the symbol cannot determine the behavior of the approximation numbers, or rather of  $\beta_N(C_\varphi)$ , of the associated composition operator  $C_\varphi: H^2(\mathbb{D}^N) \rightarrow H^2(\mathbb{D}^N)$ .

Recall that for an operator  $T: H_1 \rightarrow H_2$ , we set:

$$(4.1) \quad \beta_N^-(T) = \liminf_{n \rightarrow \infty} [a_n(T)]^{1/n^{1/N}} \quad \text{and} \quad \beta_N^+(T) = \limsup_{n \rightarrow \infty} [a_n(T)]^{1/n^{1/N}},$$

and write  $\beta_N(T)$  when  $\beta_N^-(T) = \beta_N^+(T)$ .

**Theorem 4.1.** *For  $N \geq 2$ , there exist pairs of symbols  $\Phi_1, \Phi_2: \mathbb{D}^N \rightarrow \mathbb{D}^N$ , such that  $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$  and:*

- 1)  $C_{\Phi_1}$  is not bounded, but  $C_{\Phi_2}$  is compact, and even  $\beta_N(C_{\Phi_2}) = 0$ ;
- 2)  $C_{\Phi_1}$  is bounded but not compact, so  $\beta_N(C_{\Phi_1}) = 1$ , and  $C_{\Phi_2}$  is compact, with  $\beta_N(C_{\Phi_2}) = 0$ ;
- 3)  $C_{\Phi_1}$  is compact, with  $\beta_N^-(C_{\Phi_1}) > 0$  and  $\beta_N^+(C_{\Phi_1}) < 1$ , and  $C_{\Phi_2}$  is compact, with  $\beta_N(C_{\Phi_2}) = 0$ ;
- 4)  $C_{\Phi_1}$  is compact, with  $\beta_N(C_{\Phi_1}) = 1$ , and  $C_{\Phi_2}$  is compact, but with  $\beta_N(C_{\Phi_2}) = 0$ .

*Proof.* Let  $\sigma: \mathbb{D} \rightarrow \mathbb{D}$  be a surjective symbol such that  $\rho_\sigma(h) \leq h^N e^{-2/h^2}$  given by Theorem 3.1. By Proposition 2.3, we have, with  $\gamma = N - 2$ :

$$a_n(C_\sigma: \mathcal{B}_\gamma \rightarrow H^2) \lesssim \inf_{0 < h < 1} (n^{(N-1)/2} e^{-nh} + e^{-1/h^2}),$$

and, with  $h = 1/n^{1/3}$ , we get  $a_n(C_\sigma: \mathcal{B}_\gamma \rightarrow H^2) \lesssim e^{-dn^{2/3}}$ .

We choose the exponent  $2/3$  for fixing the ideas, but every exponent  $\alpha > 1/2$ , with  $\alpha < 1$ , (i.e.  $a_n(C_\sigma: \mathcal{B}_\gamma \rightarrow H^2) \lesssim e^{-dn^\alpha}$ ) would be suitable.

1) We take  $\Phi_1(z_1, z_2, z_3, \dots, z_N) = (z_1, z_1, \dots, z_1)$ . The composition operator  $C_{\Phi_1}$  is not bounded because if  $f_n(z_1, \dots, z_N) = \left(\frac{z_1 + z_2}{2}\right)^n$ , then  $\|f_n\|_2^2 = 4^{-n} \sum_{k=0}^n \binom{n}{k}^2 = 4^{-n} \binom{2n}{n} \approx 1/\sqrt{n}$ , though  $(C_{\Phi_1} f_n)(z_1, \dots, z_N) = z_1^n$  and  $\|C_{\Phi_1} f_n\|_2 = 1$ .

We define  $\Phi_2$  by:

$$\Phi_2(z_1, z_2, \dots, z_N) = (\sigma(z_1), \sigma(z_1), \dots, \sigma(z_1)).$$

Since  $\sigma$  is surjective, we have  $\Phi_2(\mathbb{D}^N) = \Phi_1(\mathbb{D}^N)$ . Now, as in the proof of Theorem 2.4, we have  $C_{\Phi_2} = JC_{\sigma}M$ , so:

$$a_n(C_{\Phi_2}) \leq a_n(C_{\sigma}: \mathcal{B}_{N-2} \rightarrow H^2) \lesssim e^{-dn^{2/3}},$$

by the ideal property. Hence  $[a_n(C_{\Phi_2})]^{1/n^{1/N}} \lesssim e^{-dn^{\frac{2}{3}-\frac{1}{N}}}$  and therefore  $\beta_N(C_{\Phi_2}) = 0$  since  $\frac{2}{3} - \frac{1}{N} > 0$ .

2) We consider the lens map  $\lambda = \lambda_{1/N}$  of parameter  $1/N$ . We define:

$$\begin{cases} \Phi_1(z_1, \dots, z_N) = (\lambda(z_1), \lambda(z_1), \dots, \lambda(z_1)) \\ \Phi_2(z_1, \dots, z_N) = (\lambda[\sigma(z_1)], \lambda[\sigma(z_1)], \dots, \lambda[\sigma(z_1)]) \end{cases}.$$

Since  $\sigma$  is surjective, we have  $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$  and we saw in Theorem 2.5 that  $C_{\Phi_1}$  is bounded but not compact.

On the other hand, we have the factorization  $C_{\Phi_2} = JC_{\sigma}C_{\lambda}M$ . Hence  $C_{\Phi_2}$  is compact, and, as in 1),  $\beta_N(C_{\Phi_2}) = 0$ .

3) For this item, the map  $\sigma$  does not suffice, and we will use another surjective symbol  $s: \mathbb{D} \rightarrow \mathbb{D}$ . By Theorem 3.1, there exists such a map  $s$  with:

$$(4.2) \quad \rho_s(t) \leq t^2 e^{-2/t^2}$$

and

$$(4.3) \quad \rho_s(t) \leq t \delta^2(t)$$

for  $t$  small enough, where  $\delta: (0, 1) \rightarrow (0, 1)$  is a non-decreasing function such that  $\delta(\varepsilon_n) \leq e^{-n\varepsilon_n}$  and:

$$(4.4) \quad \varepsilon_n = n^{-\frac{1}{4N-7}}.$$

By the proof of Theorem 3.2, (4.3) implies that:

$$(4.5) \quad a_n(C_s) \leq e^{-n\varepsilon_n}.$$

We also consider a lens map  $\lambda = \lambda_{\theta}$ , with parameter  $\theta < 1/N$ , and we set:

$$\begin{cases} \Phi_1(z_1, \dots, z_N) = \left( \lambda(z_1), \lambda(z_1), \frac{z_3}{2}, \dots, \frac{z_N}{2} \right) \\ \Phi_2(z_1, \dots, z_N) = \left( \lambda[s(z_1)], \lambda[s(z_1)], \frac{s(z_3)}{2}, \dots, \frac{s(z_N)}{2} \right). \end{cases}$$

Since  $s$  is surjective, we have  $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$ .

a) Let us prove that  $\beta_N^-(C_{\Phi_1}) > 0$  and  $\beta_N^+(C_{\Phi_1}) < 1$ .

Note that:

$$C_{\Phi_1} = C_u \otimes C_{v_3} \otimes \cdots \otimes C_{v_N},$$

where  $u: \mathbb{D}^2 \rightarrow \mathbb{D}^2$  is defined by  $u(z_1, z_2) = (\lambda(z_1), \lambda(z_1))$  and  $v_j: \mathbb{D} \rightarrow \mathbb{D}$  is defined by  $v_j(z_j) = z_j/2$ . In fact, if  $f \in H^2(\mathbb{D}^2)$  and  $g_j \in H^2(\mathbb{D})$ ,  $3 \leq j \leq N$ , we have:

$$\begin{aligned} [C_{\Phi_1}(f \otimes g_3 \otimes \cdots \otimes g_N)](z_1, z_2, z_3, \dots, z_N) \\ &= (f \otimes g_3 \otimes \cdots \otimes g_N)(u(z_1, z_2), v_3(z_3), \dots, v_N(z_N)) \\ &= f[\lambda(z_1), \lambda(z_1)] g_3[v_3(z_3)] \cdots g_N[v_N(z_N)] \\ &= (C_u f)(z_1, z_2) (C_{v_3} g_3)(z_3) \cdots (C_{v_N} g_N)(z_N) \\ &= [(C_u \otimes C_{v_3} \otimes \cdots \otimes C_{v_N})(f \otimes g_3 \otimes \cdots \otimes g_N)](z_1, z_2, z_3, \dots, z_N), \end{aligned}$$

hence the result since  $H^2(\mathbb{D}^2) \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$  is dense in  $H^2(\mathbb{D}^N)$ . That proves in particular that  $C_{\Phi_1}$  is compact since  $C_u$  and  $C_{v_3}, \dots, C_{v_N}$  are (by Theorem 2.5 for  $C_u$ ).

By the supermultiplicativity of singular numbers of tensor products (see [11, Lemma 3.2]), it ensues that:

$$a_{nN}(C_{\Phi_1}) \geq a_{n^2}(C_u) \prod_{j=3}^N a_n(C_{v_j}) = a_{n^2}(C_u) \left(\frac{1}{2}\right)^{n(N-2)}.$$

By [11, Remark at the end of Section 4], we have  $a_{n^2}(C_u) \gtrsim e^{-bn}$  for some positive constant  $b = b(\theta)$ . Indeed, if  $J = J_2: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^2)$  is the canonical injection defined by  $(Jh)(z_1, z_2) = h(z_1)$  and  $Q: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D})$  is defined by  $(Qf)(z_1) = f(z_1, 0)$ , we have  $C_\lambda = QC_u J$ . Hence  $a_k(C_u) \gtrsim a_k(C_\lambda) \gtrsim e^{-b\sqrt{k}}$ .

Therefore we get:

$$a_{nN}(C_{\Phi_1}) \gtrsim e^{-cn}$$

for some positive constant depending only on  $\theta$  and  $N$ . It follows that  $\beta_N^-(C_{\Phi_1}) > 0$ .

To see that  $\beta_N^+(C_{\Phi_1}) < 1$ , we need the following lemma, whose proof is postponed.

**Lemma 4.2.** *Let  $S: H_1 \rightarrow H_1$  and  $T: H_2 \rightarrow H_2$  be two operators between Hilbert spaces and  $A, B$  a pair of positive numbers. Then, whenever:*

$$a_{[n^A]}(S) \leq e^{-cn} \quad \text{and} \quad a_{[n^B]}(T) \leq e^{-cn},$$

where  $[.]$  stands for the integer part, we have, for some constant integer  $M = M(A, B) > 0$ :

$$a_{M[n^{A+B}]}(S \otimes T) \leq e^{-cn}.$$

Let  $S = C_u$  and  $T = C_{v_3} \otimes \cdots \otimes C_{v_N}$ . For  $c$  small enough, we have  $a_{n^{N-2}}(T) \leq C(1/2)^n \leq e^{-cn}$  and, using (2.13),  $a_{n^2}(S) \leq e^{-dn} \leq e^{-cn}$ . Hence, with  $A = 2$ ,  $B = N - 2$ , Lemma 4.2 gives:

$$a_{Mn^N}(C_{\Phi_1}) \lesssim e^{-cn}.$$

Therefore  $\beta_N^+(C_{\Phi_1}) \leq e^{-c/M^{1/N}} < 1$ .

b) Define  $\Psi: \mathbb{D}^N \rightarrow \mathbb{D}^N$  by:

$$\Psi(z_1, z_2, z_3, \dots, z_N) = (s(z_1), s(z_1), s(z_3), \dots, s(z_N)).$$

If  $\tau_1: \mathbb{D}^2 \rightarrow \mathbb{D}^2$  is defined by  $\tau_1(z_1, z_2) = (s(z_1), s(z_1))$  and the map  $\tau_2: \mathbb{D}^{N-2} \rightarrow \mathbb{D}^{N-2}$  by  $\tau_2(z_3, \dots, z_N) = (s(z_3), \dots, s(z_N))$ , we have:

$$C_\Psi = C_{\tau_1} \otimes C_{\tau_2}.$$

As in the proof of Theorem 2.4, we have the factorization:

$$\tau_1: H^2(\mathbb{D}^2) \xrightarrow{M} \mathcal{B}_0 = \mathcal{B}^2 \xrightarrow{C_s} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^2).$$

Hence  $a_n(C_{\tau_1}) \leq \|M\| \|J\| a_n(C_s: \mathcal{B}^2 \rightarrow H^2)$ .

By Proposition 2.3, we have:

$$a_n(C_s: \mathcal{B}^2 \rightarrow H^2) \lesssim \inf_{0 < h < 1} \left( \sqrt{n} e^{-nh} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_s(t)}{t^2}} \right);$$

so (4.2) implies that  $a_n(C_s: \mathcal{B}^2 \rightarrow H^2) \lesssim \inf_{0 < h < 1} (\sqrt{n} e^{-nh} + e^{-1/h^2})$  and, taking  $h = n^{-1/3}$ , we get, with some  $c$  small enough:

$$a_n(C_s: \mathcal{B}^2 \rightarrow H^2) \lesssim e^{-cn^{2/3}}.$$

It follows that  $a_n(C_{\tau_1}) \lesssim e^{-cn^{2/3}}$  and hence:

$$(4.6) \quad a_{[n^{3/2}]}(C_{\tau_1}) \lesssim e^{-cn}.$$



On the other hand, [1, Theorem 5.5] says that:

$$a_n(C_{\tau_2}) \leq 2^{N-3} \|C_s\|^{N-2} \inf_{n_3 \dots n_N \leq n} (a_{n_3}(C_s) + \dots + a_{n_N}(C_s)).$$

Taking  $n_3 = \dots = n_N = n^{\frac{1}{N-2}}$ , we get, using (4.5):

$$a_n(C_{\tau_2}) \leq K^N N \exp \left( - n^{\frac{1}{N-2}} \varepsilon_{n^{\frac{1}{N-2}}} \right).$$

Using (4.4), that gives:

$$a_n(C_{\tau_2}) \lesssim \exp \left( - n^{\frac{1}{N-2}(1-\frac{1}{4N-7})} \right) = \exp \left( - n^{\frac{4}{4N-7}} \right),$$

or:

$$(4.7) \quad a_{\lfloor n^{N-\frac{7}{4}} \rfloor}(C_{\tau_2}) \lesssim e^{-n} \leq e^{-cn}.$$

Now, (4.6) and (4.7) allow to use Lemma 4.2 with  $A = 3/2$  and  $B = N - 7/4$ , and we get:

$$a_M \lfloor n^{N-\frac{1}{4}} \rfloor(C_\Psi) \lesssim e^{-cn}.$$

Equivalently:

$$a_k(C_\Psi) \lesssim \exp \left( - c' k^{\frac{4}{4N-1}} \right)$$

and:

$$(a_k(C_\Psi))^{1/k^{1/N}} \lesssim \exp \left( - c' k^{\frac{4}{4N-1} - \frac{1}{N}} \right) = \exp \left( - c' k^{\frac{1}{N(4N-1)}} \right),$$

which gives  $\beta_N(C_\Psi) = 0$ .

To end the proof, it suffices to remark that  $C_{\Phi_2} = C_\Psi \circ C_{\Phi_1}$ , since  $\Phi_2 = \Phi_1 \circ \Psi$ , and hence  $\beta_N^+(C_{\Phi_2}) \leq \beta_N^+(C_\Psi) = 0$ , so  $\beta_N(C_{\Phi_2}) = 0$ .

4) We use a Shapiro-Taylor map. This one-parameter map  $\varsigma_\theta$ ,  $\theta > 0$ , was introduced by J. Shapiro and P. Taylor in 1973 ([15]) and was further studied, with a slightly different definition, in [5, Section 5]. J. Shapiro and P. Taylor proved that  $C_{\varsigma_\theta}: H^2 \rightarrow H^2$  is always compact, but is Hilbert-Schmidt if and only if  $\theta > 2$ . Let us recall their definition.

For  $0 < \varepsilon < 1$ , we set  $V_\varepsilon = \{z \in \mathbb{C}; \Re z > 0 \text{ and } |z| < \varepsilon\}$ . For  $\varepsilon = \varepsilon_\theta > 0$  small enough, one can define:

$$f_\theta(z) = z(-\log z)^\theta,$$

for  $z \in V_\varepsilon$ , where  $\log z$  will be the principal determination of the logarithm. Let now  $g_\theta$  be the conformal mapping from  $\mathbb{D}$  onto  $V_\varepsilon$ , which maps  $\mathbb{T} = \partial\mathbb{D}$  onto  $\partial V_\varepsilon$ , defined by  $g_\theta(z) = \varepsilon \varphi_0(z)$ , where  $\varphi_0$  is given by:

$$\varphi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1}.$$

Then, we define:

$$\varsigma_\theta = \exp(-f_\theta \circ g_\theta).$$

We proved in [9, Section 4.2] (though it is not sharp) that:

$$(4.8) \quad a_n(C_{\varsigma_\theta}) \gtrsim \frac{1}{n^{\theta/2}}.$$

We define  $\Phi_1: \mathbb{D}^N \rightarrow \mathbb{D}^N$  as:

$$(4.9) \quad \Phi_1(z_1, z_2, \dots, z_N) = (\varsigma_\theta(z_1), 0, \dots, 0).$$

If  $J = J_N: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^N)$  is the canonical injection defined by  $(Jh)(z_1, \dots, z_N) = h(z_1)$  and  $Q = Q_N: H^2(\mathbb{D}^N) \rightarrow H^2(\mathbb{D})$  is defined by  $(Qf)(z_1) = f(z_1, 0, \dots, 0)$ , then  $C_{\Phi_1} = JC_{\varsigma_\theta}Q$ ; hence  $C_{\Phi_1}$  is compact. On the other hand, we also have  $QC_{\Phi_1}J = C_{\varsigma_\theta}$ , which implies that  $a_n(C_{\Phi_1}) \gtrsim a_n(C_{\varsigma_\theta}) \gtrsim n^{-\theta/2}$ . It follows that:

$$\beta_N(C_{\Phi_1}) \geq \lim_{n \rightarrow \infty} (n^{-\theta/2})^{1/n^{1/N}} = 1,$$

and hence  $\beta_N(C_{\Phi_1}) = 1$ .

Now, if:

$$\Phi_2(z_1, \dots, z_N) = (\varsigma_\theta[\sigma(z_1)], 0, \dots, 0),$$

since  $\sigma$  is surjective, we have  $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$ . Moreover, we have  $C_{\Phi_2} = JC_{\varsigma_\theta \circ \sigma}Q = JC_\sigma C_{\varsigma_\theta}Q$ , so  $a_n(C_{\Phi_2}) \lesssim a_n(C_\sigma)$ . Since  $\rho_\sigma(h) \leq h^{N+1} e^{-2/h^2}$ , Proposition 2.2 gives, with  $h = 1/n^{1/3}$ :

$$a_n(C_\sigma) \lesssim e^{-cn^{2/3}},$$

so  $[a_n(C_{\Phi_2})]^{1/n^{1/N}} \lesssim \exp(-cn^{\frac{2}{3}-\frac{1}{N}})$  and  $\beta_N(C_{\Phi_2}) = 0$ .  $\square$

*Proof of Lemma 4.2.* In [11], we observed that the singular numbers of  $S \otimes T$  are the non-increasing rearrangement of the numbers  $s_j t_k$ , where  $s_j$  and  $t_k$  denote respectively the  $j$ -th and the  $k$ -th singular number of  $S$  and  $T$ . We

can assume  $s_1 = t_1 = 1$ . Using this observation, we will majorize the number of pairs  $(j, k)$  such that  $s_j t_k > e^{-cn}$ . Let  $(j, k)$  be such a pair. Since  $s_j \leq s_1 = 1$ , we have  $t_k \geq e^{-cn}$  so that  $k \leq [n^B] \leq n^B$ . Hence, for some  $2 \leq l \leq n$ , we have  $(l-1)^B < k \leq l^B$ . Then, due to the assumption on  $T$ ,  $t_k < e^{-c(l-1)}$  and  $s_j \geq e^{-cn} t_k^{-1} \gtrsim e^{-c(n-l+1)}$ , implying that  $j \lesssim (n-l+1)^A$ , thanks to the assumption on  $S$ . As a consequence, since the number of integers  $k$  such that  $(l-1)^B < k \leq l^B$  is dominated by  $l^{B-1}$ , the number  $\nu_n$  of pairs  $(j, k)$  such that  $s_j t_k > e^{-cn}$  is dominated by:

$$\sum_{l=1}^n (n-l+1)^A l^{B-1} \sim n^{A+B} \int_0^1 t^A (1-t)^B dt,$$

by a Riemann sum argument. Next, let  $M \in \mathbb{N}$  big enough to have:

$$\sum_{l=1}^n (n-l+1)^A l^{B-1} \leq M n^{A+B} - 1, \quad \text{for all } n.$$

By definition,  $a_{M[n^{A+B}]}(S \otimes T) \leq a_{\nu_n+1}(S \otimes T) \leq e^{-cn}$ , giving the result.  $\square$

**Acknowledgement.** This paper was made when the two first-named authors visited the University of Sevilla in February 2018. It is their pleasure to thank this university and all colleagues therein for their warm welcome.

The third-named author is partially supported by the project MTM2015-63699-P (Spanish MINECO and FEDER funds).

## References

- [1] F. Bayart, D. Li, H. Queffélec, L. Rodríguez-Piazza, Approximation numbers of composition operators on the Hardy and Bergman spaces of the ball or of the polydisk, *Mathematical Proceedings of the Cambridge Philosophical Society* 165 (1) (2018), 69–91.
- [2] C. Cowen, B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, *Studies in Advanced Mathematics*, CRC Press (1994).
- [3] P. Duren, A. Schuster, *Bergman spaces*, *Mathematical Surveys and Monographs* 100, Amer. Math. Soc. (2004).
- [4] W. W. Hastings, A Carleson measure theorem for Bergman spaces, *Proc. Amer. Math. Soc.* 52 (1975), 237–241.

- [5] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Some examples of compact composition operators on  $H^2$ , J. Funct. Anal. 255 (11) (2008), 3098–3124.
- [6] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Some revisited results about composition operators on Hardy spaces, Revista Math. Iberoamericana 28 (1) (2012), 57–76.
- [7] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Some new properties of composition operators associated with lens maps, Israel J. Math. 195 (2) (2013), 801–824.
- [8] D. Li, H. Queffélec, L. Rodríguez-Piazza On approximation numbers of composition operators, J. Approx. Theory 164 (4) (2012), 431–459.
- [9] D. Li, H. Queffélec, L. Rodríguez-Piazza Estimates for approximation numbers of some classes of composition operators on the Hardy space, Ann. Acad. Scient. Fennicae 38 (2013), 547–564.
- [10] D. Li, H. Queffélec, L. Rodríguez-Piazza, A spectral radius formula for approximation numbers of composition operators, J. Funct. Anal. 267 (12) (2015), 4753–4774.
- [11] D. Li, H. Queffélec, L. Rodríguez-Piazza, Some examples of composition operators and their approximation numbers on the Hardy space of the bidisk, Trans. Amer. Math. Soc. ,to appear.
- [12] B. MacCluer, Spectra of compact composition operators on  $H^p(B_N)$ , Analysis 4 (1984), 87–103.
- [13] B. MacCluer, H. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, Canad. J. Math. 38 (4) (1986), 878–906.
- [14] J. Shapiro, Composition operators and classical function theory, Universitext, Tracts in Mathematics, Springer-Verlag (1993).
- [15] J. H. Shapiro, P. D. Taylor, Compact, nuclear, and Hilbert-Schmidt composition operators on  $H^2$ , Indiana Univ. Math. J. 23 (1973), 471–496.
- [16] D. A. Stegenga, Multipliers of the Dirichlet space, Ill. J. Math. 24 (1) (1980), 113–139.

- [17] K. Zhu, Operator Theory in Function Spaces, Second edition, Mathematical Surveys and Monographs 138, American Mathematical Society, Providence, RI (2007).

Daniel Li

Univ. Artois, Laboratoire de Mathématiques de Lens (LML) EA 2462, & Fédération CNRS Nord-Pas-de-Calais FR 2956, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE

daniel.li@euler.univ-artois.fr

Hervé Queffélec

Univ. Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS Nord-Pas-de-Calais FR 2956 F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE

Herve.Queffelec@univ-lille.fr

Luis Rodríguez-Piazza

Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Calle Tarfia s/n

41 012 SEVILLA, SPAIN

piazza@us.es