



HAL
open science

Pluricapacity and approximation numbers of composition operators

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza

► **To cite this version:**

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza. Pluricapacity and approximation numbers of composition operators. 2018. hal-01877752

HAL Id: hal-01877752

<https://univ-artois.hal.science/hal-01877752>

Preprint submitted on 20 Sep 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Pluricapacity and approximation numbers of composition operators

Daniel Li, Hervé Queffélec, L. Rodríguez-Piazza

September 20, 2018

Abstract. For suitable bounded hyperconvex sets Ω in \mathbb{C}^N , in particular the ball or the polydisk, we give estimates for the approximation numbers of composition operators $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$ when $\varphi(\Omega)$ is relatively compact in Ω , involving the Monge-Ampère capacity of $\varphi(\Omega)$.

Key-words: approximation numbers ; composition operator; Hardy space ; hyperconvex domain ; Monge-Ampère capacity ; pluricapacity ; pluripotential theory ; Zakharyuta conjecture

MSC 2010 numbers – Primary: 47B33 – *Secondary:* 30H10 – 31B15 – 32A35 – 32U20 – 41A25 – 41A35 – 46B28 – 46E20 – 47B06

1 Introduction

Let \mathbb{D} be the unit disk in \mathbb{C} , $H^2(\mathbb{D})$ the corresponding Hardy space, φ a non-constant analytic self-map of \mathbb{D} and $C_\varphi: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ the associated composition operator. In [40], we proved a formula connecting the approximation numbers $a_n(C_\varphi)$ of C_φ , and the Green capacity of the image $\varphi(\mathbb{D})$ in \mathbb{D} , namely, when $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$, we have:

$$(1.1) \quad \beta(C_\varphi) := \lim_{n \rightarrow \infty} [a_n(C_\varphi)]^{1/n} = \exp(-1/\text{Cap}[\varphi(\mathbb{D})]),$$

where $\text{Cap}[\varphi(\mathbb{D})]$ is the *Green capacity* of $\varphi(\mathbb{D})$.

A non-trivial consequence of that formula was the following:

$$(1.2) \quad \|\varphi\|_\infty = 1 \implies a_n(C_\varphi) \geq \delta e^{-n\varepsilon_n} \text{ where } \varepsilon_n \rightarrow 0_+.$$

In other terms, as soon as $\|\varphi\|_\infty = 1$, we cannot hope better for the numbers $a_n(C_\varphi)$ than a subexponential decay, however slowly ε_n tends to 0.

In [41], we pursued that line of investigation in dimension $N \geq 2$, namely on $H^2(\mathbb{D}^N)$, and showed that in some cases the implication (1.2) still holds ([41, Theorem 3.1]):

$$(1.3) \quad \|\varphi\|_\infty = 1 \implies a_n(C_\varphi) \geq \delta e^{-n^{1/N}\varepsilon_n} \text{ where } \varepsilon_n \rightarrow 0_+,$$

(the substitution of n by $n^{1/N}$ is mandatory as shown by the results of [4]).

We show in this paper that, in general, for non-degenerate symbols, we have similar formulas to (1.1) at our disposal for the parameters:

$$(1.4) \quad \beta_N^-(C_\varphi) = \liminf_{n \rightarrow \infty} [a_{n^N}(C_\varphi)]^{1/n} \quad \text{and} \quad \beta_N^+(C_\varphi) = \limsup_{n \rightarrow \infty} [a_{n^N}(C_\varphi)]^{1/n}.$$

These bounds are given in terms of the Monge-Ampère (or Bedford-Taylor) capacity of $\varphi(\mathbb{D}^N)$ in \mathbb{D}^N , a notion which is the natural multidimensional extension of the Green capacity when the dimension N is ≥ 2 ([41, Theorem 6.4]). We show that we have $\beta_N^-(C_\varphi) = \beta_N^+(C_\varphi)$ for well-behaved symbols.

2 Notations and background

2.1 Complex analysis

Let Ω be a domain in \mathbb{C}^N ; a function $u: \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ is said *plurisubharmonic* (*psh*) if it is u.s.c. and if for every complex line $L = \{a + zw; z \in \mathbb{C}\}$ ($a \in \Omega, w \in \mathbb{C}^N$), the function $z \mapsto u(a + zw)$ is subharmonic in $\Omega \cap L$. We denote $\mathcal{PSH}(\Omega)$ the set of plurisubharmonic functions in Ω . If $f: \Omega \rightarrow \mathbb{C}$ is holomorphic, then $\log |f|$ and $|f|^\alpha$, $\alpha > 0$, are *psh*. Every real-valued convex function is *psh* (convex functions are those whose composition with all \mathbb{R} -linear isomorphisms are subharmonic, though plurisubharmonic functions are those whose composition with all \mathbb{C} -linear isomorphisms are subharmonic: see [30, Theorem 2.9.12]).

Let $dd^c = 2i\partial\bar{\partial}$, and $(dd^c)^N = dd^c \wedge \dots \wedge dd^c$ (N times). When $u \in \mathcal{PSH}(\Omega) \cap C^2(\Omega)$, we have:

$$(dd^c u)^N = 4^N N! \det \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) d\lambda_{2N}(z),$$

where $d\lambda_{2N}(z) = (i/2)^N dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_N \wedge d\bar{z}_N$ is the usual volume in \mathbb{C}^N . In general, the current $(dd^c u)^N$ can be defined for all locally bounded $u \in \mathcal{PSH}(\Omega)$ and is actually a positive measure on Ω ([5]).

Given $p_1, \dots, p_J \in \Omega$, the pluricomplex Green function with poles p_1, \dots, p_J and weights $c_1, \dots, c_N > 0$ is defined as:

$$g(z) = g(z, p_1, \dots, p_J) = \sup\{v(z); v \in \mathcal{PSH}(\Omega), v \leq 0 \text{ and} \\ v(z) \leq c_j \log |z - p_j| + O(1), \forall j = 1, \dots, J\}.$$

In particular, for $J = 1$ and $p_1 = a$, $c_1 = 1$, $g(\cdot, a)$ is the *pluricomplex Green function* of Ω with pole $a \in \Omega$. If $0 \in \Omega$ and $a = 0$, we denote it by g_Ω and call it the *pluricomplex Green function of Ω* ; hence:

$$g_a(z) = g(z, a) = \sup\{u(z); u \in \mathcal{PSH}(\Omega), u \leq 0 \text{ and } u(z) \leq \log |z - a| + O(1)\}.$$

Let Ω be an open subset of \mathbb{C}^N . A continuous function $\rho: \Omega \rightarrow \mathbb{R}$ is an exhaustion function if there exists $a \in (-\infty, +\infty]$ such that $\rho(z) < a$ for all $z \in \Omega$, and the set $\Omega_c = \{z \in \Omega; \rho(z) < c\}$ is relatively compact in Ω for every $c < a$.

A domain Ω in \mathbb{C}^N is said *hyperconvex* if there exists a continuous *psh* exhaustion function $\rho: \Omega \rightarrow (-\infty, 0)$ (see [30, p. 80]). We may of course replace the upper bound 0 by any other real number. Without this upper bound, Ω is said *pseudoconvex*.

Let Ω be a hyperconvex domain, with negative continuous *psh* exhaustion function ρ and $\mu_{\rho,r}$ the associated Demailly-Monge-Ampère measures, defined as:

$$(2.1) \quad \mu_{u,r} = (dd^c u_r)^N - \mathbf{1}_{\Omega \setminus B_{\Omega,u}(r)} (dd^c u)^N,$$

for $r < 0$, where $u_r = \max(u, r)$ and:

$$B_{\Omega,u}(r) = \{z \in \Omega; u(z) < r\}.$$

The nonnegative measure $\mu_{u,r}$ is supported by $S_{\Omega,u}(r) := \{z \in \Omega; u(z) = r\}$.

If:

$$\int_{\Omega} (dd^c \rho)^N < \infty,$$

these measures, considered as measures on $\bar{\Omega}$, weak-* converge, as r goes to 0, to a positive measure $\mu = \mu_{\Omega,\rho}$ supported by $\partial\Omega$ and with total mass $\int_{\Omega} (dd^c \rho)^N$ ([16, Théorème 3.1], or [30, Lemma 6.5.10]).

For the pluricomplex Green function g_a with pole a , we have $(dd^c g_a)^N = (2\pi)^N \delta_a$ ([16, Théorème 4.3]) and $g_a(a) = -\infty$, so $a \in B_{\Omega,g_a}(r)$ for every $r < 0$ and $\mathbf{1}_{\Omega \setminus B_{\Omega,g_a}(r)} (dd^c g_a)^N = 0$. Hence the Demailly-Monge-Ampère measure $\mu_{g_a,r}$ is equal to $(dd^c (g_a)_r)^N$. By [51, Lemma 1], we have $(1/|r|) (dd^c (g_a)_r)^N = u_{\bar{B}_{\Omega,g_a}(r),\Omega}$, the relative extremal function of $\bar{B}_{\Omega,g_a}(r) = \{z \in \Omega; g_a(z) \leq r\}$ in Ω (see (3.2) for the definition), and this measure is supported, not only by $S_{\Omega,g_a}(r)$, but merely by the Shilov boundary of $\bar{B}_{\Omega,g_a}(r)$ (see Section 2.2.1 for the definition).

Since $(dd^c g_a)^N = (2\pi)^N \delta_a$ has mass $(2\pi)^N < \infty$, these measures weak-* converge, as r goes to 0, to a positive measure $\mu = \mu_{\Omega,g_a}$ supported by $\partial\Omega$ with mass $(2\pi)^N$. Demailly ([16, Définition 5.2] call the measure $\frac{1}{(2\pi)^N} \mu_{\Omega,g_a}$ the *pluriharmonic measure of a* . When Ω is balanced ($az \in \Omega$ for every $z \in \Omega$ and $|a| = 1$), the support of this pluriharmonic measure is the Shilov boundary of $\bar{\Omega}$ ([51, very end of the paper]).

A *bounded symmetric domain* of \mathbb{C}^N is a bounded open and convex subset Ω of \mathbb{C}^N which is circled ($az \in \Omega$ for $z \in \Omega$ and $|a| \leq 1$) and such that for every point $a \in \Omega$, there is an involutive bi-holomorphic map $\gamma: \Omega \rightarrow \Omega$ such that a is an isolated fixed point of γ (equivalently, $\gamma(a) = a$ and $\gamma'(a) = -id$: see [52, Proposition 3.1.1]). For this definition, see [13, Definition 16 and Theorem 17], or [14, Definition 5 and Theorem 4]. Note that the convexity is automatic

(Hermann Convexity Theorem; see [27, p. 503 and Corollary 4.10]). É. Cartan showed that every bounded symmetric domain of \mathbb{C}^N is homogeneous, i.e. the group Γ of automorphisms of Ω acts transitively on Ω : for every $a, b \in \Omega$, there is an automorphism γ of Ω such that $\gamma(a) = b$ (see [52, p. 250]). Conversely, every homogeneous bounded convex domain is symmetric, since $\sigma(z) = -z$ is a symmetry about 0 (see [52, p. 250] or [26, Remark 2.1.2 (e)]). Moreover, each automorphism extends continuously to $\bar{\Omega}$ (see [22]).

The unit ball \mathbb{B}_N and the polydisk \mathbb{D}^N are examples of bounded symmetric domains. Another example is, for $N = pq$, bi-holomorphic to the open unit ball of $\mathcal{M}(p, q) = \mathcal{L}(\mathbb{C}^q, \mathbb{C}^p)$ for the operator norm (see [27, Theorem 4.9]). Every product of bounded symmetric domains is still a bounded symmetric domain. In particular, every product of balls $\Omega = \mathbb{B}_{l_1} \times \cdots \times \mathbb{B}_{l_m}$, $l_1 + \cdots + l_m = N$, is a bounded symmetric domain.

If Ω is a bounded symmetric domain, its gauge is a norm $\|\cdot\|$ on \mathbb{C}^N whose open unit ball is Ω . Hence every bounded symmetric domain is hyperconvex (take $\rho(z) = \|z\| - 1$).

2.2 Hardy spaces on hyperconvex domains

2.2.1 Hardy spaces on bounded symmetric domains

We begin by defining the Hardy space on a bounded symmetric domain, because this is easier.

The *Shilov boundary* (also called the Bergman-Shilov boundary or the distinguished boundary) $\partial_S \Omega$ of a bounded domain Ω is the smallest closed set $F \subseteq \partial \Omega$ such that $\sup_{z \in \bar{\Omega}} |f(z)| = \sup_{z \in F} |f(z)|$ for every function f holomorphic in some neighborhood of $\bar{\Omega}$ (see [13, § 4.1]).

When Ω is a bounded symmetric domain, it is also, since Ω is convex, the Shilov boundary of the algebra $A(\Omega)$ of the continuous functions on Ω which are holomorphic in Ω (because every function $f \in A(\Omega)$ can be approximated by f_ε with $f_\varepsilon(z) = f(\varepsilon z_0 + (1 - \varepsilon)z)$, where $z_0 \in \Omega$ is given: see [20, pp. 152–154]).

The Shilov boundary of the ball \mathbb{B}_N is equal to its topological boundary, but the Shilov boundary of the bidisk is $\partial_S \mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| = |z_2| = 1\}$, whereas, its usual boundary $\partial \mathbb{D}^2$ is $(\mathbb{T} \times \mathbb{D}) \cup (\mathbb{D} \times \mathbb{T})$; for the unit ball B_N , the Shilov boundary is equal to the usual boundary \mathbb{S}^{N-1} ([13, § 4.1]). Another example of a bounded symmetric domain, in \mathbb{C}^3 , is the set $\Omega = \{(z_1, z_2, z_3) \in \mathbb{C}^3; |z_1|^2 + |z_2|^2 < 1, |z_3| < 1\}$ and its Shilov boundary is $\partial_S \Omega = \{(z_1, z_2, z_3); |z_1|^2 + |z_2|^2 = 1, |z_3| = 1\}$. For $p \geq q$, the matrix A is in the topological boundary of $\mathcal{M}(p, q)$ if and only if $\|A\| = 1$, but A is in the Shilov boundary if and only if $A^*A = I_q$; therefore the two boundaries coincide if and only if $q = 1$, i.e. $\Omega = \mathbb{B}_N$ (see [14, Example 2, p. 30]).

Equivalently (see [24, Corollary 9], or [13, Theorem 33], [14, Theorem 10]), $\partial_S \Omega$ is the set of the extreme points of the convex set $\bar{\Omega}$.

The Shilov boundary $\partial_S \Omega$ is invariant by the group Γ of automorphisms of Ω and the subgroup $\Gamma_0 = \{\gamma \in \Gamma; \gamma(0) = 0\}$ act transitively on $\partial_S \Omega$ (see [22]). A theorem of H. Cartan states that the elements of Γ_0 are linear trans-

formations of \mathbb{C}^N and commute with the rotations (see [24, Theorem 1] or [26, Proposition 2.1.8]). It follows that the Shilov boundary of a bounded symmetric domain Ω coincides with its topological boundary only for $\Omega = \mathbb{B}_N$ (see [35, p. 572] or [36, p. 367]); in particular the open unit ball of \mathbb{C}^N for the norm $\|\cdot\|_p$, $1 < p < \infty$, is never a bounded symmetric domain, unless $p = 2$.

The unique Γ_0 -invariant probability measure σ on $\partial_S \Omega$ is the normalized surface area (see [22]). Then the *Hardy space* $H^2(\Omega)$ is the space of all complex-valued holomorphic functions f on Ω such that:

$$\|f\|_{H^2(\Omega)} := \left(\sup_{0 < r < 1} \int_{\partial_S \Omega} |f(r\xi)|^2 d\sigma(\xi) \right)^{1/2}$$

is finite (see [22] and [23]). It is known that the integrals in this formula are non-decreasing as r increases to 1, so we can replace the supremum by a limit. The same definition can be given when Ω is a bounded complete Reinhardt domain (see [1]).

The space $H^2(\Omega)$ is a Hilbert space (see [22, Theorem 5]) and for every $z \in \Omega$, the evaluation map $f \in H^2(\Omega) \mapsto f(z)$ is uniformly bounded on compact subsets of Ω , by a depending only on that compact set, and of Ω ([22, Lemma 3]).

For every $f \in H^2(\Omega)$, there exists a boundary values function f^* such that $\|f_r - f^*\|_{L^2(\partial_S \Omega)} \xrightarrow{r \rightarrow 1} 0$, where $f_r(z) = f(rz)$ ([9, Theorem 3]), and the map $f \in H^2(\Omega) \mapsto f^* \in L^2(\partial_S \Omega)$ is an isometric embedding ([22, Theorem 6]).

2.2.2 Hardy spaces on hyperconvex domains

For hyperconvex domains, the definition of Hardy spaces is more involved. It was done by E. Poletsky and M. Stessin ([47, Theorem 6]). Those domains are associated to a continuous negative *psh* exhaustion function u on Ω and the definition of the Hardy spaces uses the Demailly-Monge-Ampère measures. The space $H_u^2(\Omega)$ is the space of all holomorphic functions $f: \Omega \rightarrow \mathbb{C}$ such that:

$$\sup_{r < 0} \int_{S_{\Omega, u}(\Omega)} |f|^2 d\mu_{u, r} < \infty$$

and its norm is defined by:

$$\|f\|_{H_u^2(\Omega)} = \sup_{r < 0} \left(\frac{1}{(2\pi)^N} \int_{S_{\Omega, u}(\Omega)} |f|^2 d\mu_{u, r} \right)^{1/2}.$$

We can replace the supremum by a limit since the integrals are non-decreasing as r increases to 0 ([16, Corollaire 1.9]).

The space $H^\infty(\Omega)$ of bounded holomorphic functions in Ω is contained in $H_u^2(\Omega)$ (see [47, remark before Lemma 3.4]).

These spaces $H_u^2(\Omega)$ are Hilbert spaces ([47, Theorem 4.1]), but depends on the exhaustion function u (even when $N = 1$: see for instance [49]). Nevertheless, they all coincide, with equivalent norms, for the functions u for which the measure $(dd^c u)^N$ is compactly supported ([47, Lemma 3.4]); this is the case

when $u(z) = g(z, a)$ is the pluricomplex Green function with pole $a \in \Omega$ (because then $(dd^c u)^N = (2\pi)^N \delta_a$: see [16, Théorème 4.3], or [30, Theorem 6.3.6]).

When Ω is the ball \mathbb{B}_N and $u(z) = \log \|z\|_2$, then $(dd^c u)^N = C \delta_0$ and $\mu_{u,r} = (2\pi)^N d\sigma_t$, where $d\sigma_t$ is the normalized surface area on the sphere of radius $t := e^r$ (see [47, Section 4] or [17, Example 3.3]). When Ω is the polydisk \mathbb{D}^N and $u(z) = \log \|z\|_\infty$, then $(dd^c u)^N = (2\pi)^N \delta_0$ ([18, Corollary 5.4]) and $\frac{1}{(2\pi)^N} \mu_{u,r}$ is the Lebesgue measure of the torus $r\mathbb{T}^N$ (see [17, Example 3.10]). Note that in [17] and [18], the operator d^c is defined as $\frac{i}{2\pi}(\bar{\partial} - \partial)$ instead of $i(\bar{\partial} - \partial)$, as usually used.

In these two cases, the Hardy spaces are the same as the usual ones (see [2, Remark 5.2.1]).

In the sequel, we only consider the exhausting function $u = g_\Omega$; hence we will write $B_\Omega(r)$, $S_\Omega(r)$ and $H^2(\Omega)$ instead of $B_{\Omega,u}(r)$, $S_{\Omega,u}(r)$ and $H_u^2(\Omega)$.

The two notions of Hardy spaces for a bounded symmetric domain are the same:

Proposition 2.1. *Let Ω be a bounded symmetric domain in \mathbb{C}^N . Then the Hardy space $H^2(\Omega)$ coincides with the subspace of the Poletski-Stessin Hardy space $H_{g_\Omega}(\Omega)$, with equality of the norms.*

Proof. First let us note that if $\|\cdot\|$ is the norm whose open unit ball is Ω , then $g_\Omega(z) = \log \|z\|$ (see [7, Proposition 3.3.2]).

Let μ_Ω be the measure which is the $*$ -weak limit of the Demailly-Monge-Ampère measures $\mu_r = (dd^c(g_\Omega)_r)^N$. We saw that it is supported by $\partial_S \Omega$. By the remark made in [16, pp. 536-537], since the automorphisms of Ω continuously extend on $\partial\Omega$, the measure μ_Ω is Γ -invariant. By unicity, the harmonic measure $\tilde{\mu}_\Omega = (2\pi)^{-N} \mu_\Omega$ at 0 hence coincides with the normalized area measure on $\partial_S \Omega$. We have, for $f: \Omega \rightarrow \mathbb{C}$ holomorphic and $0 < s < 1$:

$$\int_{\partial_S \Omega} |f(sz)|^2 d\tilde{\mu}_\Omega(z) = \int_{\partial\Omega} |f(sz)|^2 d\tilde{\mu}_\Omega(z) = \lim_{r \rightarrow 0} \frac{1}{(2\pi)^N} \int_{S_\Omega(r)} |f(sz)|^2 d\mu_r(z),$$

because $z \mapsto |f(sz)|^2$ is continuous on $\bar{\Omega}$. Now, since $g_\Omega(z) = \log \|z\|$, we have $S_\Omega(r) = e^r \partial\Omega$ and $(g_\Omega)_r(z) + t = (g_\Omega)_{r+t}(sz)$; hence $\mu_r(sA) = \mu_{r+t}(A)$ for every Borel subset A of $\partial\Omega$, where $t = \log s$. It follows that:

$$\int_{S_\Omega(r)} |f(sz)|^2 d\mu_r(z) = \int_{S_\Omega(r+t)} |f(\zeta)|^2 d\mu_{r+t}(\zeta).$$

By letting r and t going to 0, we get:

$$\|f\|_{H^2(\Omega)}^2 = \lim_{r,t \rightarrow 0} \frac{1}{(2\pi)^N} \int_{S_\Omega(r+t)} |f(\zeta)|^2 d\mu_{r+t}(\zeta) = \|f\|_{H_{g_\Omega}^2}^2;$$

hence $f \in H^2(\Omega)$ if and only if $f \in H_{g_\Omega}^2(\Omega)$, with the same norms. \square

We have ([47, Theorem 3.6]):

Proposition 2.2 (Poletsky-Stessin). *For every $z \in \Omega$, the evaluation map $f \in H^2(\Omega) \mapsto f(z)$ is uniformly bounded on compact subsets of Ω , by a constant depending only on that compact set, and of Ω .*

Hence $H^2(\Omega)$ has a reproducing kernel, defined by:

$$(2.2) \quad f(a) = \langle f, K_a \rangle, \quad \text{for } f \in H^2(\Omega),$$

and for each $r < 0$:

$$(2.3) \quad L_r := \sup_{a \in \overline{B_\Omega(r)}} \|K_a\|_2 < \infty.$$

2.3 Composition operators

A Schur map, associated with the bounded hyperconvex domain Ω , is a *non-constant* analytic map of Ω into itself. It is said *non degenerate* if its Jacobian is not identically null. It is equivalent to say that the differential $\varphi'(a): \mathbb{C}^N \rightarrow \mathbb{C}^N$ is an invertible linear map for at least one point $a \in \Omega$. In [4], we used the terminology *truly N -dimensional*. Then, by the implicit function theorem, this is equivalent to saying that $\varphi(\Omega)$ has non-void interior. We say that the Schur map φ is a *symbol* if it defines a *bounded* composition operator $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$ by $C_\varphi(f) = f \circ \varphi$.

Let us recall that although any Schur function generates a bounded composition operator on $H^2(\mathbb{D})$, this is no longer the case on $H^2(\mathbb{D}^N)$ as soon as $N \geq 2$, as shown for example by the Schur map $\varphi(z_1, z_2) = (z_1, z_1)$. Indeed (see [3]), if say $N = 2$, taking $f(z) = \sum_{j=0}^n z_1^j z_2^{n-j}$, we see that:

$$\|f\|_2 = \sqrt{n+1} \quad \text{while} \quad \|C_\varphi f\|_2 = \|(n+1)z_1^n\|_2 = n+1.$$

The same phenomenon occurs on $H^2(\mathbb{B}_N)$ ([43]; see also [11], [12], and [15]; see also [47]).

2.4 s -numbers of operators on a Hilbert space

We begin by recalling a few operator-theoretic facts. Let H be a Hilbert space. The approximation numbers $a_n(T) = a_n$ of an operator $T: H \rightarrow H$ are defined as:

$$(2.4) \quad a_n = \inf_{\text{rank } R < n} \|T - R\|, \quad n = 1, 2, \dots$$

The operator T is compact if and only if $\lim_{n \rightarrow \infty} a_n(T) = 0$.

According to a result of Allahverdiev [10, p. 155], $a_n = s_n$, the n -th singular number of T , i.e. the n -th eigenvalue of $|T| := \sqrt{T^*T}$ when those eigenvalues are rearranged in non-increasing order.

The n -th width $d_n(K)$ of a subset K of a Banach space Y measures the defect of flatness of K and is by definition:

$$(2.5) \quad d_n(K) = \inf_{\dim E < n} \left[\sup_{f \in K} \text{dist}(f, E) \right],$$

where E runs over all subspaces of Y with dimension $< n$ and where $\text{dist}(f, E)$ denotes the distance of f to E . If $T: X \rightarrow Y$ is an operator between Banach spaces, the n -th Kolmogorov number $d_n(T)$ of T is the n th-width in Y of $T(B_X)$ where B_X is the closed unit ball of X , namely:

$$(2.6) \quad d_n(T) = \inf_{\dim E < n} \left[\sup_{f \in B_X} \text{dist}(Tf, E) \right].$$

In the case where $X = Y = H$, a Hilbert space, we have:

$$(2.7) \quad a_n(T) = d_n(T) \quad \text{for all } n \geq 1,$$

and ([40]) the following alternative definition of $a_n(T)$:

$$(2.8) \quad a_n(T) = \inf_{\dim E < n} \left[\sup_{f \in B_H} \text{dist}(Tf, TE) \right].$$

In this work, we use, for an operator $T: H \rightarrow H$, the following notation:

$$(2.9) \quad \beta_N^-(T) = \liminf_{n \rightarrow \infty} [a_{n^N}(T)]^{1/n}$$

and:

$$(2.10) \quad \beta_N^+(T) = \limsup_{n \rightarrow \infty} [a_{n^N}(T)]^{1/n}.$$

When these two quantities are equal, we write them $\beta_N(T)$.

3 Pluripotential theory

3.1 Monge-Ampère capacity

Let K be a compact subset of an open subset Ω of \mathbb{C}^N . The *Monge-Ampère capacity* of K has been defined by Bedford and Taylor ([5]; see also [30, Part II, Chapter 1]) as:

$$\text{Cap}(K) = \sup \left\{ \int_K (dd^c u)^N; u \in \mathcal{PSH}(\Omega) \text{ and } 0 \leq u \leq 1 \text{ on } \Omega \right\}.$$

When Ω is bounded and hyperconvex, we have a more convenient formula ([5, Proposition 5.3], [30, Proposition 4.6.1]):

$$(3.1) \quad \text{Cap}(K) = \int_{\Omega} (dd^c u_K^*)^N = \int_K (dd^c u_K^*)^N,$$

(the positive measure $(dd^c u_K^*)^N$ is supported by K ; actually by ∂K : see [17, Properties 8.1 (c)]), where $u_K = u_{K, \Omega}$ is the *relative extremal function* of K , defined, for any subset $E \subseteq \Omega$, as:

$$(3.2) \quad u_{E, \Omega} = \sup \{ v \in \mathcal{PSH}(\Omega); v \leq 0 \text{ and } v \leq -1 \text{ on } E \},$$

and $u_{E,\Omega}^*$ is its upper semi-continuous regularization:

$$u_{E,\Omega}^*(z) = \limsup_{\zeta \rightarrow z} u_{E,\Omega}(\zeta), \quad z \in \Omega,$$

called the *regularized relative extremal function* of E .

For an open subset ω of Ω , its capacity is defined as:

$$\text{Cap}(\omega) = \sup\{\text{Cap}(K); K \text{ is a compact subset of } \omega\}.$$

When $\bar{\omega} \subset \Omega$ is a compact subset of Ω , we have ([5, equation (6.2)], [30, Corollary 4.6.2]):

$$(3.3) \quad \text{Cap}(\omega) = \int_{\Omega} (dd^c u_{\omega})^N.$$

The *outer capacity* of a subset $E \subseteq \Omega$ is:

$$\text{Cap}^*(E) = \inf\{\text{Cap}(\omega); \omega \supseteq E \text{ and } \omega \text{ open}\}.$$

If Ω is hyperconvex and E relatively compact in Ω , then ([30, Proposition 4.7.2]):

$$\text{Cap}^*(E) = \int_{\Omega} (dd^c u_{E,\Omega}^*)^N.$$

Remark. A. Zeriahi ([57]) pointed out to us the following result.

Proposition 3.1. *Let K be a compact subset of Ω . Then:*

$$\text{Cap}(K) = \text{Cap}(\partial K).$$

Proof. Of course $u_K \leq u_{\partial K}$ since $\partial K \subseteq K$. Conversely, let $v \in \mathcal{PSH}(\Omega)$ non-positive such that $v \leq -1$ on ∂K . By the maximum principle (see [30, Corollary 2.9.6]), we get that $v \leq -1$ on K . Hence $v \leq u_K$. Taking the supremum over all those v , we obtain $u_{\partial K} \leq u_K$, and therefore $u_{\partial K} = u_K$.

By (3.1), it follows that:

$$(3.4) \quad \text{Cap}(K) = \int_{\Omega} (dd^c u_K^*)^N = \int_{\Omega} (dd^c u_{\partial K}^*)^N = \text{Cap}(\partial K). \quad \square$$

3.2 Regular sets

Let $E \subseteq \mathbb{C}^N$ be bounded. Recall that the polynomial convex hull of E is:

$$\widehat{E} = \{z \in \mathbb{C}; |P(z)| \leq \sup_E |P| \text{ for every polynomial } P\}.$$

A point $a \in \widehat{E}$ is called *regular* if $u_{E,\Omega}^*(a) = -1$ for an open set $\Omega \supseteq \widehat{E}$ (note that we always have $u_{E,\Omega} = u_{E,\Omega} = -1$ on the interior of E : see [17, Properties 8.1 (c)]). The set E is said to be *regular* if all points of \widehat{E} are regular.

The *pluricomplex Green function* of E , also called the *L-extremal function* of E , is defined, for $z \in \mathbb{C}^N$, as:

$$V_E(z) = \sup\{v(z); v \in \mathcal{L}, v \leq 0 \text{ on } E\},$$

where \mathcal{L} is the *Lelong class* of all functions $v \in \mathcal{PSH}(\mathbb{C}^N)$ such that, for some constant $C > 0$:

$$v(z) \leq C + \log(1 + |z|) \quad \text{for all } z \in \mathbb{C}^N.$$

A point $a \in \widehat{E}$ is called *L-regular* if $V_E^*(a) = 0$, where V_E^* is the upper semi-continuous regularization of V_E . The set E is *L-regular* if all points of \widehat{E} are *L-regular*.

By [28, Proposition 2.2] (see also [30, Proposition 5.3.3, and Corollary 5.3.4]), for E bounded and non pluripolar, and Ω a bounded open neighbourhood of \widehat{E} , we have:

$$(3.5) \quad m(u_{E,\Omega} + 1) \leq V_E \leq M(u_{E,\Omega} + 1)$$

for some positive constants m, M . Hence the regularity of $a \in \widehat{E}$ is equivalent to its *L-regularity*.

Recall that E is pluripolar if there exists an open set Ω containing E and $v \in \mathcal{PSH}(\Omega)$ such that $E \subseteq \{v = -\infty\}$. This is equivalent to say that there exists a hyperconvex domain Ω of \mathbb{C}^N containing E such that $u_{E,\Omega}^* \equiv 0$ (see [30, Corollary 4.7.3 and Theorem 4.7.5]). By Josefson's theorem ([30, Theorem 4.7.4]), E is pluripolar if and only if there exists $v \in \mathcal{PSH}(\mathbb{C}^N)$ such that $E \subseteq \{v = -\infty\}$. Recall also that E is pluripolar if and only if its outer capacity $\text{Cap}^*(E)$ is null ([30, Theorem 4.7.5]).

When Ω is hyperconvex and E is compact, non pluripolar, the regularity of E implies that $u_{E,\Omega}$ and V_E are continuous, on Ω and \mathbb{C}^N respectively ([30, Proposition 4.5.3 and Corollary 5.1.4]). Conversely, if $u_{E,\Omega}$ is continuous, for some hyperconvex neighbourhood Ω of E , then $u_{E,\Omega}(z) = -1$ for all $z \in E$; hence $V_E(z) = 0$ for all $z \in E$, by (3.5); but $V_E = V_{\widehat{E}}$ when E is compact ([30, Theorem 5.1.7]), so $V_E(z) = 0$ for all $z \in \widehat{E}$; by (3.5) again, we obtain that $u_{E,\Omega}(z) = -1$ for all $z \in \widehat{E}$; therefore E is regular. In the same way, the continuity of V_E implies the regularity of E . These results are due to Siciak ([50, Proposition 6.1 and Proposition 6.2]).

Every closed ball $B = B(a, r)$ of an arbitrary norm $\|\cdot\|$ on \mathbb{C}^N is regular since its *L-extremal function* is:

$$V_B(z) = \log^+(\|z - a\|/r)$$

([50, p. 179, § 2.6]).

3.3 Zakharyuta's formula

We will need a formula that Zakharyuta, in order to solve a problem raised by Kolmogorov, proved, conditionally to a conjecture, called Zakharyuta's conjecture, on the uniform approximation of the relative extremal function $u_{K,\Omega}$

by pluricomplex Green functions. This conjecture has been proved by Nivoche ([45, Theorem A]), in a more general setting that we state below:

Theorem 3.2 (Nivoche). *Let K be a regular compact subset of a bounded hyperconvex domain Ω of \mathbb{C}^N . Then for every $\varepsilon > 0$ and δ small enough, there exists a pluricomplex Green function g on Ω with a finite number of logarithmic poles such that:*

- 1) *the poles of g lie in $W = \{z \in \Omega; u_K(z) < -1 + \delta\}$;*
- 2) *we have, for every $z \in \bar{\Omega} \setminus W$:*

$$(1 + \varepsilon)g(z) \leq u_K(z) \leq (1 - \varepsilon)g(z).$$

In order to state Zakharyuta's formula, we need some additional notations.

Let K be a compact subset of Ω with non-empty interior, and A_K the set of restrictions to K of those functions that are analytic and bounded by 1, i.e. those functions belonging to the unit ball $B_{H^\infty(\Omega)}$ of the space $H^\infty(\Omega)$ of the bounded analytic functions in Ω , considered as a subset of the space $\mathcal{C}(K)$ of complex functions defined on K , equipped with the sup-norm on K .

Let $d_n(A_K)$ be the n th-width of A_K in $\mathcal{C}(K)$, namely:

$$(3.6) \quad d_n(A_K) = \inf_L \left[\sup_{f \in A_K} \text{dist}(f, L) \right],$$

where L runs over all k -dimensional subspaces of $\mathcal{C}(K)$, with $k < n$.

Equivalently, $d_n(A_K)$ is the n th-Kolmogorov number of the natural injection J of $H^\infty(\Omega)$ into $\mathcal{C}(K)$ (recall that K has non-empty interior). It is convenient to set, as in [56]:

$$(3.7) \quad \tau_N(K) = \frac{1}{(2\pi)^N} \text{Cap}(K)$$

and:

$$(3.8) \quad \Gamma_N(K) = \exp \left[- \left(\frac{N!}{\tau_N(K)} \right)^{1/N} \right],$$

i.e.:

$$(3.9) \quad \Gamma_N(K) = \exp \left[- 2\pi \left(\frac{N!}{\text{Cap}(K)} \right)^{1/N} \right].$$

Observe that $\text{Cap}(K) > 0$ since we assumed that K has non-empty interior. Now, we have ([56, Theorem 5.6]; see also [55, Theorem 5] or [54, pages 30–32], for a detailed proof):

Theorem 3.3 (Zakharyuta-Nivoche). *Let Ω be a bounded hyperconvex domain and K a regular compact subset of Ω with non-empty interior, which is holomorphically convex in Ω (i.e. $K = \tilde{K}_\Omega$). Then:*

$$(3.10) \quad -\log d_n(A_K) \sim \left(\frac{N!}{\tau_N(K)} \right)^{1/N} n^{1/N}.$$

Here \tilde{K}_Ω is the holomorphic convex hull of K in Ω , that is:

$$\tilde{K}_\Omega = \{z \in \Omega; |f(z)| \leq \sup_K |f| \text{ for every } f \in \mathcal{O}(\Omega)\},$$

where $\mathcal{O}(\Omega)$ is the set of all functions holomorphic in Ω .

Relying on that theorem, which may be seen as the extension of a result of Erokhin, proved in 1958 (see [19]; see also Widom [53] which proved a more general result, with a different proof), to dimension $N > 1$, and as a result on the approximation of functions, we will give an application to the study of approximation numbers of a composition operator on $H^2(\Omega)$ for a bounded symmetric domain of \mathbb{C}^N .

4 The spectral radius type formula

In [41, Section 6.2], we proved the following result.

Theorem 4.1. *Let $\varphi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ be given by $\varphi(z_1, \dots, z_N) = (r_1 z_1, \dots, r_N z_N)$ where $0 < r_j < 1$. Then:*

$$\beta_N(C_\varphi) = \Gamma_N[\overline{\varphi(\mathbb{D}^N)}] = \Gamma_N[\varphi(\mathbb{D}^N)].$$

The proof was simple, based on result of Blocki [8] on the Monge-Ampère capacity of a cartesian product, and on the estimation, when $A \rightarrow \infty$, of the number ν_A of N -tuples $\alpha = (\alpha_1, \dots, \alpha_N)$ of non-negative integers α_j such that $\sum_{j=1}^N \alpha_j \sigma_j \leq A$, where the numbers $\sigma_j > 0$ are fixed. The estimation was:

$$(4.1) \quad \nu_A \sim \frac{A^N}{N! \sigma_1 \cdots \sigma_N}.$$

As J. F. Burnol pointed out to us, this is a consequence of the following elementary fact. Let λ_N be the Lebesgue measure on \mathbb{R}^N , and let E be a compact subset of \mathbb{R}^N such that $\lambda_N(\partial E) = 0$. Then:

$$\lambda_N(E) = \lim_{A \rightarrow \infty} A^{-N} |(A \times E) \cap \mathbb{Z}^N|.$$

Then, just take $E = \{(x_1, \dots, x_N); x_j \geq 0 \text{ and } \sum_{j=1}^N x_j \sigma_j \leq 1\}$.

In any case, this lets us suspect that the formula of Theorem 4.1 holds in much more general cases. This is not quite true, as evidenced by our counterexample of [41, Theorem 5.12]. Nevertheless, in good cases, this formula holds, as we will see in the next sections.

In remaining of this section, we consider functions $\varphi: \Omega \rightarrow \Omega$ such that $\overline{\varphi(\Omega)} \subseteq \Omega$. If ρ is an exhaustion function for Ω , there is some $R_0 < 0$ such that $\overline{\varphi(\Omega)} \subseteq B_\Omega(R_0)$, and that implies that C_φ maps $H^2(\Omega)$ into itself and is a compact operator (see [47, Theorem 8.3], since, with their notations, for $r > R_0$, we have $T(r) = \emptyset$ and hence $\delta_\varphi(r) = 0$).

4.1 Minoration

Recall that every hyperconvex domain Ω is pseudoconvex. By H. Cartan-Thullen and Oka-Bremermann-Norguet theorems, being pseudoconvex is equivalent to being a domain of holomorphy, and equivalent to being holomorphically convex (meaning that if K is a compact subset in Ω , then its holomorphic hull \widetilde{K} is also contained in Ω): see [33, Corollaire 7.7]. Now (see [32, Chapter 5, Exercise 11]), a domain of holomorphy Ω is said a *Runge domain* if every holomorphic function in Ω can be approximated uniformly on its compact subsets by polynomials, and that is equivalent to saying that the polynomial hull and the holomorphic hull of every compact subset of Ω agree. By [32, Chapter 5, Exercise 13], every circled domain (in particular every bounded symmetric domain) is a Runge domain.

Definition 4.2. *A hyperconvex domain Ω is said strongly regular if there exists a continuous psh exhaustion function ρ such that all the sub-level sets:*

$$\Omega_c = \{z \in \Omega; \rho(z) < c\}$$

($c < 0$) have a regular closure.

For example, every bounded symmetric domain Ω is strongly regular since if $\|\cdot\|$ is the associated norm, its sub-level sets Ω_c (with $\rho(z) = \log \|z\|$) are the open balls $B(0, e^c)$, and the closed balls are regular, as said above.

Theorem 4.3. *Let Ω be a strongly regular bounded hyperconvex and Runge domain in \mathbb{C}^N , and let $\varphi: \Omega \rightarrow \mathbb{C}$ be an analytic function such that $\overline{\varphi(\Omega)} \subseteq \Omega$, and which is non-degenerate. Then:*

$$(4.2) \quad \Gamma_N[\varphi(\Omega)] \leq \beta_N^-(C_\varphi).$$

Recall that if Ω is a domain in \mathbb{C}^N , a holomorphic function $\varphi: \Omega \rightarrow \mathbb{C}^M$ ($M \leq N$) is *non-degenerate* if there exists $a \in \Omega$ such that $\text{rank}_a \varphi = M$. Then $\varphi(\Omega)$ has a non-empty interior.

Proof. Let $(r_j)_{j \geq 1}$ be an increasing sequence of negative numbers tending to 0. The set $H_j = \overline{\Omega_{r_j}}$ is a regular compact subset of Ω , with non-void interior (hence non pluripolar). Let \widehat{H}_j its polynomial convex hull; this compact set is contained in Ω , since Ω being a Runge domain, we have $\widehat{H}_j = \widetilde{H}_j$, and since $\widetilde{H}_j \subseteq \Omega$, because Ω is holomorphically convex (being hyperconvex). Moreover \widehat{H}_j is regular since $V_E = V_{\widehat{E}}$ for every compact subset of \mathbb{C}^N ([50, Corollary 4.14]).

Let $K_j = \varphi(\widehat{H}_j)$ and let G be a subspace of $H^2(\Omega)$ with dimension $< n^N$.

The set K_j is regular because of the following result (see [30, Theorem 5.3.9], [46, top of page 40], [29, Theorem 1.3], or [44, Theorem 4], with a detailed proof).

Theorem 4.4 (Pleśniak). *Let E be a compact, polynomially convex, regular and non pluripolar, subset of \mathbb{C}^N . Then if Ω is a hyperconvex domain such that $E \subseteq \Omega$ and if $\varphi: \Omega \rightarrow \mathbb{C}^N$ is a non-degenerate holomorphic function, the set $\varphi(E)$ is regular.*

As before, the polynomial convex hull \widehat{K}_j of K_j is contained in Ω and is also regular. Since φ is non-degenerate, K_j has a non-void interior; hence \widehat{K}_j also. We can hence use Zakharyuta's formula (Theorem 3.3) for the compact set \widehat{K}_j .

By restriction, the subspace G can be viewed as a subspace of $\mathcal{C}(\widehat{K}_j)$. By Zakharyuta's formula, for $0 < \varepsilon < 1$, there is $n_\varepsilon \geq 1$ such that, for $n \geq n_\varepsilon$:

$$d_{n^N}(A_{\widehat{K}_j}) \geq \exp \left[- (1 + \varepsilon) (2\pi) n \left(\frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

Hence, there exists $f \in B_{H^\infty} \subseteq B_{H^2}$ such that, for all $g \in G$:

$$\|g - f\|_{\mathcal{C}(\widehat{K}_j)} \geq (1 - \varepsilon) \exp \left[- (1 + \varepsilon) (2\pi) n \left(\frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

Since $\widehat{K}_j = \widetilde{K}_j$ and, by definition $\|\cdot\|_{\mathcal{C}(\widetilde{K}_j)} = \|\cdot\|_{\mathcal{C}(K_j)}$, we have:

$$\|g - f\|_{\mathcal{C}(\widehat{K}_j)} = \|g - f\|_{\mathcal{C}(K_j)} = \|C_\varphi(g) - C_\varphi(f)\|_{\mathcal{C}(\widetilde{H}_j)}.$$

Equivalently, since, by definition $\|\cdot\|_{\mathcal{C}(\widetilde{H}_j)} = \|\cdot\|_{\mathcal{C}(H_j)}$, we have, for all $g \in G$:

$$\|C_\varphi(g) - C_\varphi(f)\|_{\mathcal{C}(H_j)} \geq (1 - \varepsilon) \exp \left[- (1 + \varepsilon) (2\pi) n \left(\frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

This implies, thanks to (2.3), that, for all $g \in G$:

$$\|C_\varphi(g) - C_\varphi(f)\|_{H^2(\Omega)} \geq L_{r_j}^{-1} (1 - \varepsilon) \exp \left[- (1 + \varepsilon) (2\pi) n \left(\frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

Using (2.8), we get, since the subspace G is arbitrary:

$$a_{n^N}(C_\varphi) \geq L_{r_j}^{-1} (1 - \varepsilon) \exp \left[- (1 + \varepsilon) (2\pi) n \left(\frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

Taking the n th-roots and passing to the limit, we obtain:

$$\beta_N^-(C_\varphi) \geq \exp \left[- (1 + \varepsilon) (2\pi) \left(\frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

and then, letting ε go to 0:

$$\beta_N^-(C_\varphi) \geq \exp \left[- (2\pi) \left(\frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right] = \Gamma_N(\widehat{K}_j).$$

Now, the sequence $(\widehat{K}_j)_{j \geq 1}$ is increasing and $\bigcup_{j \geq 1} \widehat{K}_j \supseteq \varphi(\Omega)$; hence, by [5, Theorem 8.2 (8.3)], we have $\text{Cap}(\widehat{K}_j) \xrightarrow{j \rightarrow \infty} \text{Cap}(\bigcup_{j \geq 1} \widehat{K}_j) \geq \text{Cap}[\varphi(\Omega)]$, so:

$$\beta_N^-(C_\varphi) \geq \Gamma_N[\varphi(\Omega)],$$

and the proof of Theorem 4.3 is finished. \square

4.2 Majorization

For the majorization, we assume different hypotheses on the domain Ω . Nevertheless these assumptions agree with that of Theorem 4.3 when Ω is a bounded symmetric domain.

4.2.1 Preliminaries

Recall that a domain Ω of \mathbb{C}^N is a *Reinhardt domain* (resp. *complete Reinhardt domain*) if $z = (z_1, \dots, z_N) \in \Omega$ implies that $(\zeta_1 z_1, \dots, \zeta_N z_N) \in \Omega$ for all complex numbers ζ_1, \dots, ζ_N of modulus 1 (resp. of modulus ≤ 1). A complete bounded Reinhardt domain is hyperconvex if and only if $\log j_\Omega$ is *psh* and continuous in $\mathbb{C}^N \setminus \{0\}$, where j_Ω is the Minkowski functional of Ω (see [7, Exercise following Proposition 3.3.3]). In general, the Minkowski functional j_Ω of a bounded complete Reinhardt domain Ω is *usc* and $\log j_\Omega$ is *psh* if and only if Ω is pseudoconvex ([7, Theorem 1.4.8]). Other conditions for a bounded complete Reinhardt domain to being hyperconvex can found in [34, Theorem 3.10].

For a bounded hyperconvex and complete Reinhardt domain Ω , its pluricomplex Green function with pole 0 is $g_\Omega(z) = \log j_\Omega(z)$, where j_Ω is the Minkowski functional of Ω ([7, Proposition 3.3.2]), and $S_\Omega(r) = e^r \partial\Omega$. Since $\partial\Omega$ is in particular invariant by the pluri-rotations $z = (z_1, \dots, z_N) \mapsto (e^{i\theta_1} z_1, \dots, e^{i\theta_N} z_N)$, with $\theta_1, \dots, \theta_N \in \mathbb{R}$, the harmonic measure $\tilde{\mu}_\Omega$ at 0 (see the proof of Proposition 2.1) is also invariant by the pluri-rotations (note that it is supported by the Shilov boundary of $\bar{\Omega}$: see [51, very end of the paper]). We have, as in the proof of Proposition 2.1, for $f \in H^2(\Omega)$:

$$\sup_{0 < s < 1} \int_{\partial\Omega} |f(sz)|^2 d\tilde{\mu}_\Omega(z) = \|f\|_{H^2(\Omega)}^2 < \infty.$$

Since $\tilde{\mu}_\Omega$ is in particular invariant by the rotations $z \mapsto e^{i\theta} z$, $\theta \in \mathbb{R}$, there exists, by [9, Theorem 3], a function $f^* \in L^2(\partial\Omega, \tilde{\mu}_\Omega)$ such that:

$$\int_{\partial\Omega} |f(sz) - f^*(z)|^2 d\tilde{\mu}_\Omega(z) \xrightarrow{s \rightarrow 1} 0.$$

It ensues that the map $f \in H^2(\Omega) \mapsto f^* \in L^2(\partial\Omega, \tilde{\mu}_\Omega)$ is an isometric embedding (in fact, f^* is the radial limit of f : see [21, Lemma 2]). Therefore, we can consider $H^2(\Omega)$ as a complemented subspace of $L^2(\partial\Omega, \tilde{\mu}_\Omega)$, and we call P the orthogonal projection of $L^2(\partial\Omega, \tilde{\mu}_\Omega)$ onto $H^2(\Omega)$.

Every holomorphic function f in a Reinhardt domain Ω containing 0 (in particular if Ω is a complete Reinhardt domain) has a power series expansion about 0:

$$f(z) = \sum_{\alpha} b_{\alpha} z^{\alpha}$$

which converges normally on compact subsets of Ω ([32, Proposition 2.3.14]). Recall that if $z = (z_1, \dots, z_N)$ and $\alpha = (\alpha_1, \dots, \alpha_N)$, then $z^{\alpha} = z_1^{\alpha_1} \dots z_N^{\alpha_N}$, $|\alpha| = \alpha_1 + \dots + \alpha_N$, and $\alpha! = \alpha_1! \dots \alpha_N!$.

We have:

Proposition 4.5. *Let Ω be a bounded hyperconvex and complete Reinhardt domain, and set $e_\alpha(z) = z^\alpha$. Then the system $(e_\alpha)_\alpha$ is orthogonal in $H^2(\Omega)$.*

Proof. We use the fact that the level sets $S(r)$ and the Demailly-Monge-Ampère measures $\mu_r = (dd^c(g_\Omega)_r)^N$ are pluri-rotation invariant. For $\alpha \neq \beta$, we choose $\theta_1, \dots, \theta_N \in \mathbb{R}$ such that $1, (\theta_1/2\pi), \dots, (\theta_N/2\pi)$ are rationally independent. Then $\exp[i(\sum_{j=1}^N (\alpha_j - \beta_j)\theta_j)] \neq 1$. Hence, as in [25, p. 78], we have, making the change of variables $z = (e^{i\theta_1}w_1, \dots, e^{i\theta_N}w_N)$:

$$\int_{S(r)} z^\alpha \overline{z^\beta} d\mu_r(z) = \exp\left[i\left(\sum_{j=1}^N (\alpha_j - \beta_j)\theta_j\right)\right] \int_{S(r)} w^\alpha \overline{w^\beta} d\mu_r(w),$$

which implies that:

$$\int_{S(r)} z^\alpha \overline{z^\beta} d\mu_r(z) = 0,$$

and hence:

$$(z^\alpha | z^\beta) := \lim_{r \rightarrow 0} \int_{S(r)} z^\alpha \overline{z^\beta} d\mu_r(z) = 0. \quad \square$$

For the polydisk, we have $\|e_\alpha\|_{H^2(\mathbb{D}^N)} = 1$, and for the ball (see [48, Proposition 1.4.9]):

$$\|e_\alpha\|_{H^2(\mathbb{B}_N)}^2 = \frac{(N-1)! \alpha!}{(N-1+|\alpha|)!}.$$

Definition 4.6. *We say that Ω is a good complete Reinhardt domain if, for some positive constant C_N and some positive integer c , we have, for all $p \geq 0$:*

$$\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\Omega)}^2} \leq C_N p^{cN} [j_\Omega(z)]^{2p},$$

where j_Ω is the Minkowski functional of Ω .

Examples

1. The polydisk \mathbb{D}^N is a good Reinhardt domain because $\|e_\alpha\|_{H^2(\mathbb{D}^N)} = 1$, $|z^\alpha| \leq \|z\|_\infty^{|\alpha|}$, and the number of indices α such that $|\alpha| = p$ is $\binom{N-1+p}{p} \leq C_N p^N$ (see [35, p. 498] or [37, pp. 213–214]).

2. The ball \mathbb{B}_N is a good Reinhardt domain. In fact, observe that:

$$\frac{(N-1+p)!}{(N-1)!} = p! \frac{(p+1)(p+2)\cdots(p+N-1)}{1 \times 2 \times \cdots \times (N-1)} \leq p! (p+1)^{N-1} \leq p! (p+1)^N;$$

hence:

$$\begin{aligned} \sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\mathbb{B}_N)}^2} &= \sum_{|\alpha|=p} |z^\alpha|^2 \frac{(N-1+|\alpha|)!}{(N-1)! \alpha!} \\ &\leq (p+1)^N \sum_{|\alpha|=p} \frac{|\alpha|!}{\alpha!} |z_1|^{2\alpha_1} \cdots |z_N|^{2\alpha_N} \\ &= (p+1)^N (|z_1|^2 + \cdots + |z_N|^2)^p, \end{aligned}$$

by the multinomial formula, so:

$$\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\mathbb{B}_N)}^2} \leq (p+1)^N \|z\|_2^{2p} \leq 2^N p^N \|z\|_2^{2p}.$$

3. More generally, if $\Omega = \mathbb{B}_{l_1} \times \cdots \times \mathbb{B}_{l_m}$, $l_1 + \cdots + l_m = N$, is a product of balls, we have, writing $\alpha = (\beta_1, \dots, \beta_m)$, where each β_j is an l_j -tuple:

$$\begin{aligned} \|e_\alpha\|_{H^2(\Omega)}^2 &= \int_{\mathbb{S}_{l_1} \times \cdots \times \mathbb{S}_{l_m}} |u_1^{\beta_1}|^2 \cdots |u_m^{\beta_m}|^2 d\sigma_{l_1}(u_1) \cdots d\sigma_{l_m}(u_m) \\ &= \prod_{j=1}^m \frac{(l_j - 1)! \beta_j!}{(l_j - 1 + |\beta_j|)!}, \end{aligned}$$

and, writing $z = (z_1, \dots, z_m)$, with $z_j \in \mathbb{B}_{l_j}$:

$$\begin{aligned} \sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\Omega)}^2} &\leq \sum_{p_1 + \cdots + p_m = p} \prod_{j=1}^m (p_j + 1)^{l_j} \|z_j\|_2^{2p_j} \\ &\leq C_m p^m (p+1)^{l_1 + \cdots + l_m} [j_\Omega(z)]^{2(p_1 + \cdots + p_m)}, \end{aligned}$$

since $j_\Omega(z) = \max\{\|z_1\|_2, \dots, \|z_m\|_2\}$. Hence:

$$\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\Omega)}^2} \leq C_N p^{2N} [j_\Omega(z)]^{2p}.$$

4.2.2 The result

Theorem 4.7. *Let Ω be a bounded hyperconvex domain which is a good complete Reinhardt domain in \mathbb{C}^N , and let $\varphi: \Omega \rightarrow \Omega$ be an analytic function such that $\overline{\varphi(\Omega)} \subseteq \Omega$. Then, for every compact subset $K \supseteq \varphi(\Omega)$ of Ω with non void interior, we have:*

$$(4.3) \quad \beta_N^+(C_\varphi) \leq \Gamma_N(K).$$

In particular, if φ is moreover non-degenerate, we have:

$$(4.4) \quad \beta_N^+(C_\varphi) \leq \Gamma_N[\overline{\varphi(\Omega)}].$$

The last assertion holds because $\varphi(\Omega)$ is open if φ is non-degenerate.

Corollary 4.8. *Let Ω be a good complete bounded symmetric domain in \mathbb{C}^N , and $\varphi: \Omega \rightarrow \Omega$ a non-degenerate analytic map such that $\overline{\varphi(\Omega)} \subseteq \Omega$. Then:*

$$\Gamma_N[\varphi(\Omega)] \leq \beta_N^-(C_\varphi) \leq \beta_N^+(C_\varphi) \leq \Gamma_N[\overline{\varphi(\Omega)}].$$

For the proof of Theorem 4.7, we will use the following result ([56, Proposition 6.1]), which do not need any regularity condition on the compact set (because it may be written as a decreasing sequence of regular compact sets).

Proposition 4.9 (Zakharyuta). *If K is any compact subset of a bounded hyperconvex domain Ω of \mathbb{C}^N with non-empty interior, we have:*

$$\limsup_{n \rightarrow \infty} \frac{\log d_n(A_K)}{n^{1/N}} \leq - \left(\frac{N!}{\tau_N(K)} \right)^{1/N}.$$

Proof of Theorem 4.7. In the sequel we write $\|\cdot\|_{H^2}$ for $\|\cdot\|_{H^2(\Omega)}$. We set:

$$\Lambda_N = \limsup_{n \rightarrow \infty} [d_n(A_K)]^{n^{-1/N}}.$$

Changing n into n^N , Proposition 4.9 means that for every $\varepsilon > 0$, there exists, for n large enough, an $(n^N - 1)$ -dimensional subspace F of $\mathcal{C}(K)$ such that, for any $g \in H^\infty(\Omega)$, there exists $h \in F$ such that:

$$(4.5) \quad \|g - h\|_{\mathcal{C}(K)} \leq (1 + \varepsilon)^n \Lambda_N^n \|g\|_\infty.$$

Let l be an integer to be adjusted later, and

$$f(z) = \sum_{\alpha} b_{\alpha} z^{\alpha} \in H^2(\Omega) \quad \text{with } \|f\|_{H^2} \leq 1.$$

By Proposition 4.5, we have:

$$\|f\|_{H^2}^2 = \sum_{\alpha} |b_{\alpha}|^2 \|e_{\alpha}\|_{H^2}^2.$$

We set:

$$g(z) = \sum_{|\alpha| \leq l} b_{\alpha} z^{\alpha}.$$

By the Cauchy-Schwarz inequality:

$$|g(z)|^2 \leq \left(\sum_{|\alpha| \leq l} |b_{\alpha}|^2 \|e_{\alpha}\|_{H^2}^2 \right) \left(\sum_{|\alpha| \leq l} \frac{|z^{\alpha}|^2}{\|e_{\alpha}\|_{H^2}^2} \right) \leq \sum_{|\alpha| \leq l} \frac{|z^{\alpha}|^2}{\|e_{\alpha}\|_{H^2}^2}.$$

Since Ω is a good complete Reinhardt domain and since $j_{\Omega}(z) < 1$ for $z \in \Omega$, we have:

$$|g(z)|^2 \leq \sum_{p=0}^l p^{cN} [j_{\Omega}(z)]^{2p} \leq (l+1)^{cN+1}.$$

It follows from (4.5) that there exists $h \in F$ such that:

$$\|g - h\|_{\mathcal{C}(K)} \leq (1 + \varepsilon)^n \Lambda_N^n (l+1)^{(cN+1)/2}.$$

Since $C_{\varphi} f(z) - C_{\varphi} g(z) = f(\varphi(z)) - g(\varphi(z))$ and $\overline{\varphi(\Omega)} \subseteq K$, we have $\|C_{\varphi} f - C_{\varphi} g\|_{\infty} \leq \|f - g\|_{\mathcal{C}(K)}$; therefore:

$$(4.6) \quad \begin{aligned} \|g \circ \varphi - h \circ \varphi\|_{H^2} &\leq \|g \circ \varphi - h \circ \varphi\|_{\infty} \leq \|g - h\|_{\mathcal{C}(K)} \\ &\leq (1 + \varepsilon)^n \Lambda_N^n (l+1)^{(cN+1)/2}. \end{aligned}$$

Now, the subspace \tilde{F} formed by functions $v \circ \varphi$, for $v \in F$, can be viewed as a subspace of $L^\infty(\partial\Omega, \tilde{\mu}_\Omega) \subseteq L^2(\partial\Omega, \tilde{\mu}_\Omega)$ (indeed, since v is continuous, we can write $(v \circ \varphi)^* = v \circ \varphi^*$, where φ^* denotes the almost everywhere existing radial limits of $\varphi(rz)$, which belong to K). Let finally $E = P(\tilde{F}) \subseteq H^2(\Omega)$ where $P: L^2(\partial\Omega, \tilde{\mu}_\Omega) \rightarrow H^2(\Omega)$ is the orthogonal projection. This is a subspace of $H^2(\Omega)$ with dimension $< n^N$, and we have $\text{dist}(C_\varphi g, E) \leq \|g \circ \varphi - P(h \circ \varphi)\|_{H^2}$; hence, by (4.6):

$$(4.7) \quad \text{dist}(C_\varphi g, E) \leq (1 + \varepsilon)^n \Lambda_N^n (l + 1)^{(cN+1)/2}.$$

Now, the same calculations give that:

$$|f(z) - g(z)|^2 \leq \sum_{p>l} p^{cN} [j_\Omega(z)]^{2p};$$

hence, for some positive constant M_N :

$$|f(z) - g(z)| \leq M_N (l + 1)^{(cN+1)/2} \frac{[j_\Omega(z)]^l}{(1 - [j_\Omega(z)]^2)^{(cN+1)/2}},$$

by using the following lemma, whose proof is postponed.

Lemma 4.10. *For every non-negative integer m , there exists a positive constant A_m such that, for all integers $l \geq 0$ and all $0 < x < 1$, we have:*

$$\sum_{p \geq l} p^m x^p \leq A_m l^m \frac{x^l}{(1 - x)^{m+1}}.$$

Since K is a compact subset of Ω , there is a positive number $r_0 < 1$ such that $j_\Omega(z) \leq r_0$ for $z \in K$. Since $C_\varphi f(z) - C_\varphi g(z) = f(\varphi(z)) - g(\varphi(z))$ and $\overline{\varphi(\Omega)} \subseteq K$, we have $\|C_\varphi f - C_\varphi g\|_\infty \leq \|f - g\|_{C(K)}$, and we get:

$$(4.8) \quad \|C_\varphi f - C_\varphi g\|_{H^2} \leq \|C_\varphi f - C_\varphi g\|_\infty \leq M_N (l + 1)^{(cN+1)/2} \frac{r_0^l}{(1 - r_0^2)^{(cN+1)/2}}.$$

Now, (4.7) and (4.8) give:

$$\text{dist}(C_\varphi f, E) \leq M_N (l + 1)^{(cN+1)/2} \left(\frac{r_0^l}{(1 - r_0^2)^{(cN+1)/2}} + (1 + \varepsilon)^n \Lambda_N^n \right).$$

It ensues, thanks to (2.7), that:

$$[a_{nN}(C_\varphi)]^{1/n} \leq [M_N (l + 1)^{(cN+1)/2}]^{1/n} \left[\frac{r_0^{l/n}}{(1 - r_0^2)^{(cN+1)/2n}} + (1 + \varepsilon) \Lambda_N \right].$$

Taking now for l the integer part of $n \log n$, and passing to the upper limit as $n \rightarrow \infty$, we obtain (since $l/n \rightarrow \infty$ and $(\log l)/n \rightarrow 0$):

$$\beta_N^+(C_\varphi) \leq (1 + \varepsilon) \Lambda_N,$$

and therefore, since $\varepsilon > 0$ is arbitrary:

$$\beta_N^+(C_\varphi) \leq \Lambda_N.$$

That ends the proof, by using Proposition 4.9. \square

Proof of Lemma 4.10. We make the proof by induction on m . We set:

$$S_m = \sum_{p \geq l} p^m x^p$$

The result is obvious for $m = 0$, with $A_0 = 1$, since then $S_0 = \sum_{p \geq l} x^p = \frac{x^l}{1-x}$. Let us assume that it holds till $m - 1$ and prove it for m . We observe that, since $p^m - (p - 1)^m \leq mp^{m-1}$, we have:

$$\begin{aligned} (1-x)S_m &= \sum_{p \geq l} p^m x^p - \sum_{p \geq l} p^m x^{p+1} = \sum_{p \geq l} p^m x^p - \sum_{p \geq l+1} (p-1)^m x^p \\ &= \sum_{p \geq l+1} (p^m - (p-1)^m) x^p + l^m x^l \leq \sum_{p \geq l+1} mp^{m-1} x^p + l^m x^l \\ &\leq \sum_{p \geq l} mp^{m-1} x^p + l^m x^l \leq mA_{m-1} l^{m-1} \frac{x^l}{(1-x)^m} + l^m x^l \\ &\leq (mA_{m-1} + 1) l^m \frac{x^l}{(1-x)^m}, \end{aligned}$$

giving the result, with $A_m = mA_{m-1} + 1$. \square

4.3 Equality

Proposition 4.11. *Let Ω be a bounded hyperconvex domain and ω a relatively compact open subset of Ω . Assume that:*

$$(4.9) \quad \begin{aligned} &\text{For every } a \in \partial\omega, \text{ except on a pluripolar set } E \subseteq \partial\omega, \text{ there exists} \\ &z_0 \in \omega \text{ such that the open segment } (z_0, a) \text{ is contained in } \omega. \end{aligned}$$

Then:

$$\text{Cap}(\bar{\omega}) = \text{Cap}(\omega).$$

In particular, if $\varphi: \Omega \rightarrow \Omega$ a non-degenerate holomorphic map such that $\overline{\varphi(\Omega)} \subseteq \Omega$ and $\omega = \varphi(\Omega)$ satisfies (4.9), we have:

$$\text{Cap}[\varphi(\Omega)] = \text{Cap}[\overline{\varphi(\Omega)}].$$

Before proving Proposition 4.11, let us give an example of such a situation.

Proposition 4.12. *Let Ω be a bounded hyperconvex domain with C^1 boundary. Let U be an open neighbourhood of $\bar{\Omega}$ and $\varphi: U \rightarrow \mathbb{C}^N$ be a non-degenerate holomorphic function such that $\overline{\varphi(\Omega)} \subseteq \Omega$. Then the condition (4.9) is satisfied.*

Proof. Let $\omega = \varphi(\Omega)$.

We may assume that U is connected, hyperconvex and bounded. Let B_φ be the set of points $z \in U$ such that the complex Jacobian J_φ is null. Since J_φ is holomorphic in Ω , we have $\log |J_\varphi| \in \mathcal{PSH}(U)$ and hence (see [31, proof of Lemma 10.2]):

$$B_\varphi = \{z \in U; J_\varphi(z) = 0\} = \{z \in U; \log |J_\varphi(z)| = -\infty\}$$

is pluripolar. Therefore (see [5, Theorem 6.9]), $\text{Cap}(B_\varphi, U) = 0$. It follows (see [5, page 2, line -8]) that $\text{Cap}[\varphi(B_\varphi)] := \text{Cap}[\varphi(B_\varphi), \Omega] = 0$.

Now, for every $a \in \partial\bar{\omega} \cap [\varphi(U \setminus B_\varphi)]$, there is a tangent hyperplane H_a to $\bar{\omega}$, and hence an inward normal to $\partial\bar{\omega}$ (note that $\partial\bar{\omega} \subseteq \varphi(\partial\Omega) \subseteq \varphi(U)$). It follows that there is $z_0 \in \omega$ such that the open interval (z_0, a) is contained in ω . \square

Proof of Proposition 4.11. Let $a \in \partial\omega$ and L be a complex line containing (z_0, a) ; we have $a \in \overline{\omega \cap L}$. Assume now that this point a is a *fine* (“*effilé*”) point of ω , i.e. that there exists $u \in \mathcal{PSH}(V)$, for V a neighbourhood of a , such that:

$$\limsup_{z \rightarrow a, z \in \omega} u(z) < u(a).$$

By definition, the restriction \tilde{u} of u to $\omega \cap L$ is subharmonic and we keep the inequality:

$$\limsup_{z \rightarrow a, z \in \omega \cap L} \tilde{u}(z) < \tilde{u}(a) = u(a).$$

That means that a is a fine point of $\omega \cap L$. But $a \in \overline{\omega \cap L}$ and $\omega \cap L$ is connected, so this is not possible, by [40, Lemma 2.4]. Hence no point of $\partial\omega \setminus E$ is fine.

Let now ω^f be the closure of ω for the fine topology (i.e. the coarsest topology on U for which all the functions in $\mathcal{PSH}(U)$ are continuous; it is known: see [6, comment after Theorem 2.3], that it is the trace on U of the fine topology on \mathbb{C}^N). It is also known (see [30, Corollary 4.8.10]) that ω^f is the set of points of $\bar{\omega}$ which are not fine. By the above reasoning, we thus have:

$$\bar{\omega} \setminus \omega^f \subseteq E.$$

Since $\text{Cap}(E) = 0$, we have:

$$\text{Cap}(\bar{\omega} \setminus \omega^f) = 0,$$

and it follows that:

$$\text{Cap}(\bar{\omega}) = \text{Cap}[\omega^f \cup (\bar{\omega} \setminus \omega^f)] \leq \text{Cap}(\omega^f) + \text{Cap}(\bar{\omega} \setminus \omega^f) = \text{Cap}(\omega^f),$$

and hence $\text{Cap}(\omega^f) = \text{Cap}(\bar{\omega})$.

But, since, by definition, the *psH* functions are continuous for the fine topology, it is clear, that the relative extremal functions $u_{\omega, \Omega}$ and $u_{\omega^f, \Omega}$ are equal; hence we have, by [30, Proposition 4.7.2]:

$$\text{Cap}(\omega) = \int_{\Omega} (dd^c u_{\omega, \Omega}^*)^N = \int_{\Omega} (dd^c u_{\omega^f, \Omega}^*)^N = \text{Cap}(\omega^f).$$

Hence $\text{Cap}(\omega) = \text{Cap}(\bar{\omega})$. \square

4.4 Consequences of the spectral radius type formula

Theorem 4.3 has the following consequence.

Proposition 4.13. *Let Ω be a regular bounded symmetric domain in \mathbb{C}^N , and let $\varphi: \Omega \rightarrow \Omega$ be a non-degenerate analytic function inducing a bounded composition operator C_φ on $H^2(\Omega)$.*

Then, if $\text{Cap}[\varphi(\Omega)] = \infty$, we have $\beta_N(C_\varphi) = 1$.

In other words, if, for some constants $C, c > 0$, we have $a_n(C_\varphi) \leq C e^{-cn^{1/N}}$ for all $n \geq 1$, then $\text{Cap}[\varphi(\Omega)] < \infty$.

As a corollary, we can give a new proof of [41, Theorem 3.1].

Corollary 4.14. *Let $\tau: \mathbb{D} \rightarrow \mathbb{D}$ be an analytic map such that $\|\tau\|_\infty = 1$ and $\psi: \mathbb{D}^{N-1} \rightarrow \mathbb{D}^{N-1}$ such that the map $\varphi: \mathbb{D}^N \rightarrow \mathbb{D}^N$, defined as:*

$$\varphi(z_1, z_2, \dots, z_N) = (\tau(z_1), \psi(z_2, \dots, z_N)),$$

is non-degenerate. Then $\beta_N(C_\varphi) = 1$.

Proof. Since the map φ is non-degenerate, ψ is also non-degenerate. Hence (see [44, Proposition 2]) $\psi(\mathbb{D}^{N-1})$ is not pluripolar, i.e. $\text{Cap}_{N-1}[\psi(\mathbb{D}^{N-1})] > 0$. On the other hand, it follows from [40, Theorem 3.13 and Theorem 3.14] that $\text{Cap}_1[\tau(\mathbb{D})] = +\infty$. Then, by [8, Theorem 3], we have:

$$\begin{aligned} \text{Cap}_N[\varphi(\mathbb{D}^N)] &= \text{Cap}_N[\tau(\mathbb{D}) \times \psi(\mathbb{D}^{N-1})] \\ &= \text{Cap}_1[\tau(\mathbb{D})] \times \text{Cap}_{N-1}[\psi(\mathbb{D}^{N-1})] = +\infty. \end{aligned}$$

It follows from Proposition 4.13 that $\beta_N(C_\varphi) = 1$. □

Proof of Proposition 4.13. If $R: H^2(\Omega) \rightarrow H^2(\Omega)$ is a finite-rank operator, we set, for $t < 0$:

$$(R_t f)(w) = (Rf)(e^t w), \quad f \in H^2(\Omega).$$

Then the rank of the operator R_t is less or equal to that of R .

Recall that if $\|\cdot\|$ is the norm whose unit ball is Ω , then the pluricomplex Green function of Ω is $g_\Omega(z) = \log \|z\|$, and hence the level set $S(r)$ is the sphere $S(0, e^r) = e^r \partial\Omega$ for this norm. Since:

$$\int_{S(r)} |f[\varphi(e^t w)] - (Rf)(e^t w)|^2 d\mu_r(w) = \int_{S(r+t)} |f[\varphi(z)] - (Rf)(z)|^2 d\mu_{r+t}(z),$$

we have, setting $\varphi_t(w) = \varphi(e^t w)$:

$$\|C_{\varphi_t}(f) - R_t(f)\|_{H^2} \leq \|C_\varphi(f) - R(f)\|_{H^2}.$$

It follows that $a_n(C_{\varphi_t}) \leq a_n(C_\varphi)$ for every $n \geq 1$. Therefore $\beta_N^-(C_{\varphi_t}) \leq \beta_N^-(C_\varphi)$.

By Theorem 4.3, we have:

$$\exp \left[-2\pi \left(\frac{N!}{\text{Cap}[\varphi_t(\Omega)]} \right)^{1/N} \right] \leq \beta_N^-(C_{\varphi_t}).$$

Since $\varphi_t(\Omega) = \varphi(e^t\Omega)$ increases to $\varphi(\Omega)$ as $t \uparrow 0$, we have (see [30, Corollary 4.7.11]):

$$\text{Cap}[\varphi(\Omega)] = \lim_{t \rightarrow 0} \text{Cap}[\varphi_t(\Omega)].$$

As $\text{Cap}[\varphi(\Omega)] = \infty$, we get:

$$\beta_N^-(C_\varphi) \geq \limsup_{t \rightarrow 0} \beta_N^-(C_{\varphi_t}) = 1. \quad \square$$

Remark 1. In [41, Theorem 5.12], we construct a non-degenerate analytic function $\varphi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ such that $\overline{\varphi(\mathbb{D}^2)} \cap \partial\mathbb{D}^2 \neq \emptyset$ and for which $\beta_2^+(C_\varphi) < 1$. We hence have $\text{Cap}[\varphi(\mathbb{D}^2)] < \infty$.

Remark 2. The capacity cannot tend to infinity too fast when the compact set approaches the boundary of Ω ; in fact, we have the following result, that we state for the ball, but which holds more generally.

Proposition 4.15. *For every compact set K of \mathbb{B}_N , we have, for some constant C_N :*

$$\text{Cap}(K) \leq \frac{C_N}{[\text{dist}(K, \mathbb{S}_N)]^N}.$$

Proof. We know that:

$$\text{Cap}(K) = \int_{\mathbb{B}_N} (dd^c u_K^*)^N.$$

Let $\rho(z) = |z|^2 - 1$ and $a_K := \min_{z \in K} [-\rho(z)] = -\max_{z \in K} \rho(z)$. Then ρ is in \mathcal{PSH} and is non-positive. Since $a_K > 0$, the function:

$$v(z) = \frac{\rho(z)}{a_K}$$

is in \mathcal{PSH} , non-positive on \mathbb{B}_N , and $v \leq -1$ on K . Hence $v \leq u_K \leq u_K^*$.

Since $v(w) = 0$ for all $w \in \mathbb{S}_N$ and (see [5, Proposition 6.2 (iv)], or [30, Proposition 4.5.2]):

$$\lim_{z \rightarrow w} u_K^*(z) = 0,$$

for all $w \in \mathbb{S}_N$, the comparison theorem of Bedford and Taylor ([5, Theorem 4.1]; [30, Theorem 3.7.1] gives, since $v \leq u_K^*$ and $v, u_K^* \in \mathcal{PSH}$:

$$\int_{\mathbb{B}_N} (dd^c u_K^*)^N \leq \int_{\mathbb{B}_N} (dd^c v)^N = \frac{1}{a_K^N} \int_{\mathbb{B}_N} (dd^c \rho)^N.$$

As $(dd^c \rho)^N = 4^N N! d\lambda_{2N}$, we get, with $C_N := 4^N N! \lambda_{2N}(\mathbb{B}_N)$:

$$\text{Cap}(K) \leq \frac{C_N}{a_K^N}.$$

That ends the proof since:

$$a_K = \min_{z \in K} (1 - |z|^2) \geq \min_{z \in K} (1 - |z|) = \text{dist}(K, \mathbb{S}_N) \quad \square$$

We have assumed that the symbol φ is non-degenerate. For a degenerate symbol φ , we have:

Proposition 4.16. *Let Ω be a bounded hyperconvex and good complete Reinhardt domain in \mathbb{C}^N , and let $\varphi: \Omega \rightarrow \Omega$ be an analytic function such that $\overline{\varphi(\Omega)} \subseteq \Omega$ is pluripolar. Then $\beta_N(C_\varphi) = 0$.*

Recall that $\varphi(\Omega)$ is pluripolar when φ is degenerate (see [44, Proposition 2]); its closure is also pluripolar if it satisfies the condition (4.9).

Proof. Let $K = \overline{\varphi(\Omega)}$. By hypothesis, we have $\text{Cap}(K) = 0$. For every $\varepsilon > 0$, let $K_\varepsilon = \{z \in \Omega; \text{dist}(z, K) \leq \varepsilon\}$. By Theorem 4.7, we have $\beta_N^+(C_\varphi) \leq \Gamma_N(K_\varepsilon)$. As $\lim_{\varepsilon \rightarrow 0} \text{Cap}(K_\varepsilon) = \text{Cap}(K) = 0$ ([30, Proposition 4.7.1(iv)]), we get $\beta_N(C_\varphi) = 0$. \square

Remark 1. In [41, Section 4], we construct a degenerate symbol φ on the bi-disk \mathbb{D}^2 , defined by $\varphi(z_1, z_2) = (\lambda_\theta(z_1), \lambda_\theta(z_1))$, where λ_θ is a lens map, for which $\beta^-(C_\varphi) > 0$. For this function $\overline{\varphi(\mathbb{D}^2)} \cap \partial\mathbb{D}^2 \neq \emptyset$ and hence $\overline{\varphi(\mathbb{D}^2)}$ is not a compact subset of \mathbb{D}^2 .

Remark 2. In the one dimensional case, for any (non constant) analytic map $\varphi: \mathbb{D} \rightarrow \mathbb{D}$, the parameter $\beta(C_\varphi) = \beta_1(C_\varphi)$ is determined by its range $\varphi(\mathbb{D})$, as shown by the formula:

$$\beta(C_\varphi) = e^{-1/\text{Cap}[\varphi(\mathbb{D})]}$$

proved in [40]. This is no longer true in dimension $N \geq 2$. In [42], we construct pairs of (degenerate) symbols $\varphi_1, \varphi_2: \mathbb{D}^2 \rightarrow \mathbb{D}^2$, such that $\varphi_1(\mathbb{D}^2) = \varphi_2(\mathbb{D}^2)$ and:

- 1) C_{φ_1} is not bounded, but C_{φ_2} is compact, and even $\beta_2(C_{\varphi_2}) = 0$;
- 2) C_{φ_1} is bounded but not compact, so $\beta_2(C_{\varphi_1}) = 1$, and C_{φ_2} is compact, with $\beta_2(C_{\varphi_2}) = 0$;
- 3) C_{φ_1} is compact, with $0 < \beta_2(C_{\varphi_1}) < 1$, and C_{φ_2} is compact, with $\beta_2(C_{\varphi_2}) = 0$.

Acknowledgements. We thank S. Nivoche and A. Zeriahi for useful discussions and informations, and Y. Tiba, who send us his paper [51]. We than specially S. Nivoche, who carefully read a preliminary version of this paper.

The third-named author is partially supported by the project MTM2015-63699-P (Spanish MINECO and FEDER funds).

References

- [1] L. Aizenberg and E. Liflyand, *Hardy spaces in Reinhardt domains, and Hausdorff operators*, Illinois J. Math. 53, no. 4 (2009), 1033–1049.
- [2] M. A. Alan, *Hardy spaces on hyperconvex domains*, Master thesis, Middle East Technical University, Ankara, Turkey (2003), <https://etd.lib.metu.tr/upload/1046101/index.pdf>
- [3] F. Bayart, *Composition operators on the polydisc induced by affine maps*, J. Funct. Anal. 260, no. 7 (2011), 1969–2003.
- [4] F. Bayart, D. Li, H. Queffélec and L. Rodríguez-Piazza, *Approximation numbers of composition operators on the Hardy and Bergman spaces of the ball and of the polydisk*, Math. Proc. Cambridge Philos. Soc., to appear (DOI: <https://doi.org/10.1017/S0305004117000263>).
- [5] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Mathematica 149 (1982), 1–40.
- [6] E. Bedford and B. A. Taylor, *Fine topology, Šilov boundary and $(dd^c)^n$* , J. Funct. Anal. 72, no. 2 (1987), 225–251.
- [7] Z. Błocki, *The complex Monge-Ampère operator in pluripotential theory*, Course given at the Jagiellonian University in 1997/98, <http://gamma.im.uj.edu.pl/blocki/publ/ln/wykl.pdf>
- [8] Z. Błocki, *Equilibrium measure of a product subset of \mathbb{C}^n* , Proc. Amer. Math. Soc. 128, no. 12 (2000), 3595–3599.
- [9] S. Bochner, *Classes of holomorphic functions of several variables in circular domains*, Proc. Nat. Acad. Sci. U.S.A. 46, no.5 (1960), 721–723.
- [10] B. Carl and I. Stephani, *Entropy, compactness and the approximation of operators*, Cambridge Tracts in Mathematics 98, Cambridge University Press, Cambridge (1990).
- [11] J. A. Cima, C. S. Stanton and W. R. Wogen, *On boundedness of composition operators on $H^2(B_2)$* , Proc. Amer. Math. Soc. 91, no. 2 (1984), 217–222.
- [12] J. A. Cima and W. R. Wogen, *Unbounded composition operators on $H^2(B_2)$* , Proc. Amer. Math. Soc. 99, no. 3 (1987), 477–483.
- [13] J.-L. Clerc, *Geometry of the Shilov boundary of a bounded symmetric domain*, J. Geom. Symmetry Phys. 13 (2009), 25–74.
- [14] J.-L. Clerc, *Geometry of the Shilov boundary of a bounded symmetric domain*, Geometry, integrability and quantization, 11–55, Avangard Prima, Sofia (2009).

- [15] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL (1995).
- [16] J.-P. Demailly, *Mesures de Monge-Ampère et mesures pluriharmoniques*, Math. Z. 194, no. 4 (1987), 519–564.
- [17] J.-P. Demailly, *Potential theory in several complex variables*, Course given at the CIMPA Summer School in Complex Analysis, Nice, France (1989), https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/nice_cimpa.pdf.
- [18] J.-P. Demailly, *Monge-Ampère operators, Lelong numbers and intersection theory*, Complex analysis and geometry, 115–193, Univ. Ser. Math., Plenum, New York (1993), available at <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/trento.pdf>.
- [19] V. D. Erokhin, *Best linear approximations of functions analytically continuable from a given continuum into a given region*, with an appendix by A. L. Levin and V. M. Tikhomirov, Russ. Math. Surv. 23 (1) (1968), 93–135.
- [20] A. Guichardet. *Leçons sur certaines algèbres topologiques*, Gordon & Breach, Paris-London-New York, distributed by Dunod Editeur (1967).
- [21] K. T. Hahn, *Properties of holomorphic functions of bounded characteristic on star-shaped circular domains*, J. Reine Angew. Math. 254 (1972), 33–40.
- [22] K. T. Hahn and J. Mitchell. *H^p spaces on bounded symmetric domains*, Trans. Amer. Math. Soc. 146 (1969), 521–531.
- [23] K. T. Hahn and J. Mitchell, *H^p spaces on bounded symmetric domains*, Ann. Polon. Math. 28 (1973), 89–95.
- [24] L. A. Harris, *Bounded symmetric homogeneous domains in infinite dimensional spaces*, Proceedings on Infinite Dimensional Holomorphy (Internat. Conf., Univ. Kentucky, Lexington, Ky., 1973), pp. 13–40, Lecture Notes in Math., Vol. 364, Springer, Berlin (1974).
- [25] L. K. Hua *Harmonic analysis of functions of several complex variables in the classical domains*, American Mathematical Society, Providence, R.I. (1963).
- [26] M. Jarnicki and P. Pflug, *First steps in several complex variables: Reinhardt domains*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich (2008).
- [27] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. 183, no. 4 (1983), 503–529.

- [28] M. Klimek, *Extremal plurisubharmonic functions and L -regularity in \mathbb{C}^n* , Proc. Roy. Irish Acad. Sect. A 82, no. 2 (1982), 217–230.
- [29] M. Klimek, *On the invariance of the L -regularity under holomorphic mappings*, Zeszyty Nauk. Uniw. Jagiellon, Prace Mat. No. 23 (1982), 27–38.
- [30] M. Klimek, *Pluripotential theory*, London Mathematical Society Monographs, New Series 6, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York (1991).
- [31] M. Koskenoja, *Pluripotential theory and capacity inequalities*, Ann. Acad. Sci. Fenn. Math. Diss. No. 127 (2002).
- [32] S. G. Krantz, *Function theory of several complex variables*, Pure and Applied Mathematics. A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York (1982).
- [33] C. Laurent-Thiébaud, *Théorie des fonctions holomorphes de plusieurs variables*, Savoirs actuels, InterEditions/ CNRS Éditions (1997).
- [34] J.-S. Lee, J. J. Kim, C.-Y. Oh and S.-H. Park, *Hyperconvexity and neighborhood basis of Reinhardt domains*, Kyushu J. Math. 63, no. 1 (2009), 103–111.
- [35] D. Li and H. Queffélec, *Introduction à l'étude des espaces de Banach \tilde{U} Analyse et probabilités*, Cours Spécialisés 12, Société Mathématique de France (2004).
- [36] D. Li and H. Queffélec, *Introduction to Banach Spaces: Analysis and Probability*, Vol. 1, Cambridge studies in advanced mathematics 166, Cambridge University Press (2017).
- [37] D. Li and H. Queffélec, *Introduction to Banach Spaces: Analysis and Probability*, Vol. 2, Cambridge studies in advanced mathematics 167, Cambridge University Press (2017).
- [38] D. Li, H. Queffélec and L. Rodríguez-Piazza, *On approximation numbers of composition operators*, J. Approx. Theory 164 (4) (2012), 431–459.
- [39] D. Li, H. Queffélec and L. Rodríguez-Piazza, *Estimates for approximation numbers of some classes of composition operators on the Hardy space*, Ann. Acad. Scient. Fennicae 38 (2013), 547–564.
- [40] D. Li, H. Queffélec and L. Rodríguez-Piazza, *A spectral radius formula for approximation numbers of composition operators*, J. Funct. Anal. 160 (12) (2015), 430–454.
- [41] D. Li, H. Queffélec and L. Rodríguez-Piazza, *Some examples of composition operators and their approximation numbers on the Hardy space of the bidisk*, Trans. Amer. Math. Soc., to appear.

- [42] D. Li, H. Queffélec and L. Rodríguez-Piazza, Composition operators with surjective symbol and small approximation numbers, *in preparation*.
- [43] B. MacCluer, Spectra of compact composition operators on $H^p(B_N)$, *Analysis* 4 (1984), 87–103.
- [44] Nguyen Thanh Van and W. Pleśniak, Invariance of L -regularity and Leja’s polynomial condition under holomorphic mappings, *Proc. R. Ir. Acad.* 84A, No. 2 (1984), 111–115.
- [45] S. Nivoche, Proof of a conjecture of Zahariuta concerning a problem of Kolmogorov on the ε -entropy, *Invent. Math.* 158, no. 2 (2004), 413–450.
- [46] W. Pleśniak, Siciak’s extremal function in complex and real analysis, *Ann. Polon. Math.* 80 (2003), 37–46.
- [47] E. A. Poletsky and M. I. Stessin, Hardy and Bergman spaces on hyperconvex domains and their composition operators, *Indiana Univ. Math. J.* 57, no. 5 (2008), 2153–2201.
- [48] W. Rudin, *Function theory in the unit ball of \mathbb{C}^n* , Reprint of the 1980 edition, *Classics in Mathematics*, Springer-Verlag, Berlin (2008).
- [49] S. Şahin, Monge-Ampère measures and Poletsky-Stessin Hardy spaces on bounded hyperconvex domains, *Constructive approximation of functions*, 205–214, *Banach Center Publ.*, 107, Polish Acad. Sci. Inst. Math., Warsaw (2015).
- [50] J. Siciak, Extremal plurisubharmonic functions in \mathbb{C}^n , *Ann. Polon. Math.* 39 (1981), 175–211.
- [51] Y. Tiba, Shilov boundaries of the pluricomplex Green function’s level sets, *Int. Math. Res. Not. IMRN* 2015, no. 24 (2015), 13717–13727.
- [52] J.-P. Vigué, Le groupe des automorphismes analytiques d’un domaine borné d’un espace de Banach complexe. Application aux domaines bornés symétriques, *Ann. Sci. École Norm. Sup. (4)* 9, no. 2 (1976), 203–281.
- [53] H. Widom, Rational approximation and n -dimensional diameter, *J. Approx. Theory* 5 (1972), 342–361.
- [54] Ö. Yazıcı, Kolmogorov problem on widths asymptotics and pluripotential theory, *Master thesis, Sabanci University, Istanbul, Turkey (2008)*, <https://oyazici.expressions.syr.edu/wp-content/uploads/2014/07/tez-4-haz-2.pdf>
- [55] V. Zakharyuta, Kolmogorov problem on widths asymptotics and pluripotential theory, *Functional analysis and complex analysis*, 171–196, *Contemp. Math.* 481, Amer. Math. Soc., Providence, RI (2009).

[56] V. Zakharyuta, Extendible bases and Kolmogorov problem on asymptotics of entropy and widths of some classes of analytic functions, *Annales de la Faculté des Sciences de Toulouse Vol. XX, numéro spécial (2011)*, 211–239.

[57] A. Zeriahi, *Private Communication*.

Daniel Li

Univ. Artois, Laboratoire de Mathématiques de Lens (LML) EA 2462, & Fédération CNRS Nord-Pas-de-Calais FR 2956, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62 300 LENS, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec

Univ. Lille Nord de France, USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS Nord-Pas-de-Calais FR 2956 F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
Herve.Queffelec@univ-lille1.fr

Luis Rodríguez-Piazza

Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Calle Tarfia s/n
41 012 SEVILLA, SPAIN
piazza@us.es