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# Pluricapacity and approximation numbers of composition operators

Daniel Li, Hervé Queffélec, L. Rodríguez-Piazza

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**Abstract.** For suitable bounded hyperconvex sets  $\Omega$  in  $\mathbb{C}^N$ , in particular the ball or the polydisk, we give estimates for the approximation numbers of composition operators  $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$  when  $\varphi(\Omega)$  is relatively compact in  $\Omega$ , involving the Monge-Ampère capacity of  $\varphi(\Omega)$ .

*Key-words:* approximation numbers ; composition operator; Hardy space ; hyperconvex domain ; Monge-Ampère capacity ; pluricapacity ; pluripotential theory ; Zakharyuta conjecture

*MSC 2010 numbers – Primary:* 47B33 – *Secondary:* 30H10 – 31B15 – 32A35 – 32U20 – 41A25 – 41A35 – 46B28 – 46E20 – 47B06

## 1 Introduction

Let  $\mathbb{D}$  be the unit disk in  $\mathbb{C}$ ,  $H^2(\mathbb{D})$  the corresponding Hardy space,  $\varphi$  a non-constant analytic self-map of  $\mathbb{D}$  and  $C_\varphi: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$  the associated composition operator. In [40], we proved a formula connecting the approximation numbers  $a_n(C_\varphi)$  of  $C_\varphi$ , and the Green capacity of the image  $\varphi(\mathbb{D})$  in  $\mathbb{D}$ , namely, when  $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$ , we have:

$$(1.1) \quad \beta(C_\varphi) := \lim_{n \rightarrow \infty} [a_n(C_\varphi)]^{1/n} = \exp(-1/\text{Cap}[\varphi(\mathbb{D})]),$$

where  $\text{Cap}[\varphi(\mathbb{D})]$  is the *Green capacity* of  $\varphi(\mathbb{D})$ .

A non-trivial consequence of that formula was the following:

$$(1.2) \quad \|\varphi\|_\infty = 1 \implies a_n(C_\varphi) \geq \delta e^{-n\varepsilon_n} \text{ where } \varepsilon_n \rightarrow 0_+.$$

In other terms, as soon as  $\|\varphi\|_\infty = 1$ , we cannot hope better for the numbers  $a_n(C_\varphi)$  than a subexponential decay, however slowly  $\varepsilon_n$  tends to 0.

In [41], we pursued that line of investigation in dimension  $N \geq 2$ , namely on  $H^2(\mathbb{D}^N)$ , and showed that in some cases the implication (1.2) still holds ([41, Theorem 3.1]):

$$(1.3) \quad \|\varphi\|_\infty = 1 \implies a_n(C_\varphi) \geq \delta e^{-n^{1/N}\varepsilon_n} \text{ where } \varepsilon_n \rightarrow 0_+,$$

(the substitution of  $n$  by  $n^{1/N}$  is mandatory as shown by the results of [4]).

We show in this paper that, in general, for non-degenerate symbols, we have similar formulas to (1.1) at our disposal for the parameters:

$$(1.4) \quad \beta_N^-(C_\varphi) = \liminf_{n \rightarrow \infty} [a_{n^N}(C_\varphi)]^{1/n} \quad \text{and} \quad \beta_N^+(C_\varphi) = \limsup_{n \rightarrow \infty} [a_{n^N}(C_\varphi)]^{1/n}.$$

These bounds are given in terms of the Monge-Ampère (or Bedford-Taylor) capacity of  $\varphi(\mathbb{D}^N)$  in  $\mathbb{D}^N$ , a notion which is the natural multidimensional extension of the Green capacity when the dimension  $N$  is  $\geq 2$  ([41, Theorem 6.4]). We show that we have  $\beta_N^-(C_\varphi) = \beta_N^+(C_\varphi)$  for well-behaved symbols.

## 2 Notations and background

### 2.1 Complex analysis

Let  $\Omega$  be a domain in  $\mathbb{C}^N$ ; a function  $u: \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  is said *plurisubharmonic* (*psh*) if it is u.s.c. and if for every complex line  $L = \{a + zw; z \in \mathbb{C}\}$  ( $a \in \Omega, w \in \mathbb{C}^N$ ), the function  $z \mapsto u(a + zw)$  is subharmonic in  $\Omega \cap L$ . We denote  $\mathcal{PSH}(\Omega)$  the set of plurisubharmonic functions in  $\Omega$ . If  $f: \Omega \rightarrow \mathbb{C}$  is holomorphic, then  $\log |f|$  and  $|f|^\alpha$ ,  $\alpha > 0$ , are *psh*. Every real-valued convex function is *psh* (convex functions are those whose composition with all  $\mathbb{R}$ -linear isomorphisms are subharmonic, though plurisubharmonic functions are those whose composition with all  $\mathbb{C}$ -linear isomorphisms are subharmonic: see [30, Theorem 2.9.12]).

Let  $dd^c = 2i\partial\bar{\partial}$ , and  $(dd^c)^N = dd^c \wedge \dots \wedge dd^c$  ( $N$  times). When  $u \in \mathcal{PSH}(\Omega) \cap C^2(\Omega)$ , we have:

$$(dd^c u)^N = 4^N N! \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) d\lambda_{2N}(z),$$

where  $d\lambda_{2N}(z) = (i/2)^N dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_N \wedge d\bar{z}_N$  is the usual volume in  $\mathbb{C}^N$ . In general, the current  $(dd^c u)^N$  can be defined for all locally bounded  $u \in \mathcal{PSH}(\Omega)$  and is actually a positive measure on  $\Omega$  ([5]).

Given  $p_1, \dots, p_J \in \Omega$ , the pluricomplex Green function with poles  $p_1, \dots, p_J$  and weights  $c_1, \dots, c_N > 0$  is defined as:

$$g(z) = g(z, p_1, \dots, p_J) = \sup\{v(z); v \in \mathcal{PSH}(\Omega), v \leq 0 \text{ and} \\ v(z) \leq c_j \log |z - p_j| + O(1), \forall j = 1, \dots, J\}.$$

In particular, for  $J = 1$  and  $p_1 = a$ ,  $c_1 = 1$ ,  $g(\cdot, a)$  is the *pluricomplex Green function* of  $\Omega$  with pole  $a \in \Omega$ . If  $0 \in \Omega$  and  $a = 0$ , we denote it by  $g_\Omega$  and call it the *pluricomplex Green function of  $\Omega$* ; hence:

$$g_a(z) = g(z, a) = \sup\{u(z); u \in \mathcal{PSH}(\Omega), u \leq 0 \text{ and } u(z) \leq \log |z - a| + O(1)\}.$$

Let  $\Omega$  be an open subset of  $\mathbb{C}^N$ . A continuous function  $\rho: \Omega \rightarrow \mathbb{R}$  is an exhaustion function if there exists  $a \in (-\infty, +\infty]$  such that  $\rho(z) < a$  for all  $z \in \Omega$ , and the set  $\Omega_c = \{z \in \Omega; \rho(z) < c\}$  is relatively compact in  $\Omega$  for every  $c < a$ .

A domain  $\Omega$  in  $\mathbb{C}^N$  is said *hyperconvex* if there exists a continuous *psh* exhaustion function  $\rho: \Omega \rightarrow (-\infty, 0)$  (see [30, p. 80]). We may of course replace the upper bound 0 by any other real number. Without this upper bound,  $\Omega$  is said *pseudoconvex*.

Let  $\Omega$  be a hyperconvex domain, with negative continuous *psh* exhaustion function  $\rho$  and  $\mu_{\rho,r}$  the associated Demailly-Monge-Ampère measures, defined as:

$$(2.1) \quad \mu_{u,r} = (dd^c u_r)^N - \mathbb{1}_{\Omega \setminus B_{\Omega,u}(r)} (dd^c u)^N,$$

for  $r < 0$ , where  $u_r = \max(u, r)$  and:

$$B_{\Omega,u}(r) = \{z \in \Omega; u(z) < r\}.$$

The nonnegative measure  $\mu_{u,r}$  is supported by  $S_{\Omega,u}(r) := \{z \in \Omega; u(z) = r\}$ .

If:

$$\int_{\Omega} (dd^c \rho)^N < \infty,$$

these measures, considered as measures on  $\overline{\Omega}$ , weak-\* converge, as  $r$  goes to 0, to a positive measure  $\mu = \mu_{\Omega,\rho}$  supported by  $\partial\Omega$  and with total mass  $\int_{\Omega} (dd^c \rho)^N$  ([16, Théorème 3.1], or [30, Lemma 6.5.10]).

For the pluricomplex Green function  $g_a$  with pole  $a$ , we have  $(dd^c g_a)^N = (2\pi)^N \delta_a$  ([16, Théorème 4.3]) and  $g_a(a) = -\infty$ , so  $a \in B_{\Omega,g_a}(r)$  for every  $r < 0$  and  $\mathbb{1}_{\Omega \setminus B_{\Omega,g_a}(r)} (dd^c g_a)^N = 0$ . Hence the Demailly-Monge-Ampère measure  $\mu_{g_a,r}$  is equal to  $(dd^c (g_a)_r)^N$ . By [51, Lemma 1], we have  $(1/|r|) (dd^c (g_a)_r)^N = u_{\overline{B_{\Omega,g_a}(r)}, \Omega}$ , the relative extremal function of  $\overline{B_{\Omega,g_a}(r)} = \{z \in \Omega; g_a(z) \leq r\}$  in  $\Omega$  (see (3.2) for the definition), and this measure is supported, not only by  $S_{\Omega,g_a}(r)$ , but merely by the Shilov boundary of  $\overline{B_{\Omega,g_a}(r)}$  (see Section 2.2.1 for the definition).

Since  $(dd^c g_a)^N = (2\pi)^N \delta_a$  has mass  $(2\pi)^N < \infty$ , these measures weak-\* converge, as  $r$  goes to 0, to a positive measure  $\mu = \mu_{\Omega,g_a}$  supported by  $\partial\Omega$  with mass  $(2\pi)^N$ . Demailly ([16, Définition 5.2] call the measure  $\frac{1}{(2\pi)^N} \mu_{\Omega,g_a}$  the *pluriharmonic measure of  $a$* . When  $\Omega$  is balanced ( $az \in \Omega$  for every  $z \in \Omega$  and  $|a| = 1$ ), the support of this pluriharmonic measure is the Shilov boundary of  $\overline{\Omega}$  ([51, very end of the paper]).

A *bounded symmetric domain* of  $\mathbb{C}^N$  is a bounded open and convex subset  $\Omega$  of  $\mathbb{C}^N$  which is circled ( $az \in \Omega$  for  $z \in \Omega$  and  $|a| \leq 1$ ) and such that for every point  $a \in \Omega$ , there is an involutive bi-holomorphic map  $\gamma: \Omega \rightarrow \Omega$  such that  $a$  is an isolated fixed point of  $\gamma$  (equivalently,  $\gamma(a) = a$  and  $\gamma'(a) = -id$ : see [52, Proposition 3.1.1]). For this definition, see [13, Definition 16 and Theorem 17], or [14, Definition 5 and Theorem 4]. Note that the convexity is automatic

(Hermann Convexity Theorem; see [27, p. 503 and Corollary 4.10]). É. Cartan showed that every bounded symmetric domain of  $\mathbb{C}^N$  is homogeneous, i.e. the group  $\Gamma$  of automorphisms of  $\Omega$  acts transitively on  $\Omega$ : for every  $a, b \in \Omega$ , there is an automorphism  $\gamma$  of  $\Omega$  such that  $\gamma(a) = b$  (see [52, p. 250]). Conversely, every homogeneous bounded convex domain is symmetric, since  $\sigma(z) = -z$  is a symmetry about 0 (see [52, p. 250] or [26, Remark 2.1.2 (e)]). Moreover, each automorphism extends continuously to  $\bar{\Omega}$  (see [22]).

The unit ball  $\mathbb{B}_N$  and the polydisk  $\mathbb{D}^N$  are examples of bounded symmetric domains. Another example is, for  $N = pq$ , bi-holomorphic to the open unit ball of  $\mathcal{M}(p, q) = \mathcal{L}(\mathbb{C}^q, \mathbb{C}^p)$  for the operator norm (see [27, Theorem 4.9]). Every product of bounded symmetric domains is still a bounded symmetric domain. In particular, every product of balls  $\Omega = \mathbb{B}_{l_1} \times \cdots \times \mathbb{B}_{l_m}$ ,  $l_1 + \cdots + l_m = N$ , is a bounded symmetric domain.

If  $\Omega$  is a bounded symmetric domain, its gauge is a norm  $\|\cdot\|$  on  $\mathbb{C}^N$  whose open unit ball is  $\Omega$ . Hence every bounded symmetric domain is hyperconvex (take  $\rho(z) = \|z\| - 1$ ).

## 2.2 Hardy spaces on hyperconvex domains

### 2.2.1 Hardy spaces on bounded symmetric domains

We begin by defining the Hardy space on a bounded symmetric domain, because this is easier.

The *Shilov boundary* (also called the Bergman-Shilov boundary or the distinguished boundary)  $\partial_S \Omega$  of a bounded domain  $\Omega$  is the smallest closed set  $F \subseteq \partial \Omega$  such that  $\sup_{z \in \bar{\Omega}} |f(z)| = \sup_{z \in F} |f(z)|$  for every function  $f$  holomorphic in some neighborhood of  $\bar{\Omega}$  (see [13, § 4.1]).

When  $\Omega$  is a bounded symmetric domain, it is also, since  $\Omega$  is convex, the Shilov boundary of the algebra  $A(\Omega)$  of the continuous functions on  $\Omega$  which are holomorphic in  $\Omega$  (because every function  $f \in A(\Omega)$  can be approximated by  $f_\varepsilon$  with  $f_\varepsilon(z) = f(\varepsilon z_0 + (1 - \varepsilon)z)$ , where  $z_0 \in \Omega$  is given: see [20, pp. 152–154]).

The Shilov boundary of the ball  $\mathbb{B}_N$  is equal to its topological boundary, but the Shilov boundary of the bidisk is  $\partial_S \mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| = |z_2| = 1\}$ , whereas, its usual boundary  $\partial \mathbb{D}^2$  is  $(\mathbb{T} \times \mathbb{D}) \cup (\mathbb{D} \times \mathbb{T})$ ; for the unit ball  $B_N$ , the Shilov boundary is equal to the usual boundary  $\mathbb{S}^{N-1}$  ([13, § 4.1]). Another example of a bounded symmetric domain, in  $\mathbb{C}^3$ , is the set  $\Omega = \{(z_1, z_2, z_3) \in \mathbb{C}^3; |z_1|^2 + |z_2|^2 < 1, |z_3| < 1\}$  and its Shilov boundary is  $\partial_S \Omega = \{(z_1, z_2, z_3); |z_1|^2 + |z_2|^2 = 1, |z_3| = 1\}$ . For  $p \geq q$ , the matrix  $A$  is in the topological boundary of  $\mathcal{M}(p, q)$  if and only if  $\|A\| = 1$ , but  $A$  is in the Shilov boundary if and only if  $A^*A = I_q$ ; therefore the two boundaries coincide if and only if  $q = 1$ , i.e.  $\Omega = \mathbb{B}_N$  (see [14, Example 2, p. 30]).

Equivalently (see [24, Corollary 9], or [13, Theorem 33], [14, Theorem 10]),  $\partial_S \Omega$  is the set of the extreme points of the convex set  $\bar{\Omega}$ .

The Shilov boundary  $\partial_S \Omega$  is invariant by the group  $\Gamma$  of automorphisms of  $\Omega$  and the subgroup  $\Gamma_0 = \{\gamma \in \Gamma; \gamma(0) = 0\}$  act transitively on  $\partial_S \Omega$  (see [22]). A theorem of H. Cartan states that the elements of  $\Gamma_0$  are linear trans-

formations of  $\mathbb{C}^N$  and commute with the rotations (see [24, Theorem 1] or [26, Proposition 2.1.8]). It follows that the Shilov boundary of a bounded symmetric domain  $\Omega$  coincides with its topological boundary only for  $\Omega = \mathbb{B}_N$  (see [35, p. 572] or [36, p. 367]); in particular the open unit ball of  $\mathbb{C}^N$  for the norm  $\|\cdot\|_p$ ,  $1 < p < \infty$ , is never a bounded symmetric domain, unless  $p = 2$ .

The unique  $\Gamma_0$ -invariant probability measure  $\sigma$  on  $\partial_S \Omega$  is the normalized surface area (see [22]). Then the *Hardy space*  $H^2(\Omega)$  is the space of all complex-valued holomorphic functions  $f$  on  $\Omega$  such that:

$$\|f\|_{H^2(\Omega)} := \left( \sup_{0 < r < 1} \int_{\partial_S \Omega} |f(r\xi)|^2 d\sigma(\xi) \right)^{1/2}$$

is finite (see [22] and [23]). It is known that the integrals in this formula are non-decreasing as  $r$  increases to 1, so we can replace the supremum by a limit. The same definition can be given when  $\Omega$  is a bounded complete Reinhardt domain (see [1]).

The space  $H^2(\Omega)$  is a Hilbert space (see [22, Theorem 5]) and for every  $z \in \Omega$ , the evaluation map  $f \in H^2(\Omega) \mapsto f(z)$  is uniformly bounded on compact subsets of  $\Omega$ , by a depending only on that compact set, and of  $\Omega$  ([22, Lemma 3]).

For every  $f \in H^2(\Omega)$ , there exists a boundary values function  $f^*$  such that  $\|f_r - f^*\|_{L^2(\partial_S \Omega)} \xrightarrow{r \rightarrow 1} 0$ , where  $f_r(z) = f(rz)$  ([9, Theorem 3]), and the map  $f \in H^2(\Omega) \mapsto f^* \in L^2(\partial_S \Omega)$  is an isometric embedding ([22, Theorem 6]).

### 2.2.2 Hardy spaces on hyperconvex domains

For hyperconvex domains, the definition of Hardy spaces is more involved. It was done by E. Poletsky and M. Stessin ([47, Theorem 6]). Those domains are associated to a continuous negative *psh* exhaustion function  $u$  on  $\Omega$  and the definition of the Hardy spaces uses the Demailly-Monge-Ampère measures. The space  $H_u^2(\Omega)$  is the space of all holomorphic functions  $f: \Omega \rightarrow \mathbb{C}$  such that:

$$\sup_{r < 0} \int_{S_{\Omega, u}(\Omega)} |f|^2 d\mu_{u, r} < \infty$$

and its norm is defined by:

$$\|f\|_{H_u^2(\Omega)} = \sup_{r < 0} \left( \frac{1}{(2\pi)^N} \int_{S_{\Omega, u}(\Omega)} |f|^2 d\mu_{u, r} \right)^{1/2}.$$

We can replace the supremum by a limit since the integrals are non-decreasing as  $r$  increases to 0 ([16, Corollaire 1.9]).

The space  $H^\infty(\Omega)$  of bounded holomorphic functions in  $\Omega$  is contained in  $H_u^2(\Omega)$  (see [47, remark before Lemma 3.4]).

These spaces  $H_u^2(\Omega)$  are Hilbert spaces ([47, Theorem 4.1]), but depends on the exhaustion function  $u$  (even when  $N = 1$ : see for instance [49]). Nevertheless, they all coincide, with equivalent norms, for the functions  $u$  for which the measure  $(dd^c u)^N$  is compactly supported ([47, Lemma 3.4]); this is the case

when  $u(z) = g(z, a)$  is the pluricomplex Green function with pole  $a \in \Omega$  (because then  $(dd^c u)^N = (2\pi)^N \delta_a$ : see [16, Théorème 4.3], or [30, Theorem 6.3.6]).

When  $\Omega$  is the ball  $\mathbb{B}_N$  and  $u(z) = \log \|z\|_2$ , then  $(dd^c u)^N = C \delta_0$  and  $\mu_{u,r} = (2\pi)^N d\sigma_t$ , where  $d\sigma_t$  is the normalized surface area on the sphere of radius  $t := e^r$  (see [47, Section 4] or [17, Example 3.3]). When  $\Omega$  is the polydisk  $\mathbb{D}^N$  and  $u(z) = \log \|z\|_\infty$ , then  $(dd^c u)^N = (2\pi)^N \delta_0$  ([18, Corollary 5.4]) and  $\frac{1}{(2\pi)^N} \mu_{u,r}$  is the Lebesgue measure of the torus  $r\mathbb{T}^N$  (see [17, Example 3.10]). Note that in [17] and [18], the operator  $d^c$  is defined as  $\frac{i}{2\pi}(\bar{\partial} - \partial)$  instead of  $i(\bar{\partial} - \partial)$ , as usually used.

In these two cases, the Hardy spaces are the same as the usual ones (see [2, Remark 5.2.1]).

In the sequel, we only consider the exhausting function  $u = g_\Omega$ ; hence we will write  $B_\Omega(r)$ ,  $S_\Omega(r)$  and  $H^2(\Omega)$  instead of  $B_{\Omega,u}(r)$ ,  $S_{\Omega,u}(r)$  and  $H_u^2(\Omega)$ .

The two notions of Hardy spaces for a bounded symmetric domain are the same:

**Proposition 2.1.** *Let  $\Omega$  be a bounded symmetric domain in  $\mathbb{C}^N$ . Then the Hardy space  $H^2(\Omega)$  coincides with the subspace of the Poletski-Stessin Hardy space  $H_{g_\Omega}(\Omega)$ , with equality of the norms.*

*Proof.* First let us note that if  $\|\cdot\|$  is the norm whose open unit ball is  $\Omega$ , then  $g_\Omega(z) = \log \|z\|$  (see [7, Proposition 3.3.2]).

Let  $\mu_\Omega$  be the measure which is the  $*$ -weak limit of the Demailly-Monge-Ampère measures  $\mu_r = (dd^c(g_\Omega)_r)^N$ . We saw that it is supported by  $\partial_S \Omega$ . By the remark made in [16, pp. 536-537], since the automorphisms of  $\Omega$  continuously extend on  $\partial\Omega$ , the measure  $\mu_\Omega$  is  $\Gamma$ -invariant. By unicity, the harmonic measure  $\tilde{\mu}_\Omega = (2\pi)^{-N} \mu_\Omega$  at 0 hence coincides with the normalized area measure on  $\partial_S \Omega$ . We have, for  $f: \Omega \rightarrow \mathbb{C}$  holomorphic and  $0 < s < 1$ :

$$\int_{\partial_S \Omega} |f(sz)|^2 d\tilde{\mu}_\Omega(z) = \int_{\partial\Omega} |f(sz)|^2 d\tilde{\mu}_\Omega(z) = \lim_{r \rightarrow 0} \frac{1}{(2\pi)^N} \int_{S_\Omega(r)} |f(sz)|^2 d\mu_r(z),$$

because  $z \mapsto |f(sz)|^2$  is continuous on  $\bar{\Omega}$ . Now, since  $g_\Omega(z) = \log \|z\|$ , we have  $S_\Omega(r) = e^r \partial\Omega$  and  $(g_\Omega)_r(z) + t = (g_\Omega)_{r+t}(sz)$ ; hence  $\mu_r(sA) = \mu_{r+t}(A)$  for every Borel subset  $A$  of  $\partial\Omega$ , where  $t = \log s$ . It follows that:

$$\int_{S_\Omega(r)} |f(sz)|^2 d\mu_r(z) = \int_{S_\Omega(r+t)} |f(\zeta)|^2 d\mu_{r+t}(\zeta).$$

By letting  $r$  and  $t$  going to 0, we get:

$$\|f\|_{H^2(\Omega)}^2 = \lim_{r,t \rightarrow 0} \frac{1}{(2\pi)^N} \int_{S_\Omega(r+t)} |f(\zeta)|^2 d\mu_{r+t}(\zeta) = \|f\|_{H_{g_\Omega}^2}^2;$$

hence  $f \in H^2(\Omega)$  if and only if  $f \in H_{g_\Omega}^2(\Omega)$ , with the same norms.  $\square$

We have ([47, Theorem 3.6]):

**Proposition 2.2** (Poletsky-Stessin). *For every  $z \in \Omega$ , the evaluation map  $f \in H^2(\Omega) \mapsto f(z)$  is uniformly bounded on compact subsets of  $\Omega$ , by a constant depending only on that compact set, and of  $\Omega$ .*

Hence  $H^2(\Omega)$  has a reproducing kernel, defined by:

$$(2.2) \quad f(a) = \langle f, K_a \rangle, \quad \text{for } f \in H^2(\Omega),$$

and for each  $r < 0$ :

$$(2.3) \quad L_r := \sup_{a \in \overline{B_\Omega(r)}} \|K_a\|_2 < \infty.$$

### 2.3 Composition operators

A Schur map, associated with the bounded hyperconvex domain  $\Omega$ , is a *non-constant* analytic map of  $\Omega$  into itself. It is said *non degenerate* if its Jacobian is not identically null. It is equivalent to say that the differential  $\varphi'(a): \mathbb{C}^N \rightarrow \mathbb{C}^N$  is an invertible linear map for at least one point  $a \in \Omega$ . In [4], we used the terminology *truly  $N$ -dimensional*. Then, by the implicit function theorem, this is equivalent to saying that  $\varphi(\Omega)$  has non-void interior. We say that the Schur map  $\varphi$  is a *symbol* if it defines a *bounded* composition operator  $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$  by  $C_\varphi(f) = f \circ \varphi$ .

Let us recall that although any Schur function generates a bounded composition operator on  $H^2(\mathbb{D})$ , this is no longer the case on  $H^2(\mathbb{D}^N)$  as soon as  $N \geq 2$ , as shown for example by the Schur map  $\varphi(z_1, z_2) = (z_1, z_1)$ . Indeed (see [3]), if say  $N = 2$ , taking  $f(z) = \sum_{j=0}^n z_1^j z_2^{n-j}$ , we see that:

$$\|f\|_2 = \sqrt{n+1} \quad \text{while} \quad \|C_\varphi f\|_2 = \|(n+1)z_1^n\|_2 = n+1.$$

The same phenomenon occurs on  $H^2(\mathbb{B}_N)$  ([43]; see also [11], [12], and [15]; see also [47]).

### 2.4 $s$ -numbers of operators on a Hilbert space

We begin by recalling a few operator-theoretic facts. Let  $H$  be a Hilbert space. The approximation numbers  $a_n(T) = a_n$  of an operator  $T: H \rightarrow H$  are defined as:

$$(2.4) \quad a_n = \inf_{\text{rank } R < n} \|T - R\|, \quad n = 1, 2, \dots$$

The operator  $T$  is compact if and only if  $\lim_{n \rightarrow \infty} a_n(T) = 0$ .

According to a result of Allahverdiev [10, p. 155],  $a_n = s_n$ , the  $n$ -th singular number of  $T$ , i.e. the  $n$ -th eigenvalue of  $|T| := \sqrt{T^*T}$  when those eigenvalues are rearranged in non-increasing order.

The  $n$ -th width  $d_n(K)$  of a subset  $K$  of a Banach space  $Y$  measures the defect of flatness of  $K$  and is by definition:

$$(2.5) \quad d_n(K) = \inf_{\dim E < n} \left[ \sup_{f \in K} \text{dist}(f, E) \right],$$



where  $E$  runs over all subspaces of  $Y$  with dimension  $< n$  and where  $\text{dist}(f, E)$  denotes the distance of  $f$  to  $E$ . If  $T: X \rightarrow Y$  is an operator between Banach spaces, the  $n$ -th Kolmogorov number  $d_n(T)$  of  $T$  is the  $n$ th-width in  $Y$  of  $T(B_X)$  where  $B_X$  is the closed unit ball of  $X$ , namely:

$$(2.6) \quad d_n(T) = \inf_{\dim E < n} \left[ \sup_{f \in B_X} \text{dist}(Tf, E) \right].$$

In the case where  $X = Y = H$ , a Hilbert space, we have:

$$(2.7) \quad a_n(T) = d_n(T) \quad \text{for all } n \geq 1,$$

and ([40]) the following alternative definition of  $a_n(T)$ :

$$(2.8) \quad a_n(T) = \inf_{\dim E < n} \left[ \sup_{f \in B_H} \text{dist}(Tf, TE) \right].$$

In this work, we use, for an operator  $T: H \rightarrow H$ , the following notation:

$$(2.9) \quad \beta_N^-(T) = \liminf_{n \rightarrow \infty} [a_{n^N}(T)]^{1/n}$$

and:

$$(2.10) \quad \beta_N^+(T) = \limsup_{n \rightarrow \infty} [a_{n^N}(T)]^{1/n}.$$

When these two quantities are equal, we write them  $\beta_N(T)$ .

### 3 Pluripotential theory

#### 3.1 Monge-Ampère capacity

Let  $K$  be a compact subset of an open subset  $\Omega$  of  $\mathbb{C}^N$ . The *Monge-Ampère capacity* of  $K$  has been defined by Bedford and Taylor ([5]; see also [30, Part II, Chapter 1]) as:

$$\text{Cap}(K) = \sup \left\{ \int_K (dd^c u)^N; u \in \mathcal{PSH}(\Omega) \text{ and } 0 \leq u \leq 1 \text{ on } \Omega \right\}.$$

When  $\Omega$  is bounded and hyperconvex, we have a more convenient formula ([5, Proposition 5.3], [30, Proposition 4.6.1]):

$$(3.1) \quad \text{Cap}(K) = \int_{\Omega} (dd^c u_K^*)^N = \int_K (dd^c u_K^*)^N,$$

(the positive measure  $(dd^c u_K^*)^N$  is supported by  $K$ ; actually by  $\partial K$ : see [17, Properties 8.1 (c)]), where  $u_K = u_{K, \Omega}$  is the *relative extremal function* of  $K$ , defined, for any subset  $E \subseteq \Omega$ , as:

$$(3.2) \quad u_{E, \Omega} = \sup \{ v \in \mathcal{PSH}(\Omega); v \leq 0 \text{ and } v \leq -1 \text{ on } E \},$$

and  $u_{E,\Omega}^*$  is its upper semi-continuous regularization:

$$u_{E,\Omega}^*(z) = \limsup_{\zeta \rightarrow z} u_{E,\Omega}(\zeta), \quad z \in \Omega,$$

called the *regularized relative extremal function* of  $E$ .

For an open subset  $\omega$  of  $\Omega$ , its capacity is defined as:

$$\text{Cap}(\omega) = \sup\{\text{Cap}(K); K \text{ is a compact subset of } \omega\}.$$

When  $\bar{\omega} \subset \Omega$  is a compact subset of  $\Omega$ , we have ([5, equation (6.2)], [30, Corollary 4.6.2]):

$$(3.3) \quad \text{Cap}(\omega) = \int_{\Omega} (dd^c u_{\omega})^N.$$

The *outer capacity* of a subset  $E \subseteq \Omega$  is:

$$\text{Cap}^*(E) = \inf\{\text{Cap}(\omega); \omega \supseteq E \text{ and } \omega \text{ open}\}.$$

If  $\Omega$  is hyperconvex and  $E$  relatively compact in  $\Omega$ , then ([30, Proposition 4.7.2]):

$$\text{Cap}^*(E) = \int_{\Omega} (dd^c u_{E,\Omega}^*)^N.$$

**Remark.** A. Zeriahi ([57]) pointed out to us the following result.

**Proposition 3.1.** *Let  $K$  be a compact subset of  $\Omega$ . Then:*

$$\text{Cap}(K) = \text{Cap}(\partial K).$$

*Proof.* Of course  $u_K \leq u_{\partial K}$  since  $\partial K \subseteq K$ . Conversely, let  $v \in \mathcal{PSH}(\Omega)$  non-positive such that  $v \leq -1$  on  $\partial K$ . By the maximum principle (see [30, Corollary 2.9.6]), we get that  $v \leq -1$  on  $K$ . Hence  $v \leq u_K$ . Taking the supremum over all those  $v$ , we obtain  $u_{\partial K} \leq u_K$ , and therefore  $u_{\partial K} = u_K$ .

By (3.1), it follows that:

$$(3.4) \quad \text{Cap}(K) = \int_{\Omega} (dd^c u_K^*)^N = \int_{\Omega} (dd^c u_{\partial K}^*)^N = \text{Cap}(\partial K). \quad \square$$

### 3.2 Regular sets

Let  $E \subseteq \mathbb{C}^N$  be bounded. Recall that the polynomial convex hull of  $E$  is:

$$\widehat{E} = \{z \in \mathbb{C}; |P(z)| \leq \sup_E |P| \text{ for every polynomial } P\}.$$

A point  $a \in \widehat{E}$  is called *regular* if  $u_{E,\Omega}^*(a) = -1$  for an open set  $\Omega \supseteq \widehat{E}$  (note that we always have  $u_{E,\Omega} = u_{E,\Omega} = -1$  on the interior of  $E$ : see [17, Properties 8.1 (c)]). The set  $E$  is said to be *regular* if all points of  $\widehat{E}$  are regular.

The *pluricomplex Green function* of  $E$ , also called the *L-extremal function* of  $E$ , is defined, for  $z \in \mathbb{C}^N$ , as:

$$V_E(z) = \sup\{v(z); v \in \mathcal{L}, v \leq 0 \text{ on } E\},$$

where  $\mathcal{L}$  is the *Lelong class* of all functions  $v \in \mathcal{PSH}(\mathbb{C}^N)$  such that, for some constant  $C > 0$ :

$$v(z) \leq C + \log(1 + |z|) \quad \text{for all } z \in \mathbb{C}^N.$$

A point  $a \in \widehat{E}$  is called *L-regular* if  $V_E^*(a) = 0$ , where  $V_E^*$  is the upper semi-continuous regularization of  $V_E$ . The set  $E$  is *L-regular* if all points of  $\widehat{E}$  are *L-regular*.

By [28, Proposition 2.2] (see also [30, Proposition 5.3.3, and Corollary 5.3.4]), for  $E$  bounded and non pluripolar, and  $\Omega$  a bounded open neighbourhood of  $\widehat{E}$ , we have:

$$(3.5) \quad m(u_{E,\Omega} + 1) \leq V_E \leq M(u_{E,\Omega} + 1)$$

for some positive constants  $m, M$ . Hence the regularity of  $a \in \widehat{E}$  is equivalent to its *L-regularity*.

Recall that  $E$  is pluripolar if there exists an open set  $\Omega$  containing  $E$  and  $v \in \mathcal{PSH}(\Omega)$  such that  $E \subseteq \{v = -\infty\}$ . This is equivalent to say that there exists a hyperconvex domain  $\Omega$  of  $\mathbb{C}^N$  containing  $E$  such that  $u_{E,\Omega}^* \equiv 0$  (see [30, Corollary 4.7.3 and Theorem 4.7.5]). By Josefson's theorem ([30, Theorem 4.7.4]),  $E$  is pluripolar if and only if there exists  $v \in \mathcal{PSH}(\mathbb{C}^N)$  such that  $E \subseteq \{v = -\infty\}$ . Recall also that  $E$  is pluripolar if and only if its outer capacity  $\text{Cap}^*(E)$  is null ([30, Theorem 4.7.5]).

When  $\Omega$  is hyperconvex and  $E$  is compact, non pluripolar, the regularity of  $E$  implies that  $u_{E,\Omega}$  and  $V_E$  are continuous, on  $\Omega$  and  $\mathbb{C}^N$  respectively ([30, Proposition 4.5.3 and Corollary 5.1.4]). Conversely, if  $u_{E,\Omega}$  is continuous, for some hyperconvex neighbourhood  $\Omega$  of  $E$ , then  $u_{E,\Omega}(z) = -1$  for all  $z \in E$ ; hence  $V_E(z) = 0$  for all  $z \in E$ , by (3.5); but  $V_E = V_{\widehat{E}}$  when  $E$  is compact ([30, Theorem 5.1.7]), so  $V_E(z) = 0$  for all  $z \in \widehat{E}$ ; by (3.5) again, we obtain that  $u_{E,\Omega}(z) = -1$  for all  $z \in \widehat{E}$ ; therefore  $E$  is regular. In the same way, the continuity of  $V_E$  implies the regularity of  $E$ . These results are due to Siciak ([50, Proposition 6.1 and Proposition 6.2]).

Every closed ball  $B = B(a, r)$  of an arbitrary norm  $\|\cdot\|$  on  $\mathbb{C}^N$  is regular since its *L-extremal function* is:

$$V_B(z) = \log^+(\|z - a\|/r)$$

([50, p. 179, § 2.6]).

### 3.3 Zakharyuta's formula

We will need a formula that Zakharyuta, in order to solve a problem raised by Kolmogorov, proved, conditionally to a conjecture, called Zakharyuta's conjecture, on the uniform approximation of the relative extremal function  $u_{K,\Omega}$

by pluricomplex Green functions. This conjecture has been proved by Nivoche ([45, Theorem A]), in a more general setting that we state below:

**Theorem 3.2** (Nivoche). *Let  $K$  be a regular compact subset of a bounded hyperconvex domain  $\Omega$  of  $\mathbb{C}^N$ . Then for every  $\varepsilon > 0$  and  $\delta$  small enough, there exists a pluricomplex Green function  $g$  on  $\Omega$  with a finite number of logarithmic poles such that:*

- 1) *the poles of  $g$  lie in  $W = \{z \in \Omega; u_K(z) < -1 + \delta\}$ ;*
- 2) *we have, for every  $z \in \bar{\Omega} \setminus W$ :*

$$(1 + \varepsilon)g(z) \leq u_K(z) \leq (1 - \varepsilon)g(z).$$

In order to state Zakharyuta's formula, we need some additional notations.

Let  $K$  be a compact subset of  $\Omega$  with non-empty interior, and  $A_K$  the set of restrictions to  $K$  of those functions that are analytic and bounded by 1, i.e. those functions belonging to the unit ball  $B_{H^\infty(\Omega)}$  of the space  $H^\infty(\Omega)$  of the bounded analytic functions in  $\Omega$ , considered as a subset of the space  $\mathcal{C}(K)$  of complex functions defined on  $K$ , equipped with the sup-norm on  $K$ .

Let  $d_n(A_K)$  be the  $n$ th-width of  $A_K$  in  $\mathcal{C}(K)$ , namely:

$$(3.6) \quad d_n(A_K) = \inf_L \left[ \sup_{f \in A_K} \text{dist}(f, L) \right],$$

where  $L$  runs over all  $k$ -dimensional subspaces of  $\mathcal{C}(K)$ , with  $k < n$ .

Equivalently,  $d_n(A_K)$  is the  $n$ th-Kolmogorov number of the natural injection  $J$  of  $H^\infty(\Omega)$  into  $\mathcal{C}(K)$  (recall that  $K$  has non-empty interior). It is convenient to set, as in [56]:

$$(3.7) \quad \tau_N(K) = \frac{1}{(2\pi)^N} \text{Cap}(K)$$

and:

$$(3.8) \quad \Gamma_N(K) = \exp \left[ - \left( \frac{N!}{\tau_N(K)} \right)^{1/N} \right],$$

i.e.:

$$(3.9) \quad \Gamma_N(K) = \exp \left[ - 2\pi \left( \frac{N!}{\text{Cap}(K)} \right)^{1/N} \right].$$

Observe that  $\text{Cap}(K) > 0$  since we assumed that  $K$  has non-empty interior. Now, we have ([56, Theorem 5.6]; see also [55, Theorem 5] or [54, pages 30–32], for a detailed proof):

**Theorem 3.3** (Zakharyuta-Nivoche). *Let  $\Omega$  be a bounded hyperconvex domain and  $K$  a regular compact subset of  $\Omega$  with non-empty interior, which is holomorphically convex in  $\Omega$  (i.e.  $K = \tilde{K}_\Omega$ ). Then:*

$$(3.10) \quad -\log d_n(A_K) \sim \left( \frac{N!}{\tau_N(K)} \right)^{1/N} n^{1/N}.$$

Here  $\tilde{K}_\Omega$  is the holomorphic convex hull of  $K$  in  $\Omega$ , that is:

$$\tilde{K}_\Omega = \{z \in \Omega; |f(z)| \leq \sup_K |f| \text{ for every } f \in \mathcal{O}(\Omega)\},$$

where  $\mathcal{O}(\Omega)$  is the set of all functions holomorphic in  $\Omega$ .

Relying on that theorem, which may be seen as the extension of a result of Erokhin, proved in 1958 (see [19]; see also Widom [53] which proved a more general result, with a different proof), to dimension  $N > 1$ , and as a result on the approximation of functions, we will give an application to the study of approximation numbers of a composition operator on  $H^2(\Omega)$  for a bounded symmetric domain of  $\mathbb{C}^N$ .

## 4 The spectral radius type formula

In [41, Section 6.2], we proved the following result.

**Theorem 4.1.** *Let  $\varphi: \mathbb{D}^N \rightarrow \mathbb{D}^N$  be given by  $\varphi(z_1, \dots, z_N) = (r_1 z_1, \dots, r_N z_N)$  where  $0 < r_j < 1$ . Then:*

$$\beta_N(C_\varphi) = \Gamma_N[\overline{\varphi(\mathbb{D}^N)}] = \Gamma_N[\varphi(\mathbb{D}^N)].$$

The proof was simple, based on result of Blocki [8] on the Monge-Ampère capacity of a cartesian product, and on the estimation, when  $A \rightarrow \infty$ , of the number  $\nu_A$  of  $N$ -tuples  $\alpha = (\alpha_1, \dots, \alpha_N)$  of non-negative integers  $\alpha_j$  such that  $\sum_{j=1}^N \alpha_j \sigma_j \leq A$ , where the numbers  $\sigma_j > 0$  are fixed. The estimation was:

$$(4.1) \quad \nu_A \sim \frac{A^N}{N! \sigma_1 \cdots \sigma_N}.$$

As J. F. Burnol pointed out to us, this is a consequence of the following elementary fact. Let  $\lambda_N$  be the Lebesgue measure on  $\mathbb{R}^N$ , and let  $E$  be a compact subset of  $\mathbb{R}^N$  such that  $\lambda_N(\partial E) = 0$ . Then:

$$\lambda_N(E) = \lim_{A \rightarrow \infty} A^{-N} |(A \times E) \cap \mathbb{Z}^N|.$$

Then, just take  $E = \{(x_1, \dots, x_N); x_j \geq 0 \text{ and } \sum_{j=1}^N x_j \sigma_j \leq 1\}$ .

In any case, this lets us suspect that the formula of Theorem 4.1 holds in much more general cases. This is not quite true, as evidenced by our counterexample of [41, Theorem 5.12]. Nevertheless, in good cases, this formula holds, as we will see in the next sections.

In remaining of this section, we consider functions  $\varphi: \Omega \rightarrow \Omega$  such that  $\overline{\varphi(\Omega)} \subseteq \Omega$ . If  $\rho$  is an exhaustion function for  $\Omega$ , there is some  $R_0 < 0$  such that  $\overline{\varphi(\Omega)} \subseteq B_\Omega(R_0)$ , and that implies that  $C_\varphi$  maps  $H^2(\Omega)$  into itself and is a compact operator (see [47, Theorem 8.3], since, with their notations, for  $r > R_0$ , we have  $T(r) = \emptyset$  and hence  $\delta_\varphi(r) = 0$ ).

## 4.1 Minoration

Recall that every hyperconvex domain  $\Omega$  is pseudoconvex. By H. Cartan-Thullen and Oka-Bremermann-Norguet theorems, being pseudoconvex is equivalent to being a domain of holomorphy, and equivalent to being holomorphically convex (meaning that if  $K$  is a compact subset in  $\Omega$ , then its holomorphic hull  $\widetilde{K}$  is also contained in  $\Omega$ ): see [33, Corollaire 7.7]. Now (see [32, Chapter 5, Exercise 11]), a domain of holomorphy  $\Omega$  is said a *Runge domain* if every holomorphic function in  $\Omega$  can be approximated uniformly on its compact subsets by polynomials, and that is equivalent to saying that the polynomial hull and the holomorphic hull of every compact subset of  $\Omega$  agree. By [32, Chapter 5, Exercise 13], every circled domain (in particular every bounded symmetric domain) is a Runge domain.

**Definition 4.2.** *A hyperconvex domain  $\Omega$  is said strongly regular if there exists a continuous psh exhaustion function  $\rho$  such that all the sub-level sets:*

$$\Omega_c = \{z \in \Omega; \rho(z) < c\}$$

( $c < 0$ ) have a regular closure.

For example, every bounded symmetric domain  $\Omega$  is strongly regular since if  $\|\cdot\|$  is the associated norm, its sub-level sets  $\Omega_c$  (with  $\rho(z) = \log \|z\|$ ) are the open balls  $B(0, e^c)$ , and the closed balls are regular, as said above.

**Theorem 4.3.** *Let  $\Omega$  be a strongly regular bounded hyperconvex and Runge domain in  $\mathbb{C}^N$ , and let  $\varphi: \Omega \rightarrow \mathbb{C}$  be an analytic function such that  $\overline{\varphi(\Omega)} \subseteq \Omega$ , and which is non-degenerate. Then:*

$$(4.2) \quad \Gamma_N[\varphi(\Omega)] \leq \beta_N^-(C_\varphi).$$

Recall that if  $\Omega$  is a domain in  $\mathbb{C}^N$ , a holomorphic function  $\varphi: \Omega \rightarrow \mathbb{C}^M$  ( $M \leq N$ ) is *non-degenerate* if there exists  $a \in \Omega$  such that  $\text{rank}_a \varphi = M$ . Then  $\varphi(\Omega)$  has a non-empty interior.

*Proof.* Let  $(r_j)_{j \geq 1}$  be an increasing sequence of negative numbers tending to 0. The set  $H_j = \overline{\Omega_{r_j}}$  is a regular compact subset of  $\Omega$ , with non-void interior (hence non pluripolar). Let  $\widehat{H}_j$  its polynomial convex hull; this compact set is contained in  $\Omega$ , since  $\Omega$  being a Runge domain, we have  $\widehat{H}_j = \widetilde{H}_j$ , and since  $\widetilde{H}_j \subseteq \Omega$ , because  $\Omega$  is holomorphically convex (being hyperconvex). Moreover  $\widehat{H}_j$  is regular since  $V_E = V_{\widehat{E}}$  for every compact subset of  $\mathbb{C}^N$  ([50, Corollary 4.14]).

Let  $K_j = \varphi(\widehat{H}_j)$  and let  $G$  be a subspace of  $H^2(\Omega)$  with dimension  $< n^N$ .

The set  $K_j$  is regular because of the following result (see [30, Theorem 5.3.9], [46, top of page 40], [29, Theorem 1.3], or [44, Theorem 4], with a detailed proof).

**Theorem 4.4** (Pleśniak). *Let  $E$  be a compact, polynomially convex, regular and non pluripolar, subset of  $\mathbb{C}^N$ . Then if  $\Omega$  is a hyperconvex domain such that  $E \subseteq \Omega$  and if  $\varphi: \Omega \rightarrow \mathbb{C}^N$  is a non-degenerate holomorphic function, the set  $\varphi(E)$  is regular.*

As before, the polynomial convex hull  $\widehat{K}_j$  of  $K_j$  is contained in  $\Omega$  and is also regular. Since  $\varphi$  is non-degenerate,  $K_j$  has a non-void interior; hence  $\widehat{K}_j$  also. We can hence use Zakharyuta's formula (Theorem 3.3) for the compact set  $\widehat{K}_j$ .

By restriction, the subspace  $G$  can be viewed as a subspace of  $\mathcal{C}(\widehat{K}_j)$ . By Zakharyuta's formula, for  $0 < \varepsilon < 1$ , there is  $n_\varepsilon \geq 1$  such that, for  $n \geq n_\varepsilon$ :

$$d_{n^N}(A_{\widehat{K}_j}) \geq \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

Hence, there exists  $f \in B_{H^\infty} \subseteq B_{H^2}$  such that, for all  $g \in G$ :

$$\|g - f\|_{\mathcal{C}(\widehat{K}_j)} \geq (1 - \varepsilon) \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

Since  $\widehat{K}_j = \widetilde{K}_j$  and, by definition  $\|\cdot\|_{\mathcal{C}(\widetilde{K}_j)} = \|\cdot\|_{\mathcal{C}(K_j)}$ , we have:

$$\|g - f\|_{\mathcal{C}(\widehat{K}_j)} = \|g - f\|_{\mathcal{C}(K_j)} = \|C_\varphi(g) - C_\varphi(f)\|_{\mathcal{C}(\widetilde{H}_j)}.$$

Equivalently, since, by definition  $\|\cdot\|_{\mathcal{C}(\widetilde{H}_j)} = \|\cdot\|_{\mathcal{C}(H_j)}$ , we have, for all  $g \in G$ :

$$\|C_\varphi(g) - C_\varphi(f)\|_{\mathcal{C}(H_j)} \geq (1 - \varepsilon) \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

This implies, thanks to (2.3), that, for all  $g \in G$ :

$$\|C_\varphi(g) - C_\varphi(f)\|_{H^2(\Omega)} \geq L_{r_j}^{-1} (1 - \varepsilon) \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

Using (2.8), we get, since the subspace  $G$  is arbitrary:

$$a_{n^N}(C_\varphi) \geq L_{r_j}^{-1} (1 - \varepsilon) \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

Taking the  $n$ th-roots and passing to the limit, we obtain:

$$\beta_N^-(C_\varphi) \geq \exp \left[ - (1 + \varepsilon) (2\pi) \left( \frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right].$$

and then, letting  $\varepsilon$  go to 0:

$$\beta_N^-(C_\varphi) \geq \exp \left[ - (2\pi) \left( \frac{N!}{\text{Cap}(\widehat{K}_j)} \right)^{1/N} \right] = \Gamma_N(\widehat{K}_j).$$

Now, the sequence  $(\widehat{K}_j)_{j \geq 1}$  is increasing and  $\bigcup_{j \geq 1} \widehat{K}_j \supseteq \varphi(\Omega)$ ; hence, by [5, Theorem 8.2 (8.3)], we have  $\text{Cap}(\widehat{K}_j) \xrightarrow{j \rightarrow \infty} \text{Cap}(\bigcup_{j \geq 1} \widehat{K}_j) \geq \text{Cap}[\varphi(\Omega)]$ , so:

$$\beta_N^-(C_\varphi) \geq \Gamma_N[\varphi(\Omega)],$$

and the proof of Theorem 4.3 is finished.  $\square$

## 4.2 Majorization

For the majorization, we assume different hypotheses on the domain  $\Omega$ . Nevertheless these assumptions agree with that of Theorem 4.3 when  $\Omega$  is a bounded symmetric domain.

### 4.2.1 Preliminaries

Recall that a domain  $\Omega$  of  $\mathbb{C}^N$  is a *Reinhardt domain* (resp. *complete Reinhardt domain*) if  $z = (z_1, \dots, z_N) \in \Omega$  implies that  $(\zeta_1 z_1, \dots, \zeta_N z_N) \in \Omega$  for all complex numbers  $\zeta_1, \dots, \zeta_N$  of modulus 1 (resp. of modulus  $\leq 1$ ). A complete bounded Reinhardt domain is hyperconvex if and only if  $\log j_\Omega$  is *psh* and continuous in  $\mathbb{C}^N \setminus \{0\}$ , where  $j_\Omega$  is the Minkowski functional of  $\Omega$  (see [7, Exercise following Proposition 3.3.3]). In general, the Minkowski functional  $j_\Omega$  of a bounded complete Reinhardt domain  $\Omega$  is *usc* and  $\log j_\Omega$  is *psh* if and only if  $\Omega$  is pseudoconvex ([7, Theorem 1.4.8]). Other conditions for a bounded complete Reinhardt domain to being hyperconvex can found in [34, Theorem 3.10].

For a bounded hyperconvex and complete Reinhardt domain  $\Omega$ , its pluricomplex Green function with pole 0 is  $g_\Omega(z) = \log j_\Omega(z)$ , where  $j_\Omega$  is the Minkowski functional of  $\Omega$  ([7, Proposition 3.3.2]), and  $S_\Omega(r) = e^r \partial\Omega$ . Since  $\partial\Omega$  is in particular invariant by the pluri-rotations  $z = (z_1, \dots, z_N) \mapsto (e^{i\theta_1} z_1, \dots, e^{i\theta_N} z_N)$ , with  $\theta_1, \dots, \theta_N \in \mathbb{R}$ , the harmonic measure  $\tilde{\mu}_\Omega$  at 0 (see the proof of Proposition 2.1) is also invariant by the pluri-rotations (note that it is supported by the Shilov boundary of  $\bar{\Omega}$ : see [51, very end of the paper]). We have, as in the proof of Proposition 2.1, for  $f \in H^2(\Omega)$ :

$$\sup_{0 < s < 1} \int_{\partial\Omega} |f(sz)|^2 d\tilde{\mu}_\Omega(z) = \|f\|_{H^2(\Omega)}^2 < \infty.$$

Since  $\tilde{\mu}_\Omega$  is in particular invariant by the rotations  $z \mapsto e^{i\theta} z$ ,  $\theta \in \mathbb{R}$ , there exists, by [9, Theorem 3], a function  $f^* \in L^2(\partial\Omega, \tilde{\mu}_\Omega)$  such that:

$$\int_{\partial\Omega} |f(sz) - f^*(z)|^2 d\tilde{\mu}_\Omega(z) \xrightarrow{s \rightarrow 1} 0.$$

It ensues that the map  $f \in H^2(\Omega) \mapsto f^* \in L^2(\partial\Omega, \tilde{\mu}_\Omega)$  is an isometric embedding (in fact,  $f^*$  is the radial limit of  $f$ : see [21, Lemma 2]). Therefore, we can consider  $H^2(\Omega)$  as a complemented subspace of  $L^2(\partial\Omega, \tilde{\mu}_\Omega)$ , and we call  $P$  the orthogonal projection of  $L^2(\partial\Omega, \tilde{\mu}_\Omega)$  onto  $H^2(\Omega)$ .

Every holomorphic function  $f$  in a Reinhardt domain  $\Omega$  containing 0 (in particular if  $\Omega$  is a complete Reinhardt domain) has a power series expansion about 0:

$$f(z) = \sum_{\alpha} b_{\alpha} z^{\alpha}$$

which converges normally on compact subsets of  $\Omega$  ([32, Proposition 2.3.14]). Recall that if  $z = (z_1, \dots, z_N)$  and  $\alpha = (\alpha_1, \dots, \alpha_N)$ , then  $z^{\alpha} = z_1^{\alpha_1} \dots z_N^{\alpha_N}$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_N$ , and  $\alpha! = \alpha_1! \dots \alpha_N!$ .

We have:



**Proposition 4.5.** *Let  $\Omega$  be a bounded hyperconvex and complete Reinhardt domain, and set  $e_\alpha(z) = z^\alpha$ . Then the system  $(e_\alpha)_\alpha$  is orthogonal in  $H^2(\Omega)$ .*

*Proof.* We use the fact that the level sets  $S(r)$  and the Demailly-Monge-Ampère measures  $\mu_r = (dd^c(g_\Omega)_r)^N$  are pluri-rotation invariant. For  $\alpha \neq \beta$ , we choose  $\theta_1, \dots, \theta_N \in \mathbb{R}$  such that  $1, (\theta_1/2\pi), \dots, (\theta_N/2\pi)$  are rationally independent. Then  $\exp[i(\sum_{j=1}^N (\alpha_j - \beta_j)\theta_j)] \neq 1$ . Hence, as in [25, p. 78], we have, making the change of variables  $z = (e^{i\theta_1} w_1, \dots, e^{i\theta_N} w_N)$ :

$$\int_{S(r)} z^\alpha \overline{z^\beta} d\mu_r(z) = \exp\left[i\left(\sum_{j=1}^N (\alpha_j - \beta_j)\theta_j\right)\right] \int_{S(r)} w^\alpha \overline{w^\beta} d\mu_r(w),$$

which implies that:

$$\int_{S(r)} z^\alpha \overline{z^\beta} d\mu_r(z) = 0,$$

and hence:

$$(z^\alpha | z^\beta) := \lim_{r \rightarrow 0} \int_{S(r)} z^\alpha \overline{z^\beta} d\mu_r(z) = 0. \quad \square$$

For the polydisk, we have  $\|e_\alpha\|_{H^2(\mathbb{D}^N)} = 1$ , and for the ball (see [48, Proposition 1.4.9]):

$$\|e_\alpha\|_{H^2(\mathbb{B}_N)}^2 = \frac{(N-1)! \alpha!}{(N-1+|\alpha|)!}.$$

**Definition 4.6.** *We say that  $\Omega$  is a good complete Reinhardt domain if, for some positive constant  $C_N$  and some positive integer  $c$ , we have, for all  $p \geq 0$ :*

$$\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\Omega)}^2} \leq C_N p^{cN} [j_\Omega(z)]^{2p},$$

where  $j_\Omega$  is the Minkowski functional of  $\Omega$ .

### Examples

**1.** The polydisk  $\mathbb{D}^N$  is a good Reinhardt domain because  $\|e_\alpha\|_{H^2(\mathbb{D}^N)} = 1$ ,  $|z^\alpha| \leq \|z\|_\infty^{|\alpha|}$ , and the number of indices  $\alpha$  such that  $|\alpha| = p$  is  $\binom{N-1+p}{p} \leq C_N p^N$  (see [35, p. 498] or [37, pp. 213–214]).

**2.** The ball  $\mathbb{B}_N$  is a good Reinhardt domain. In fact, observe that:

$$\frac{(N-1+p)!}{(N-1)!} = p! \frac{(p+1)(p+2)\cdots(p+N-1)}{1 \times 2 \times \cdots \times (N-1)} \leq p! (p+1)^{N-1} \leq p! (p+1)^N;$$

hence:

$$\begin{aligned} \sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\mathbb{B}_N)}^2} &= \sum_{|\alpha|=p} |z^\alpha|^2 \frac{(N-1+|\alpha|)!}{(N-1)! \alpha!} \\ &\leq (p+1)^N \sum_{|\alpha|=p} \frac{|\alpha|!}{\alpha!} |z_1|^{2\alpha_1} \cdots |z_N|^{2\alpha_N} \\ &= (p+1)^N (|z_1|^2 + \cdots + |z_N|^2)^p, \end{aligned}$$

by the multinomial formula, so:

$$\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\mathbb{B}_N)}^2} \leq (p+1)^N \|z\|_2^{2p} \leq 2^N p^N \|z\|_2^{2p}.$$

**3.** More generally, if  $\Omega = \mathbb{B}_{l_1} \times \cdots \times \mathbb{B}_{l_m}$ ,  $l_1 + \cdots + l_m = N$ , is a product of balls, we have, writing  $\alpha = (\beta_1, \dots, \beta_m)$ , where each  $\beta_j$  is an  $l_j$ -tuple:

$$\begin{aligned} \|e_\alpha\|_{H^2(\Omega)}^2 &= \int_{\mathbb{S}_{l_1} \times \cdots \times \mathbb{S}_{l_m}} |u_1^{\beta_1}|^2 \cdots |u_m^{\beta_m}|^2 d\sigma_{l_1}(u_1) \cdots d\sigma_{l_m}(u_m) \\ &= \prod_{j=1}^m \frac{(l_j - 1)! \beta_j!}{(l_j - 1 + |\beta_j|)!}, \end{aligned}$$

and, writing  $z = (z_1, \dots, z_m)$ , with  $z_j \in \mathbb{B}_{l_j}$ :

$$\begin{aligned} \sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\Omega)}^2} &\leq \sum_{p_1 + \cdots + p_m = p} \prod_{j=1}^m (p_j + 1)^{l_j} \|z_j\|_2^{2p_j} \\ &\leq C_m p^m (p+1)^{l_1 + \cdots + l_m} [j_\Omega(z)]^{2(p_1 + \cdots + p_m)}, \end{aligned}$$

since  $j_\Omega(z) = \max\{\|z_1\|_2, \dots, \|z_m\|_2\}$ . Hence:

$$\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|_{H^2(\Omega)}^2} \leq C_N p^{2N} [j_\Omega(z)]^{2p}.$$

#### 4.2.2 The result

**Theorem 4.7.** *Let  $\Omega$  be a bounded hyperconvex domain which is a good complete Reinhardt domain in  $\mathbb{C}^N$ , and let  $\varphi: \Omega \rightarrow \Omega$  be an analytic function such that  $\overline{\varphi(\Omega)} \subseteq \Omega$ . Then, for every compact subset  $K \supseteq \varphi(\Omega)$  of  $\Omega$  with non void interior, we have:*

$$(4.3) \quad \beta_N^+(C_\varphi) \leq \Gamma_N(K).$$

*In particular, if  $\varphi$  is moreover non-degenerate, we have:*

$$(4.4) \quad \beta_N^+(C_\varphi) \leq \Gamma_N[\overline{\varphi(\Omega)}].$$

The last assertion holds because  $\varphi(\Omega)$  is open if  $\varphi$  is non-degenerate.

**Corollary 4.8.** *Let  $\Omega$  be a good complete bounded symmetric domain in  $\mathbb{C}^N$ , and  $\varphi: \Omega \rightarrow \Omega$  a non-degenerate analytic map such that  $\overline{\varphi(\Omega)} \subseteq \Omega$ . Then:*

$$\Gamma_N[\varphi(\Omega)] \leq \beta_N^-(C_\varphi) \leq \beta_N^+(C_\varphi) \leq \Gamma_N[\overline{\varphi(\Omega)}].$$

For the proof of Theorem 4.7, we will use the following result ([56, Proposition 6.1]), which do not need any regularity condition on the compact set (because it may be written as a decreasing sequence of regular compact sets).

**Proposition 4.9** (Zakharyuta). *If  $K$  is any compact subset of a bounded hyperconvex domain  $\Omega$  of  $\mathbb{C}^N$  with non-empty interior, we have:*

$$\limsup_{n \rightarrow \infty} \frac{\log d_n(A_K)}{n^{1/N}} \leq - \left( \frac{N!}{\tau_N(K)} \right)^{1/N}.$$

*Proof of Theorem 4.7.* In the sequel we write  $\|\cdot\|_{H^2}$  for  $\|\cdot\|_{H^2(\Omega)}$ . We set:

$$\Lambda_N = \limsup_{n \rightarrow \infty} [d_n(A_K)]^{n^{-1/N}}.$$

Changing  $n$  into  $n^N$ , Proposition 4.9 means that for every  $\varepsilon > 0$ , there exists, for  $n$  large enough, an  $(n^N - 1)$ -dimensional subspace  $F$  of  $\mathcal{C}(K)$  such that, for any  $g \in H^\infty(\Omega)$ , there exists  $h \in F$  such that:

$$(4.5) \quad \|g - h\|_{\mathcal{C}(K)} \leq (1 + \varepsilon)^n \Lambda_N^n \|g\|_\infty.$$

Let  $l$  be an integer to be adjusted later, and

$$f(z) = \sum_{\alpha} b_{\alpha} z^{\alpha} \in H^2(\Omega) \quad \text{with } \|f\|_{H^2} \leq 1.$$

By Proposition 4.5, we have:

$$\|f\|_{H^2}^2 = \sum_{\alpha} |b_{\alpha}|^2 \|e_{\alpha}\|_{H^2}^2.$$

We set:

$$g(z) = \sum_{|\alpha| \leq l} b_{\alpha} z^{\alpha}.$$

By the Cauchy-Schwarz inequality:

$$|g(z)|^2 \leq \left( \sum_{|\alpha| \leq l} |b_{\alpha}|^2 \|e_{\alpha}\|_{H^2}^2 \right) \left( \sum_{|\alpha| \leq l} \frac{|z^{\alpha}|^2}{\|e_{\alpha}\|_{H^2}^2} \right) \leq \sum_{|\alpha| \leq l} \frac{|z^{\alpha}|^2}{\|e_{\alpha}\|_{H^2}^2}.$$

Since  $\Omega$  is a good complete Reinhardt domain and since  $j_{\Omega}(z) < 1$  for  $z \in \Omega$ , we have:

$$|g(z)|^2 \leq \sum_{p=0}^l p^{cN} [j_{\Omega}(z)]^{2p} \leq (l+1)^{cN+1}.$$

It follows from (4.5) that there exists  $h \in F$  such that:

$$\|g - h\|_{\mathcal{C}(K)} \leq (1 + \varepsilon)^n \Lambda_N^n (l+1)^{(cN+1)/2}.$$

Since  $C_{\varphi} f(z) - C_{\varphi} g(z) = f(\varphi(z)) - g(\varphi(z))$  and  $\overline{\varphi(\Omega)} \subseteq K$ , we have  $\|C_{\varphi} f - C_{\varphi} g\|_{\infty} \leq \|f - g\|_{\mathcal{C}(K)}$ ; therefore:

$$(4.6) \quad \begin{aligned} \|g \circ \varphi - h \circ \varphi\|_{H^2} &\leq \|g \circ \varphi - h \circ \varphi\|_{\infty} \leq \|g - h\|_{\mathcal{C}(K)} \\ &\leq (1 + \varepsilon)^n \Lambda_N^n (l+1)^{(cN+1)/2}. \end{aligned}$$

Now, the subspace  $\tilde{F}$  formed by functions  $v \circ \varphi$ , for  $v \in F$ , can be viewed as a subspace of  $L^\infty(\partial\Omega, \tilde{\mu}_\Omega) \subseteq L^2(\partial\Omega, \tilde{\mu}_\Omega)$  (indeed, since  $v$  is continuous, we can write  $(v \circ \varphi)^* = v \circ \varphi^*$ , where  $\varphi^*$  denotes the almost everywhere existing radial limits of  $\varphi(rz)$ , which belong to  $K$ ). Let finally  $E = P(\tilde{F}) \subseteq H^2(\Omega)$  where  $P: L^2(\partial\Omega, \tilde{\mu}_\Omega) \rightarrow H^2(\Omega)$  is the orthogonal projection. This is a subspace of  $H^2(\Omega)$  with dimension  $< n^N$ , and we have  $\text{dist}(C_\varphi g, E) \leq \|g \circ \varphi - P(h \circ \varphi)\|_{H^2}$ ; hence, by (4.6):

$$(4.7) \quad \text{dist}(C_\varphi g, E) \leq (1 + \varepsilon)^n \Lambda_N^n (l + 1)^{(cN+1)/2}.$$

Now, the same calculations give that:

$$|f(z) - g(z)|^2 \leq \sum_{p>l} p^{cN} [j_\Omega(z)]^{2p};$$

hence, for some positive constant  $M_N$ :

$$|f(z) - g(z)| \leq M_N (l + 1)^{(cN+1)/2} \frac{[j_\Omega(z)]^l}{(1 - [j_\Omega(z)]^2)^{(cN+1)/2}},$$

by using the following lemma, whose proof is postponed.

**Lemma 4.10.** *For every non-negative integer  $m$ , there exists a positive constant  $A_m$  such that, for all integers  $l \geq 0$  and all  $0 < x < 1$ , we have:*

$$\sum_{p \geq l} p^m x^p \leq A_m l^m \frac{x^l}{(1 - x)^{m+1}}.$$

Since  $K$  is a compact subset of  $\Omega$ , there is a positive number  $r_0 < 1$  such that  $j_\Omega(z) \leq r_0$  for  $z \in K$ . Since  $C_\varphi f(z) - C_\varphi g(z) = f(\varphi(z)) - g(\varphi(z))$  and  $\overline{\varphi(\Omega)} \subseteq K$ , we have  $\|C_\varphi f - C_\varphi g\|_\infty \leq \|f - g\|_{C(K)}$ , and we get:

$$(4.8) \quad \|C_\varphi f - C_\varphi g\|_{H^2} \leq \|C_\varphi f - C_\varphi g\|_\infty \leq M_N (l + 1)^{(cN+1)/2} \frac{r_0^l}{(1 - r_0^2)^{(cN+1)/2}}.$$

Now, (4.7) and (4.8) give:

$$\text{dist}(C_\varphi f, E) \leq M_N (l + 1)^{(cN+1)/2} \left( \frac{r_0^l}{(1 - r_0^2)^{(cN+1)/2}} + (1 + \varepsilon)^n \Lambda_N^n \right).$$

It ensues, thanks to (2.7), that:

$$[a_{nN}(C_\varphi)]^{1/n} \leq [M_N (l + 1)^{(cN+1)/2}]^{1/n} \left[ \frac{r_0^{l/n}}{(1 - r_0^2)^{(cN+1)/2n}} + (1 + \varepsilon) \Lambda_N \right].$$

Taking now for  $l$  the integer part of  $n \log n$ , and passing to the upper limit as  $n \rightarrow \infty$ , we obtain (since  $l/n \rightarrow \infty$  and  $(\log l)/n \rightarrow 0$ ):

$$\beta_N^+(C_\varphi) \leq (1 + \varepsilon) \Lambda_N,$$

and therefore, since  $\varepsilon > 0$  is arbitrary:

$$\beta_N^+(C_\varphi) \leq \Lambda_N.$$

That ends the proof, by using Proposition 4.9.  $\square$

*Proof of Lemma 4.10.* We make the proof by induction on  $m$ . We set:

$$S_m = \sum_{p \geq l} p^m x^p$$

The result is obvious for  $m = 0$ , with  $A_0 = 1$ , since then  $S_0 = \sum_{p \geq l} x^p = \frac{x^l}{1-x}$ . Let us assume that it holds till  $m - 1$  and prove it for  $m$ . We observe that, since  $p^m - (p - 1)^m \leq mp^{m-1}$ , we have:

$$\begin{aligned} (1-x)S_m &= \sum_{p \geq l} p^m x^p - \sum_{p \geq l} p^m x^{p+1} = \sum_{p \geq l} p^m x^p - \sum_{p \geq l+1} (p-1)^m x^p \\ &= \sum_{p \geq l+1} (p^m - (p-1)^m) x^p + l^m x^l \leq \sum_{p \geq l+1} mp^{m-1} x^p + l^m x^l \\ &\leq \sum_{p \geq l} mp^{m-1} x^p + l^m x^l \leq mA_{m-1} l^{m-1} \frac{x^l}{(1-x)^m} + l^m x^l \\ &\leq (mA_{m-1} + 1) l^m \frac{x^l}{(1-x)^m}, \end{aligned}$$

giving the result, with  $A_m = mA_{m-1} + 1$ .  $\square$

### 4.3 Equality

**Proposition 4.11.** *Let  $\Omega$  be a bounded hyperconvex domain and  $\omega$  a relatively compact open subset of  $\Omega$ . Assume that:*

$$(4.9) \quad \begin{aligned} &\text{For every } a \in \partial\omega, \text{ except on a pluripolar set } E \subseteq \partial\omega, \text{ there exists} \\ &z_0 \in \omega \text{ such that the open segment } (z_0, a) \text{ is contained in } \omega. \end{aligned}$$

Then:

$$\text{Cap}(\bar{\omega}) = \text{Cap}(\omega).$$

*In particular, if  $\varphi: \Omega \rightarrow \Omega$  a non-degenerate holomorphic map such that  $\overline{\varphi(\Omega)} \subseteq \Omega$  and  $\omega = \varphi(\Omega)$  satisfies (4.9), we have:*

$$\text{Cap}[\varphi(\Omega)] = \text{Cap}[\overline{\varphi(\Omega)}].$$

Before proving Proposition 4.11, let us give an example of such a situation.

**Proposition 4.12.** *Let  $\Omega$  be a bounded hyperconvex domain with  $C^1$  boundary. Let  $U$  be an open neighbourhood of  $\bar{\Omega}$  and  $\varphi: U \rightarrow \mathbb{C}^N$  be a non-degenerate holomorphic function such that  $\overline{\varphi(\Omega)} \subseteq \Omega$ . Then the condition (4.9) is satisfied.*

*Proof.* Let  $\omega = \varphi(\Omega)$ .

We may assume that  $U$  is connected, hyperconvex and bounded. Let  $B_\varphi$  be the set of points  $z \in U$  such that the complex Jacobian  $J_\varphi$  is null. Since  $J_\varphi$  is holomorphic in  $\Omega$ , we have  $\log |J_\varphi| \in \mathcal{PSH}(U)$  and hence (see [31, proof of Lemma 10.2]):

$$B_\varphi = \{z \in U; J_\varphi(z) = 0\} = \{z \in U; \log |J_\varphi(z)| = -\infty\}$$

is pluripolar. Therefore (see [5, Theorem 6.9]),  $\text{Cap}(B_\varphi, U) = 0$ . It follows (see [5, page 2, line -8]) that  $\text{Cap}[\varphi(B_\varphi)] := \text{Cap}[\varphi(B_\varphi), \Omega] = 0$ .

Now, for every  $a \in \partial\bar{\omega} \cap [\varphi(U \setminus B_\varphi)]$ , there is a tangent hyperplane  $H_a$  to  $\bar{\omega}$ , and hence an inward normal to  $\partial\bar{\omega}$  (note that  $\partial\bar{\omega} \subseteq \varphi(\partial\Omega) \subseteq \varphi(U)$ ). It follows that there is  $z_0 \in \omega$  such that the open interval  $(z_0, a)$  is contained in  $\omega$ .  $\square$

*Proof of Proposition 4.11.* Let  $a \in \partial\omega$  and  $L$  be a complex line containing  $(z_0, a)$ ; we have  $a \in \overline{\omega \cap L}$ . Assume now that this point  $a$  is a *fine* (“*effilé*”) point of  $\omega$ , i.e. that there exists  $u \in \mathcal{PSH}(V)$ , for  $V$  a neighbourhood of  $a$ , such that:

$$\limsup_{z \rightarrow a, z \in \omega} u(z) < u(a).$$

By definition, the restriction  $\tilde{u}$  of  $u$  to  $\omega \cap L$  is subharmonic and we keep the inequality:

$$\limsup_{z \rightarrow a, z \in \omega \cap L} \tilde{u}(z) < \tilde{u}(a) = u(a).$$

That means that  $a$  is a fine point of  $\omega \cap L$ . But  $a \in \overline{\omega \cap L}$  and  $\omega \cap L$  is connected, so this is not possible, by [40, Lemma 2.4]. Hence no point of  $\partial\omega \setminus E$  is fine.

Let now  $\omega^f$  be the closure of  $\omega$  for the fine topology (i.e. the coarsest topology on  $U$  for which all the functions in  $\mathcal{PSH}(U)$  are continuous; it is known: see [6, comment after Theorem 2.3], that it is the trace on  $U$  of the fine topology on  $\mathbb{C}^N$ ). It is also known (see [30, Corollary 4.8.10]) that  $\omega^f$  is the set of points of  $\bar{\omega}$  which are not fine. By the above reasoning, we thus have:

$$\bar{\omega} \setminus \omega^f \subseteq E.$$

Since  $\text{Cap}(E) = 0$ , we have:

$$\text{Cap}(\bar{\omega} \setminus \omega^f) = 0,$$

and it follows that:

$$\text{Cap}(\bar{\omega}) = \text{Cap}[\omega^f \cup (\bar{\omega} \setminus \omega^f)] \leq \text{Cap}(\omega^f) + \text{Cap}(\bar{\omega} \setminus \omega^f) = \text{Cap}(\omega^f),$$

and hence  $\text{Cap}(\omega^f) = \text{Cap}(\bar{\omega})$ .

But, since, by definition, the *psH* functions are continuous for the fine topology, it is clear, that the relative extremal functions  $u_{\omega, \Omega}$  and  $u_{\omega^f, \Omega}$  are equal; hence we have, by [30, Proposition 4.7.2]:

$$\text{Cap}(\omega) = \int_{\Omega} (dd^c u_{\omega, \Omega}^*)^N = \int_{\Omega} (dd^c u_{\omega^f, \Omega}^*)^N = \text{Cap}(\omega^f).$$

Hence  $\text{Cap}(\omega) = \text{Cap}(\bar{\omega})$ .  $\square$

## 4.4 Consequences of the spectral radius type formula

Theorem 4.3 has the following consequence.

**Proposition 4.13.** *Let  $\Omega$  be a regular bounded symmetric domain in  $\mathbb{C}^N$ , and let  $\varphi: \Omega \rightarrow \Omega$  be a non-degenerate analytic function inducing a bounded composition operator  $C_\varphi$  on  $H^2(\Omega)$ .*

*Then, if  $\text{Cap}[\varphi(\Omega)] = \infty$ , we have  $\beta_N(C_\varphi) = 1$ .*

*In other words, if, for some constants  $C, c > 0$ , we have  $a_n(C_\varphi) \leq C e^{-cn^{1/N}}$  for all  $n \geq 1$ , then  $\text{Cap}[\varphi(\Omega)] < \infty$ .*

As a corollary, we can give a new proof of [41, Theorem 3.1].

**Corollary 4.14.** *Let  $\tau: \mathbb{D} \rightarrow \mathbb{D}$  be an analytic map such that  $\|\tau\|_\infty = 1$  and  $\psi: \mathbb{D}^{N-1} \rightarrow \mathbb{D}^{N-1}$  such that the map  $\varphi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ , defined as:*

$$\varphi(z_1, z_2, \dots, z_N) = (\tau(z_1), \psi(z_2, \dots, z_N)),$$

*is non-degenerate. Then  $\beta_N(C_\varphi) = 1$ .*

*Proof.* Since the map  $\varphi$  is non-degenerate,  $\psi$  is also non-degenerate. Hence (see [44, Proposition 2])  $\psi(\mathbb{D}^{N-1})$  is not pluripolar, i.e.  $\text{Cap}_{N-1}[\psi(\mathbb{D}^{N-1})] > 0$ . On the other hand, it follows from [40, Theorem 3.13 and Theorem 3.14] that  $\text{Cap}_1[\tau(\mathbb{D})] = +\infty$ . Then, by [8, Theorem 3], we have:

$$\begin{aligned} \text{Cap}_N[\varphi(\mathbb{D}^N)] &= \text{Cap}_N[\tau(\mathbb{D}) \times \psi(\mathbb{D}^{N-1})] \\ &= \text{Cap}_1[\tau(\mathbb{D})] \times \text{Cap}_{N-1}[\psi(\mathbb{D}^{N-1})] = +\infty. \end{aligned}$$

It follows from Proposition 4.13 that  $\beta_N(C_\varphi) = 1$ . □

*Proof of Proposition 4.13.* If  $R: H^2(\Omega) \rightarrow H^2(\Omega)$  is a finite-rank operator, we set, for  $t < 0$ :

$$(R_t f)(w) = (Rf)(e^t w), \quad f \in H^2(\Omega).$$

Then the rank of the operator  $R_t$  is less or equal to that of  $R$ .

Recall that if  $\|\cdot\|$  is the norm whose unit ball is  $\Omega$ , then the pluricomplex Green function of  $\Omega$  is  $g_\Omega(z) = \log \|z\|$ , and hence the level set  $S(r)$  is the sphere  $S(0, e^r) = e^r \partial\Omega$  for this norm. Since:

$$\int_{S(r)} |f[\varphi(e^t w)] - (Rf)(e^t w)|^2 d\mu_r(w) = \int_{S(r+t)} |f[\varphi(z)] - (Rf)(z)|^2 d\mu_{r+t}(z),$$

we have, setting  $\varphi_t(w) = \varphi(e^t w)$ :

$$\|C_{\varphi_t}(f) - R_t(f)\|_{H^2} \leq \|C_\varphi(f) - R(f)\|_{H^2}.$$

It follows that  $a_n(C_{\varphi_t}) \leq a_n(C_\varphi)$  for every  $n \geq 1$ . Therefore  $\beta_N^-(C_{\varphi_t}) \leq \beta_N^-(C_\varphi)$ .

By Theorem 4.3, we have:

$$\exp \left[ -2\pi \left( \frac{N!}{\text{Cap}[\varphi_t(\Omega)]} \right)^{1/N} \right] \leq \beta_N^-(C_{\varphi_t}).$$

Since  $\varphi_t(\Omega) = \varphi(e^t\Omega)$  increases to  $\varphi(\Omega)$  as  $t \uparrow 0$ , we have (see [30, Corollary 4.7.11]):

$$\text{Cap}[\varphi(\Omega)] = \lim_{t \rightarrow 0} \text{Cap}[\varphi_t(\Omega)].$$

As  $\text{Cap}[\varphi(\Omega)] = \infty$ , we get:

$$\beta_N^-(C_\varphi) \geq \limsup_{t \rightarrow 0} \beta_N^-(C_{\varphi_t}) = 1. \quad \square$$

**Remark 1.** In [41, Theorem 5.12], we construct a non-degenerate analytic function  $\varphi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$  such that  $\overline{\varphi(\mathbb{D}^2)} \cap \partial\mathbb{D}^2 \neq \emptyset$  and for which  $\beta_2^+(C_\varphi) < 1$ . We hence have  $\text{Cap}[\varphi(\mathbb{D}^2)] < \infty$ .

**Remark 2.** The capacity cannot tend to infinity too fast when the compact set approaches the boundary of  $\Omega$ ; in fact, we have the following result, that we state for the ball, but which holds more generally.

**Proposition 4.15.** *For every compact set  $K$  of  $\mathbb{B}_N$ , we have, for some constant  $C_N$ :*

$$\text{Cap}(K) \leq \frac{C_N}{[\text{dist}(K, \mathbb{S}_N)]^N}.$$

*Proof.* We know that:

$$\text{Cap}(K) = \int_{\mathbb{B}_N} (dd^c u_K^*)^N.$$

Let  $\rho(z) = |z|^2 - 1$  and  $a_K := \min_{z \in K} [-\rho(z)] = -\max_{z \in K} \rho(z)$ . Then  $\rho$  is in  $\mathcal{PSH}$  and is non-positive. Since  $a_K > 0$ , the function:

$$v(z) = \frac{\rho(z)}{a_K}$$

is in  $\mathcal{PSH}$ , non-positive on  $\mathbb{B}_N$ , and  $v \leq -1$  on  $K$ . Hence  $v \leq u_K \leq u_K^*$ .

Since  $v(w) = 0$  for all  $w \in \mathbb{S}_N$  and (see [5, Proposition 6.2 (iv)], or [30, Proposition 4.5.2]):

$$\lim_{z \rightarrow w} u_K^*(z) = 0,$$

for all  $w \in \mathbb{S}_N$ , the comparison theorem of Bedford and Taylor ([5, Theorem 4.1]; [30, Theorem 3.7.1] gives, since  $v \leq u_K^*$  and  $v, u_K^* \in \mathcal{PSH}$ :

$$\int_{\mathbb{B}_N} (dd^c u_K^*)^N \leq \int_{\mathbb{B}_N} (dd^c v)^N = \frac{1}{a_K^N} \int_{\mathbb{B}_N} (dd^c \rho)^N.$$

As  $(dd^c \rho)^N = 4^N N! d\lambda_{2N}$ , we get, with  $C_N := 4^N N! \lambda_{2N}(\mathbb{B}_N)$ :

$$\text{Cap}(K) \leq \frac{C_N}{a_K^N}.$$



That ends the proof since:

$$a_K = \min_{z \in K} (1 - |z|^2) \geq \min_{z \in K} (1 - |z|) = \text{dist}(K, \mathbb{S}_N) \quad \square$$

We have assumed that the symbol  $\varphi$  is non-degenerate. For a degenerate symbol  $\varphi$ , we have:

**Proposition 4.16.** *Let  $\Omega$  be a bounded hyperconvex and good complete Reinhardt domain in  $\mathbb{C}^N$ , and let  $\varphi: \Omega \rightarrow \Omega$  be an analytic function such that  $\overline{\varphi(\Omega)} \subseteq \Omega$  is pluripolar. Then  $\beta_N(C_\varphi) = 0$ .*

Recall that  $\varphi(\Omega)$  is pluripolar when  $\varphi$  is degenerate (see [44, Proposition 2]); its closure is also pluripolar if it satisfies the condition (4.9).

*Proof.* Let  $K = \overline{\varphi(\Omega)}$ . By hypothesis, we have  $\text{Cap}(K) = 0$ . For every  $\varepsilon > 0$ , let  $K_\varepsilon = \{z \in \Omega; \text{dist}(z, K) \leq \varepsilon\}$ . By Theorem 4.7, we have  $\beta_N^+(C_\varphi) \leq \Gamma_N(K_\varepsilon)$ . As  $\lim_{\varepsilon \rightarrow 0} \text{Cap}(K_\varepsilon) = \text{Cap}(K) = 0$  ([30, Proposition 4.7.1(iv)]), we get  $\beta_N(C_\varphi) = 0$ .  $\square$

**Remark 1.** In [41, Section 4], we construct a degenerate symbol  $\varphi$  on the bi-disk  $\mathbb{D}^2$ , defined by  $\varphi(z_1, z_2) = (\lambda_\theta(z_1), \lambda_\theta(z_1))$ , where  $\lambda_\theta$  is a lens map, for which  $\beta^-(C_\varphi) > 0$ . For this function  $\overline{\varphi(\mathbb{D}^2)} \cap \partial\mathbb{D}^2 \neq \emptyset$  and hence  $\overline{\varphi(\mathbb{D}^2)}$  is not a compact subset of  $\mathbb{D}^2$ .

**Remark 2.** In the one dimensional case, for any (non constant) analytic map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ , the parameter  $\beta(C_\varphi) = \beta_1(C_\varphi)$  is determined by its range  $\varphi(\mathbb{D})$ , as shown by the formula:

$$\beta(C_\varphi) = e^{-1/\text{Cap}[\varphi(\mathbb{D})]}$$

proved in [40]. This is no longer true in dimension  $N \geq 2$ . In [42], we construct pairs of (degenerate) symbols  $\varphi_1, \varphi_2: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ , such that  $\varphi_1(\mathbb{D}^2) = \varphi_2(\mathbb{D}^2)$  and:

- 1)  $C_{\varphi_1}$  is not bounded, but  $C_{\varphi_2}$  is compact, and even  $\beta_2(C_{\varphi_2}) = 0$ ;
- 2)  $C_{\varphi_1}$  is bounded but not compact, so  $\beta_2(C_{\varphi_1}) = 1$ , and  $C_{\varphi_2}$  is compact, with  $\beta_2(C_{\varphi_2}) = 0$ ;
- 3)  $C_{\varphi_1}$  is compact, with  $0 < \beta_2(C_{\varphi_1}) < 1$ , and  $C_{\varphi_2}$  is compact, with  $\beta_2(C_{\varphi_2}) = 0$ .

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