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# Some examples of composition operators and their approximation numbers on the Hardy space of the bi-disk

Daniel Li, Hervé Queffélec, L. Rodríguez-Piazza

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**Abstract.** We give examples of composition operators  $C_\Phi$  on  $H^2(\mathbb{D}^2)$  showing that the condition  $\|\Phi\|_\infty = 1$  is not sufficient for their approximation numbers  $a_n(C_\Phi)$  to satisfy  $\lim_{n \rightarrow \infty} [a_n(C_\Phi)]^{1/\sqrt{n}} = 1$ , contrary to the 1-dimensional case. We also give a situation where this implication holds. We make a link with the Monge-Ampère capacity of the image of  $\Phi$ .

*Key-words:* approximation numbers; Bergman space; bidisk; composition operator; Green capacity; Hardy space; Monge-Ampère capacity; weighted composition operator.

*MSC 2010 numbers – Primary:* 47B33 – *Secondary:* 30H10 – 30H20 – 31B15 – 32A35 – 32U20 – 41A35 – 46B28

## 1 Introduction and notation

### 1.1 Introduction

The purpose of this paper is to continue the study of composition operators on the polydisk initiated in [2], and in particular to examine to what extent one of the main results of [21] still holds.

Let  $H$  be a Hilbert space and  $T: H \rightarrow H$  a bounded operator. Recall that the *approximation numbers* of  $T$  are defined as:

$$a_n(T) = \inf_{\text{rank } R < n} \|T - R\|, \quad n \geq 1,$$

and we have:

$$\|T\| = a_1(T) \geq a_2(T) \geq \cdots \geq a_n(T) \geq \cdots$$

The operator  $T$  is compact if and only if  $a_n(T) \xrightarrow{n \rightarrow \infty} 0$ .

For  $d \geq 1$ , we define:

$$\begin{cases} \beta_d^-(T) &= \liminf_{n \rightarrow \infty} [a_{n^d}(T)]^{1/n} \\ \beta_d^+(T) &= \limsup_{n \rightarrow \infty} [a_{n^d}(T)]^{1/n} \end{cases}$$

We have:

$$0 \leq \beta_d^-(T) \leq \beta_d^+(T) \leq 1,$$

and we simply write  $\beta_d(T)$  in case of equality.

It may well happen in general (consider diagonal operators) that  $\beta_d^-(T) = 0$  and  $\beta_d^+(T) = 1$ .

When  $H = H^2(\mathbb{D})$  is the Hardy space on the open unit disk  $\mathbb{D}$  of  $\mathbb{C}$ , and  $T = C_\Phi$  is a composition operator, with  $\Phi: \mathbb{D} \rightarrow \mathbb{D}$  a non-constant analytic function, we always have ([19]):

$$\beta_1^-(C_\Phi) > 0,$$

and one of the main results of [19] is the equivalence:

$$(1.1) \quad \beta_1^+(C_\Phi) < 1 \iff \|\Phi\|_\infty < 1.$$

An alternative proof was given in [21], as a consequence of a so-called ‘‘spectral radius formula’’, which moreover shows that:

$$\beta_1^-(C_\Phi) = \beta_1^+(C_\Phi).$$

In [2], for  $d \geq 2$ , it is proved that, for a bounded symmetric domain  $\Omega \subseteq \mathbb{C}^d$ , if  $\Phi: \Omega \rightarrow \Omega$  is analytic, such that  $\Phi(\Omega)$  has a non-void interior, and the composition operator  $C_\Phi: H^2(\Omega) \rightarrow H^2(\Omega)$  is compact, then:

$$\beta_d^-(C_\Phi) > 0.$$

On the other hand, if  $\Omega$  is a product of balls, then:

$$\|\Phi\|_\infty < 1 \implies \beta_d^+(C_\Phi) < 1.$$

We do not know whether the converse holds and the purpose of this paper is to study some examples towards an answer.

The paper is organized as follows. Section 1 is this short introduction, as well as some notations and definitions on singular numbers of operators and Hardy spaces of the polydisk to follow. Section 2 contains preliminary results on weighted composition operators in one variable, which surprisingly play an important role in the study of non-weighted composition operators in two variables. Section 3 studies the case of symbols with ‘‘separated’’ variables. Our main one variable result extends in this case. Section 4 studies the ‘‘glued case’’  $\Phi(z_1, z_2) = (\phi(z_1), \phi(z_1))$  for which even boundedness is an issue. Here, the

Bergman space  $B^2(\mathbb{D})$  enters the picture. Section 5 studies the case of “triangularly separated” variables. This section lets direct Hilbertian sums of weighted composition operators in one variable appear, and it contains our main result: an example of a symbol  $\Phi$  satisfying  $\|\Phi\|_\infty = 1$  and yet  $\beta_2^+(C_\Phi) < 1$ . The final Section 6 discusses the role of the Monge-Ampère pluricapacity, which is a multivariate extension of the Green capacity in the disk. Even though, as evidenced by our counterexample of Section 5, this capacity will not capture all the behavior of the parameter  $\beta_m(C_\Phi)$ , some partial results are obtained, relying on theorems of S. Nivoche and V. Zakharyuta.

## 1.2 Notation

We denote by  $\mathbb{D}$  the open unit disk of the complex plane and by  $\mathbb{T}$  its boundary, the 1-dimensional torus.

The Hardy space  $H^2(\mathbb{D}^d)$  is the space of holomorphic functions  $f: \mathbb{D}^d \rightarrow \mathbb{C}$  whose boundary values  $f^*$  on  $\mathbb{T}^d$  are square-integrable with respect to the Haar measure  $m_d$  of  $\mathbb{T}^d$ , and normed with:

$$\|f\|_2^2 = \|f\|_{H^2(\mathbb{D}^d)}^2 = \int_{\mathbb{T}^d} |f^*(\xi_1, \dots, \xi_d)|^2 dm_d(\xi_1, \dots, \xi_d).$$

If  $f(z_1, \dots, z_d) = \sum_{\alpha_1, \dots, \alpha_d \geq 0} a_{\alpha_1, \dots, \alpha_d} z_1^{\alpha_1} \dots z_d^{\alpha_d}$ , then:

$$\|f\|_2^2 = \sum_{\alpha_1, \dots, \alpha_d \geq 0} |a_{\alpha_1, \dots, \alpha_d}|^2.$$

We say that an analytic map  $\Phi: \mathbb{D}^d \rightarrow \mathbb{D}^d$  is a *symbol* if its associated composition operator  $C_\Phi: H^2(\mathbb{D}^d) \rightarrow H^2(\mathbb{D}^d)$ , defined by  $C_\Phi(f) = f \circ \Phi$ , is bounded.

We say that  $\Phi$  is *truly  $d$ -dimensional* if  $\Phi(\mathbb{D}^d)$  has a non-void interior.

We will make use of two kinds of symbols defined on  $\mathbb{D}$ .

The *lens map*  $\lambda_\theta: \mathbb{D} \rightarrow \mathbb{D}$  is defined, for  $\theta \in (0, 1)$ , by:

$$(1.2) \quad \lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}$$

(see [26], p. 27, or [16], for more information), and corresponds to  $u \mapsto u^\theta$  in the right half-plane.

The *cusp map*  $\chi: \mathbb{D} \rightarrow \mathbb{D}$  was first defined in [15] and in a slightly different form in [20]; we actually use here the modified form introduced in [17], and then used in [18]. We first define:

$$\chi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1};$$

we note that  $\chi_0(1) = 0$ ,  $\chi_0(-1) = 1$ ,  $\chi_0(i) = -i$ ,  $\chi_0(-i) = i$ , and  $\chi_0(0) = \sqrt{2}-1$ . Then we set:

$$\chi_1(z) = \log \chi_0(z), \quad \chi_2(z) = -\frac{2}{\pi} \chi_1(z) + 1, \quad \chi_3(z) = \frac{a}{\chi_2(z)},$$

and finally:

$$\chi(z) = 1 - \chi_3(z),$$

where:

$$(1.3) \quad a = 1 - \frac{2}{\pi} \log(\sqrt{2} - 1) \in (1, 2)$$

is chosen in order that  $\chi(0) = 0$ . The image  $\Omega$  of the (univalent) cusp map is formed by the intersection of the inside of the disk  $D(1 - \frac{a}{2}, \frac{a}{2})$  and the outside of the two disks  $D(1 + \frac{ia}{2}, \frac{a}{2})$  and  $D(1 - \frac{ia}{2}, \frac{a}{2})$ .

Besides the approximation numbers, we need other singular numbers for an operator  $S: X \rightarrow Y$  between Banach spaces  $X$  and  $Y$ .

The *Bernstein numbers*  $b_n(S)$ ,  $n \geq 1$ , which are defined by:

$$(1.4) \quad b_n(S) = \sup_E \min_{x \in S_E} \|Sx\|,$$

where the supremum is taken over all  $n$ -dimensional subspaces of  $X$  and  $S_E$  is the unit sphere of  $E$ .

The *Gelfand numbers*  $c_n(S)$ ,  $n \geq 1$ , which are defined by:

$$(1.5) \quad c_n(S) = \inf\{\|S|_M\|; \text{codim } M < n\}.$$

The *Kolmogorov numbers*  $d_n(S)$ ,  $n \geq 1$ , which are defined by:

$$(1.6) \quad d_n(S) = \inf_{\dim E < n} \left[ \sup_{x \in \bar{B}_X} \text{dist}(Sx, E) \right].$$

Pietsch showed that all  $s$ -numbers on Hilbert spaces are equal (see [24], § 2, Corollary, or [25], Theorem 11.3.4); hence:

$$(1.7) \quad a_n(S) = b_n(S) = c_n(S) = d_n(S).$$

We denote  $m$  the normalized Lebesgue measure on  $\mathbb{T} = \partial\mathbb{D}$ . If  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ ,  $m_\varphi$  is the pull-back measure on  $\bar{\mathbb{D}}$  defined by  $m_\varphi(E) = m[\varphi^{*-1}(E)]$ , where  $\varphi^*$  stands for the non-tangential boundary values of  $\varphi$ .

The notation  $A \lesssim B$  means that  $A \leq CB$  for some positive constant  $C$  and we write  $A \approx B$  if we have both  $A \lesssim B$  and  $B \lesssim A$ .

## 2 Preliminary results on weighted composition operators on $H^2(\mathbb{D})$

We see in this section that the presence of a “rapidly decaying” weight allows simpler estimates for the approximation numbers of a corresponding weighted composition operator. Such a study, but a bit different, is made in [14].

Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  a non-constant analytic self-map in the disk algebra  $A(\mathbb{D})$  such that, for some constant  $C > 1$  and for all  $z \in \mathbb{D}$ :

$$(2.1) \quad \varphi(1) = 1, \quad |1 - \varphi(z)| \leq 1, \quad |1 - \varphi(z)| \leq C(1 - |\varphi(z)|)$$

as well as  $\varphi(z) \neq 1$  for  $z \neq 1$ . We can take for example  $\varphi = \frac{1+\lambda_\theta}{2}$  where  $\lambda_\theta$  is the lens map with parameter  $\theta$ .

Let  $w \in H^\infty$  and let  $T$  be the weighted composition operator

$$T = M_{w \circ \varphi} C_\varphi: H^2 \rightarrow H^2.$$

Note that  $M_{w \circ \varphi} C_\varphi = C_\varphi M_w$ . We first show that:

**Theorem 2.1.** *Let  $T = M_{w \circ \varphi} C_\varphi: H^2 \rightarrow H^2$  be as above and let  $B$  be a Blaschke product with length  $< N$ . Then, with the implied constant depending only on the number  $C$  in (2.1) (and of  $\varphi$ ):*

$$a_N(T) \lesssim \sup_{|z-1| \leq 1, z \in \varphi(\mathbb{D})} |B(z)| |w(z)|.$$

*Proof.* The following preliminary observation (see also [16], p. 809), in which we denote by  $S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\}$  the Carleson window with center  $\xi \in \mathbb{T}$  and size  $h$ , and by  $K_\varphi$  the support of the pull-back measure  $m_\varphi$ , will be useful.

$$(2.2) \quad u \in S(\xi, h) \cap K_\varphi \implies u \in S(1, Ch) \cap K_\varphi.$$

Indeed, if  $|u - \xi| \leq h$  and  $u \in K_\varphi$ , (2.1) implies:

$$1 - |u| \leq |u - \xi| \leq h \quad \text{and} \quad |u - 1| \leq C(1 - |u|) \leq Ch.$$

Set  $E = BH^2$ . This is a subspace of codimension  $< N$ . If  $f = Bg \in E$ , with  $\|g\| = \|f\|$  (isometric division by  $B$  in  $BH^2$ ), we have  $Tf = (wBg) \circ \varphi$ , whence:

$$\|T(f)\|^2 = \int_{\mathbb{D}} |B|^2 |w|^2 |g|^2 dm_\varphi,$$

implying  $\|T(f)\|^2 \leq \|f\|^2 \|J\|^2$  where  $J: H^2 \rightarrow L^2(\sigma)$  is the natural embedding and where

$$\sigma = |B|^2 |w|^2 dm_\varphi.$$

Now, Carleson's embedding theorem for the measure  $\sigma$  and (2.2) show that (the implied constants being absolute):

$$\begin{aligned}
\|J\|^2 &\lesssim \sup_{\xi \in \mathbb{T}, 0 < h < 1} \frac{1}{h} \int_{S(\xi, h) \cap K_\varphi} |B|^2 |w|^2 dm_\varphi \\
&\lesssim \sup_{0 < h < 1} \frac{1}{h} \int_{S(1, Ch) \cap K_\varphi} |B|^2 |w|^2 dm_\varphi \\
&\lesssim \left( \sup_{|z-1| \leq 1, z \in \overline{\varphi(\mathbb{D})}} |B(z)|^2 |w(z)|^2 \right) \left( \sup_{0 < h < 1} \frac{1}{h} \int_{S(1, Ch) \cap K_\varphi} dm_\varphi \right) \\
&\lesssim \sup_{|z-1| \leq 1, z \in \overline{\varphi(\mathbb{D})}} |B(z)|^2 |w(z)|^2,
\end{aligned}$$

since  $m_\varphi$  is a Carleson measure for  $H^2$  and where we used that, according to (2.1):

$$K_\varphi \subseteq \overline{\varphi(\mathbb{D})} \subseteq \{z \in \mathbb{D}; |z-1| \leq 1\}.$$

This ends the proof of Theorem 2.1 with help of the equality of  $a_N(T)$  with the Gelfand number  $c_N(T)$  recalled in (1.7).  $\square$

In order to specialize efficiently the general Theorem 2.1, we recall the following simple Lemma 2.3 of [16], where:

$$(2.3) \quad \rho(a, b) = \left| \frac{a-b}{1-\bar{a}b} \right|, \quad a, b \in \mathbb{D},$$

is the *pseudo-hyperbolic distance*:

**Lemma 2.2** ([16]). *Let  $a, b \in \mathbb{D}$  such that  $|a-b| \leq L \min(1-|a|, 1-|b|)$ . Then:*

$$\rho(a, b) \leq \frac{L}{\sqrt{L^2+1}} =: \kappa < 1.$$

We can now state:

**Theorem 2.3.** *Assume that  $\varphi$  is as in (2.1) and that the weight  $w$  satisfies, for some parameters  $0 < \theta \leq 1$  and  $R > 0$ :*

$$|w(z)| \leq \exp\left(-\frac{R}{|1-z|^\theta}\right), \quad \forall z \in \mathbb{D} \text{ with } \Re z \geq 0.$$

*Then, the approximation numbers of  $T = M_{w \circ \varphi} C_\varphi$  satisfy:*

$$a_{nm+1}(T) \lesssim \max[\exp(-an), \exp(-R2^{m\theta})],$$

*for all integers  $n, m \geq 1$ , where  $a = \log[\sqrt{16C^2+1}/(4C)] > 0$  and  $C$  is as in (2.1).*

*Proof.* Let  $p_l = 1 - 2^{-l}$ ,  $0 \leq l < m$  and let  $B$  be the Blaschke product:

$$B(z) = \prod_{0 \leq l < m} \left( \frac{z - p_l}{1 - p_l z} \right)^n.$$

Let  $z \in K_\varphi \cap \mathbb{D}$  so that  $0 < |z - 1| \leq 1$ . Let  $l$  be the non-negative integer such that  $2^{-l-1} < |z - 1| \leq 2^{-l}$ . We separate two cases:

*Case 1:  $l \geq m$ .* Then, *the weight does the job.* Indeed, majorizing  $|B(z)|$  by 1 and using the assumption on  $w$ , we get:

$$\begin{aligned} |B(z)|^2 |w(z)|^2 &\leq |w(z)|^2 \leq \exp\left(-\frac{2R}{|1 - z|^\theta}\right) \\ &\leq \exp(-2R 2^{l\theta}) \leq \exp(-2R 2^{m\theta}). \end{aligned}$$

*Case 2:  $l < m$ .* Then, *the Blaschke product does the job.* Indeed, majorize  $|w(z)|$  by 1 and estimate  $|B(z)|$  more accurately with help of Lemma 2.2; we observe that

$$|z - p_l| \leq |z - 1| + 1 - p_l \leq 2 \times 2^{-l} = 2(1 - p_l) \leq 4C(1 - p_l)$$

and then, since  $z \in K_\varphi$ , we can write with  $C \geq 1$  as in (2.1):

$$1 - |z| \geq \frac{1}{C} |1 - z| \geq \frac{1}{2C} 2^{-l} \geq \frac{1}{4C} |z - p_l|,$$

so that the assumptions of Lemma 2.2 are verified with  $L = 4C$ , giving:

$$\rho(z, p_l) \leq \frac{4C}{\sqrt{16C^2 + 1}} = \exp(-a) < 1.$$

Hence, by definition, since  $l < m$ :

$$|B(z)| \leq [\rho(z, p_l)]^n \leq \exp(-an).$$

Putting both cases together, and observing that our Blaschke product has length  $nm < nm + 1$ , we get the result by applying Theorem 2.1 with  $N = nm + 1$ .  $\square$

## 2.1 Some remarks

**1.** Twisting a composition operator by a weight may improve the compactness of this composition operator, or even may make this weighted composition operator compact though the non-weighted was not (see [8] or [14]). However, this is not possible for all symbols, as seen in the following proposition.

**Proposition 2.4.** *Let  $w \in H^\infty$ . If  $\varphi$  is inner, or more generally if  $|\varphi| = 1$  on a subset of  $\mathbb{T}$  of positive measure, then  $M_w C_\varphi$  is never compact (unless  $w \equiv 0$ ).*



*Proof.* Indeed, suppose  $T = M_w C_\varphi$  compact. Since  $(z^n)_n$  converges weakly to 0 in  $H^2$  and since  $T(z^n) = w \varphi^n$ , we should have, since  $|\varphi| = 1$  on  $E$ , with  $m(E) > 0$ :

$$\int_E |w|^2 dm = \int_E |w|^2 |\varphi|^{2n} dm \leq \int_{\mathbb{T}} |w|^2 |\varphi|^{2n} dm = \|T(z^n)\|^2 \xrightarrow{n \rightarrow \infty} 0,$$

but this would imply that  $w$  is null a.e. on  $E$  and hence  $w \equiv 0$  (see [7], Theorem 2.2), which was excluded.  $\square$

Note that É. Amar and A. Lederer proved in [1] that  $|\varphi| = 1$  on a set of positive measure if and only if  $\varphi$  is an exposed point of the unit ball of  $H^\infty$ ; hence the following proposition can be viewed as the (almost) opposite case.

**Proposition 2.5.** *Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  such that  $\|\varphi\|_\infty = 1$ . Assume that:*

$$\int_{\mathbb{T}} \log(1 - |\varphi|) dm > -\infty$$

(meaning that  $\varphi$  is not an extreme point of the unit ball of  $H^\infty$ : see [7], Theorem 7.9). Then, if  $w$  is an outer function such that  $|w| = 1 - |\varphi|$ , the weighted composition operator  $T = M_w C_\varphi$  is Hilbert-Schmidt.

*Proof.* We have:

$$\sum_{n=0}^{\infty} \|T(z^n)\|^2 = \sum_{n=0}^{\infty} \int_{\mathbb{T}} (1 - |\varphi|)^2 |\varphi|^{2n} dm = \int_{\mathbb{T}} \frac{1 - |\varphi|}{1 + |\varphi|} dm < +\infty,$$

and  $T$  is Hilbert-Schmidt, as claimed.  $\square$

**2.** In [14], Theorem 2.5, it is proved that we always have, for some constants  $\delta, \rho > 0$ :

$$(2.4) \quad a_n(M_w C_\varphi) \geq \delta \rho^n, \quad n = 1, 2, \dots$$

(if  $w \neq 0$ ). We give here an alternative proof, based on a result of Gunatillake ([9]), this result holding in a wider context.

**Theorem 2.6** (Gunatillake). *Let  $T = M_w C_\varphi$  be a compact weighted composition operator on  $H^2$  and assume that  $\varphi$  has a fixed point  $a \in \mathbb{D}$ . Then the spectrum of  $T$  is the set:*

$$\sigma(T) = \{0, w(a), w(a) \varphi'(a), w(a) [\varphi'(a)]^2, \dots, w(a) [\varphi'(a)]^n, \dots\}$$

*Proof of (2.4).* First observe that, in view of Proposition 2.4,  $\varphi$  cannot be an automorphism of  $\mathbb{D}$  so that the point  $a$  is the Denjoy-Wolff point of  $\varphi$  and is attractive. Theorem 2.6 is interesting only when  $w(a) \varphi'(a) \neq 0$ .

Now, we can give a new proof Theorem 2.5 of [14] as follows. Let  $a \in \mathbb{D}$  be such that  $w(a)\varphi'(a) \neq 0$  ( $H(\mathbb{D})$  is a division ring and  $\varphi' \neq 0, w \neq 0$ ). Let  $b = \varphi(a)$  and  $\tau \in \text{Aut } \mathbb{D}$  with  $\tau(b) = a$ . We set:

$$\psi = \tau \circ \varphi \quad \text{and} \quad S = M_w C_\psi = T C_\tau.$$

This operator  $S$  is compact because  $T$  is.

Since  $\psi(a) = a$  and  $\psi'(a) = \tau'(b)\varphi'(a) \neq 0$ , Theorem 2.6 says that the non-zero eigenvalues of  $S$ , arranged in non-increasing order, are the numbers  $\lambda_n = w(a) [\psi'(a)]^{n-1}$ ,  $n \geq 1$ . As a consequence of Weyl's inequalities, we know that:

$$a_1(S) a_n(S) \geq |\lambda_{2n}|^2 \geq \delta \rho^n,$$

with:

$$\delta = |w(a)|^2 > 0 \quad \text{and} \quad \rho = |\psi'(a)|^4 > 0.$$

To finish, it is enough to observe that  $a_n(S) \leq a_n(T) \|C_\tau\|$  by the ideal property of approximation numbers.  $\square$

### 3 The splitted case

**Theorem 3.1.** *Let  $\Phi = (\phi, \psi): \mathbb{D}^d \rightarrow \mathbb{D}^d$  be a truly  $d$ -dimensional symbol with  $\phi: \mathbb{D} \rightarrow \mathbb{D}$  depending only on  $z_1$  and  $\psi: \mathbb{D}^{d-1} \rightarrow \mathbb{D}^{d-1}$  only on  $z_2, \dots, z_d$ , i.e.  $\Phi(z_1, z_2, \dots, z_d) = (\phi(z_1), \psi(z_2, \dots, z_d))$ . Then, whatever  $\psi$  behaves:*

$$\|\phi\|_\infty = 1 \quad \implies \quad \beta_d(C_\Phi) = 1.$$

*Proof.* The proof is based on the following simple lemma, certainly well-known.

**Lemma 3.2.** *Let  $S: H_1 \rightarrow H_1$  and  $T: H_2 \rightarrow H_2$  be two compact linear operators, where  $H_1$  and  $H_2$  are Hilbert spaces. Let  $S \otimes T$  be their tensor product, acting on the tensor product  $H_1 \otimes H_2$ . Then:*

$$a_{mn}(S \otimes T) \geq a_m(S) a_n(T)$$

for all positive integers  $m, n$ .

We postpone the proof of the lemma and show how to conclude.

We can assume  $C_\Phi$  to be compact, so that  $C_\phi$  is compact as well. Since  $\|\phi\|_\infty = 1$ , we have, thanks to (1.1):

$$a_m(C_\phi) \geq e^{-m \varepsilon_m} \quad \text{with} \quad \varepsilon_m \xrightarrow{m \rightarrow \infty} 0.$$

Replacing  $\varepsilon_m$  by  $\delta_m := \sup_{p \geq m} \varepsilon_p$ , we can assume that  $(\varepsilon_m)_m$  is non-increasing. Moreover,

$$m \varepsilon_m \rightarrow \infty$$

since  $C_\phi$  is compact and hence  $a_m(C_\phi) \xrightarrow{m \rightarrow \infty} 0$ . We next observe that, due to the separation of variables in the definition of  $\phi$  and  $\psi$ , we can write:

$$(3.1) \quad C_\Phi = C_\phi \otimes C_\psi.$$

Indeed, write  $z = (z_1, w)$  with  $z_1 \in \mathbb{D}$  and  $w \in \mathbb{D}^{d-1}$ . If  $f \in H^2(\mathbb{D})$  and  $g \in H^2(\mathbb{D}^{d-1})$ , we see that:

$$\begin{aligned} C_\Phi(f \otimes g)(z) &= (f \otimes g)(\phi(z_1), \psi(w)) = f(\phi(z_1)) g(\psi(w)) \\ &= [C_\phi f(z_1)] [C_\psi g(w)] = (C_\phi f \otimes C_\psi g)(z). \end{aligned}$$

Since tensor products  $f \otimes g$  generate  $H^2(\mathbb{D}^d) = H^2(\mathbb{D}) \otimes H^2(\mathbb{D}^{d-1})$ , this proves (3.1).

Let now  $m$  be a large positive integer. Set ( $[\cdot]$  stands for the integer part):

$$(3.2) \quad n_m = [m\varepsilon_m]^{d-1} \quad \text{and} \quad N_m = m n_m.$$

From what we know in dimension  $d-1$  (see [2], Theorem 3.1) and from the preceding, we can write (observe that  $\psi$  has to be truly  $(d-1)$ -dimensional since  $\Phi$  is truly  $d$ -dimensional):

$$a_m(C_\phi) \geq \exp(-m\varepsilon_m) \quad \text{and} \quad a_n(C_\psi) \geq a \exp(-C n^{1/(d-1)}),$$

for some positive constant  $C$ , which will be allowed to vary from one formula to another. Lemma 3.2 implies:

$$a_{N_m}(C_\Phi) \geq a \exp[-C(m\varepsilon_m + n_m^{1/(d-1)})].$$

Since  $n_m \lesssim (m\varepsilon_m)^{d-1}$ , we get:

$$a_{N_m}(C_\Phi) \geq a \exp(-C m \varepsilon_m).$$

Observe that  $N_m = m n_m \sim m^d \varepsilon_m^{d-1}$  and so  $N_m^{1/d} \sim m \varepsilon_m^{1-1/d}$ . As a consequence:

$$\begin{aligned} a_{N_m}(C_\Phi) &\geq a \exp(-C m \varepsilon_m) = a \exp[-(C \varepsilon_m^{1/d})(m \varepsilon_m^{1-1/d})] \\ &\geq a \exp(-\eta_m N_m^{1/d}) \end{aligned}$$

with  $\eta_m := C \varepsilon_m^{1/d}$ .

Now, for  $N > N_1$ , let  $m$  be the smallest integer satisfying  $N_m \geq N$  (so that  $N_{m-1} < N \leq N_m$ ), and set  $\delta_N = \eta_m$ . We have  $\lim_{N \rightarrow \infty} \delta_N = 0$ . Next, we note that  $\lim_{m \rightarrow \infty} N_m/N_{m-1} = 1$ , because  $N_m \geq N_{m-1}$  and:

$$\frac{N_m}{N_{m-1}} \leq \frac{m}{m-1} \left( \frac{m\varepsilon_m + 1}{(m-1)\varepsilon_{m-1}} \right)^{d-1} \sim \left( \frac{\varepsilon_m}{\varepsilon_{m-1}} \right)^{d-1} \leq 1.$$

Finally, if  $N$  is an arbitrary integer and  $N_{m-1} < N \leq N_m$ , we obtain:

$$a_N(C_\Phi) \geq a_{N_m}(C_\Phi) \geq a \exp(-\eta_m N_m^{1/d}) \geq a \exp(-C \delta_N N^{1/d}),$$

since we observed that  $\lim_{m \rightarrow \infty} N_m/N_{m-1} = 1$ .

This amounts to say that  $\beta_d(C_\Phi) = 1$ . □

*Proof of Lemma 3.2.* It is rather formal. Start from the Schmidt decompositions of  $S$  and  $T$  respectively (recall that Hilbert spaces, the approximation numbers are equal to the singular ones):

$$S = \sum_{m=1}^{\infty} a_m(S) u_m \odot v_m, \quad T = \sum_{n=1}^{\infty} a_n(T) u'_n \odot v'_n,$$

where  $(u_m), (v_m)$  are two orthonormal sequences of  $H_1$ ,  $(u'_n), (v'_n)$  two orthonormal sequences of  $H_2$ , and  $u_m \odot v_m$  and  $u'_n \odot v'_n$  denote the rank one operators defined by  $(u_m \odot v_m)(x) = \langle x, v_m \rangle u_m$ ,  $x \in H_1$ , and  $(u'_n \odot v'_n)(x) = \langle x, v'_n \rangle u'_n$ ,  $x \in H_2$ .

We clearly have:

$$(u_m \odot v_m) \otimes (u'_n \odot v'_n) = (u_m \otimes u'_n) \odot (v_m \otimes v'_n),$$

so that the Schmidt decomposition of  $S \otimes T$  is (with SOT-convergence):

$$S \otimes T = \sum_{m,n \geq 1} a_m(S) a_n(T) (u_m \otimes u'_n) \odot (v_m \otimes v'_n),$$

since the two sequences  $(u_m \otimes u'_n)_{m,n}$  and  $(v_m \otimes v'_n)_{m,n}$  are orthonormal: for instance, we have by definition:

$$\langle u_{m_1} \otimes u'_{n_1}, u_{m_2} \otimes u'_{n_2} \rangle = \langle u_{m_1}, u_{m_2} \rangle \langle u'_{n_1}, u'_{n_2} \rangle.$$

This shows that the singular values of  $S \otimes T$  are the non-increasing rearrangement of the positive numbers  $a_m(S) a_n(T)$  and ends the proof of the lemma: the  $mn$  numbers  $a_k(S) a_l(T)$ , for  $1 \leq k \leq m$ ,  $1 \leq l \leq n$  all satisfy  $a_k(S) a_l(T) \geq a_m(S) a_n(T)$ , so that  $a_{mn}(S \otimes T) \geq a_m(S) a_n(T)$ .  $\square$

## 4 The glued case

Here we consider symbols of the form:

$$(4.1) \quad \Phi(z_1, z_2) = (\phi(z_1), \phi(z_1)),$$

where  $\phi: \mathbb{D} \rightarrow \mathbb{D}$  is a non-constant analytic map.

Note that such maps  $\Phi$  are not truly 2-dimensional.

### 4.1 Preliminary

We begin by remarking the following fact.

Let  $B^2(\mathbb{D})$  be the Bergman space of all analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that:

$$\|f\|_{B^2}^2 := \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty,$$

where  $dA$  is the normalized area measure on  $\mathbb{D}$ .

**Proposition 4.1.** *Assume that the composition operator  $C_\phi$  maps boundedly  $B^2(\mathbb{D})$  into  $H^2(\mathbb{D})$ . Then  $C_\Phi: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$ , defined by (4.1), is bounded.*

*Proof.* If we write  $f \in H^2(\mathbb{D}^2)$  as:

$$f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k, \quad \text{with} \quad \sum_{j,k \geq 0} |c_{j,k}|^2 = \|f\|_{H^2}^2,$$

we formally (or assuming that  $f$  is a polynomial) have:

$$[C_\Phi f](z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} [\phi(z_1)]^j [\phi(z_2)]^k = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) [\phi(z_1)]^n.$$

Hence, if we set  $g(z) = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) z^n$ , we get:

$$[C_\Phi(f)](z_1, z_2) = [C_\phi(g)](z_1),$$

so that, by integrating:

$$\|C_\Phi(f)\|_{H^2(\mathbb{D}^2)} = \|C_\phi(g)\|_{H^2(\mathbb{D})}.$$

By hypothesis, there is a positive constant  $M$  such that:

$$\|C_\phi(g)\|_{H^2(\mathbb{D})} \leq M \|g\|_{B^2(\mathbb{D})}.$$

But, by the Cauchy-Schwarz inequality:

$$\begin{aligned} \|g\|_{B^2(\mathbb{D})}^2 &= \sum_{n=0}^{\infty} \frac{1}{n+1} \left| \sum_{j+k=n} c_{j,k} \right|^2 \\ &\leq \sum_{n=0}^{\infty} \left( \sum_{j+k=n} |c_{j,k}|^2 \right) = \sum_{j,k \geq 0} |c_{j,k}|^2 = \|f\|_{H^2(\mathbb{D}^2)}^2, \end{aligned}$$

and we obtain  $\|C_\Phi(f)\|_{H^2(\mathbb{D}^2)} \leq M \|f\|_{H^2(\mathbb{D}^2)}$ . □

## 4.2 Lens maps

Let  $\lambda_\theta$  be a lens map of parameter  $\theta$ ,  $0 < \theta < 1$ . We consider  $\Phi_\theta: \mathbb{D}^2 \rightarrow \mathbb{D}^2$  defined by:

$$(4.2) \quad \Phi_\theta(z_1, z_2) = (\lambda_\theta(z_1), \lambda_\theta(z_2)).$$

We have the following result.

**Theorem 4.2.** *The composition operator  $C_{\Phi_\theta}: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$  is:*

- 1) *not bounded for  $\theta > 1/2$ ;*
- 2) *bounded, but not compact for  $\theta = 1/2$ ;*
- 3) *compact, and even Hilbert-Schmidt, for  $0 < \theta < 1/2$ .*

*Proof.* The reproducing kernel of  $H^2(\mathbb{D}^2)$  is, for  $(a, b) \in \mathbb{D}^2$ :

$$(4.3) \quad K_{a,b}(z_1, z_2) = \frac{1}{1 - \bar{a}z_1} \frac{1}{1 - \bar{b}z_2}, \quad (z_1, z_2) \in \mathbb{D}^2,$$

and:

$$\|K_{a,b}\|^2 = \frac{1}{(1 - |a|^2)(1 - |b|^2)}.$$

1) If  $C_{\Phi_\theta}$  were bounded, we should have, for some  $M < \infty$ :

$$\|C_{\Phi_\theta}^*(K_{a,b})\|_{H^2} \leq M \|K_{a,b}\|_{H^2}, \quad \text{for all } a, b \in \mathbb{D}.$$

Since  $C_{\Phi_\theta}^*(K_{a,b}) = K_{\Phi_\theta(a,b)} = K_{\lambda_\theta(a), \lambda_\theta(b)}$ , we would get, with  $b = 0$ :

$$\left( \frac{1}{1 - |\lambda_\theta(a)|^2} \right)^2 \leq M^2 \frac{1}{1 - |a|^2};$$

but this is not possible for  $\theta > 1/2$ , since  $1 - |\lambda_\theta(a)|^2 \approx 1 - |\lambda_\theta(a)| \sim (1 - a)^\theta$  when  $a$  goes to 1, with  $0 < a < 1$ .

For 2) and 3), let us consider the pull-back measure  $m_\theta$  of the normalized Lebesgue measure on  $\mathbb{T} = \partial\mathbb{D}$  by  $\lambda_\theta$ . It is easy to see that:

$$(4.4) \quad \sup_{\xi \in \mathbb{T}} m_\theta[D(\xi, h) \cap \mathbb{D}] = m_\theta[D(1, h) \cap \mathbb{D}] \approx h^{1/\theta}.$$

In particular, for  $\theta \leq 1/2$ ,  $m_\theta$  is a 2-Carleson measure, and hence (see [15], Theorem 2.1, for example) the canonical injection  $j: B^2(\mathbb{D}) \rightarrow L^2(m_\theta)$  is bounded, meaning that, for some positive constant  $M < \infty$ :

$$\int_{\mathbb{D}} |f(z)|^2 dm_\theta(z) \leq M^2 \|f\|_{B^2}^2.$$

Since

$$\int_{\mathbb{D}} |f(z)|^2 dm_\theta(z) = \int_{\mathbb{T}} |f[\lambda_\theta(u)]|^2 dm(u) = \|C_{\lambda_\theta}(f)\|_{H^2}^2,$$

we get that  $C_{\lambda_\theta}$  maps boundedly  $B^2(\mathbb{D})$  into  $H^2(\mathbb{D})$ .

It follows from Proposition 4.1 that  $C_{\Phi_\theta}: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$  is bounded.

However,  $C_{\Phi_{1/2}}$  is not compact since  $C_{\Phi_{1/2}}^*(K_{a,b})/\|K_{a,b}\|$  does not converge to 0 as  $a, b \rightarrow 1$ , by the calculations made in 1).

For 3), let  $e_{j,k}(z_1, z_2) = z_1^j z_2^k$ ,  $j, k \geq 0$ , be the canonical orthonormal basis of  $H^2(\mathbb{D}^2)$ ; we have  $[C_{\Phi_\theta}(e_{j,k})](z_1, z_2) = [\lambda_\theta(z_1)]^{j+k}$ . Hence:

$$\sum_{j,k \geq 0} \|C_{\Phi_\theta}(e_{j,k})\|_{H^2(\mathbb{D}^2)}^2 \leq \sum_{n=0}^{\infty} (2n+1) \int_{\mathbb{T}} |\lambda_\theta|^{2n} dm \leq \int_{\mathbb{T}} \frac{2}{(1 - |\lambda_\theta|^2)^2} dm.$$

Since, by Lemma 4.3 below,  $1 - |\lambda_\theta(e^{it})|^2 \gtrsim |1 - e^{it}|^\theta \geq t^\theta$  for  $|t| \leq \pi/2$ , we get:

$$\sum_{j,k \geq 0} \|C_{\Phi_\theta}(e_{j,k})\|_{H^2(\mathbb{D}^2)}^2 \lesssim \int_0^{\pi/2} \frac{dt}{t^{2\theta}} < \infty,$$

since  $\theta < 1/2$ . Therefore  $C_{\Phi_\theta}$  is Hilbert-Schmidt for  $\theta < 1/2$ .  $\square$

For sake of completeness, we recall the following elementary fact (see [26], p. 28, or also [16], Lemma 2.5)).

**Lemma 4.3.** *With  $\delta = \cos(\theta\pi/2)$ , we have, for  $|z| \leq 1$  and  $\Re z \geq 0$ :*

$$1 - |\lambda_\theta(z)|^2 \geq \frac{\delta}{2} |1 - z|^\theta.$$

*Proof.* We can write:

$$\lambda_\theta(z) = \frac{1-w}{1+w} \quad \text{with} \quad w = \left( \frac{1-z}{1+z} \right)^\theta \quad \text{and} \quad |w| \leq 1.$$

Then:

$$\Re w \geq \delta |w| \geq \frac{\delta}{2} |1 - z|^\theta.$$

Hence:

$$1 - |\lambda_\theta(z)|^2 = \frac{4 \Re w}{|1+w|^2} \geq \delta |w| \geq \frac{\delta}{2} |1 - z|^\theta,$$

as announced □

We now improve the result 3) of Theorem 4.2 by estimating the approximation numbers of  $C_{\Phi_\theta}$  and get that  $C_{\Phi_\theta}$  is in all Schatten classes of  $H^2(\mathbb{D}^2)$  when  $\theta < 1/2$ .

**Theorem 4.4.** *For  $0 < \theta < 1/2$ , there exists  $b = b_\theta > 0$  such that:*

$$(4.5) \quad a_n(C_{\Phi_\theta}) \lesssim e^{-b\sqrt{n}}.$$

*In particular  $\beta_2^+(C_{\Phi_\theta}) \leq e^{-b} < 1$ , though  $\|\Phi_\theta\|_\infty = 1$ , and even  $\Phi_\theta(\mathbb{T}^2) \cap \mathbb{T}^2 \neq \emptyset$ .*

*Proof.* Proposition 4.1 (and its proof) can be rephrased in the following way: if  $C_\phi$  maps boundedly  $B^2(\mathbb{D})$  into  $H^2(\mathbb{D})$ , then, we have the following factorization:

$$(4.6) \quad C_\phi: H^2(\mathbb{D}^2) \xrightarrow{J} B^2(\mathbb{D}) \xrightarrow{C_\phi} H^2(\mathbb{D}) \xrightarrow{I} H^2(\mathbb{D}^2),$$

where  $I: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^2)$  is the canonical injection given by  $(If)(z_1, z_2) = f(z_1)$  for  $f \in H^2(\mathbb{D})$ , and  $J: H^2(\mathbb{D}^2) \rightarrow B^2(\mathbb{D})$  is the contractive map defined by:

$$(Jf)(z) = \sum_{n=0}^{\infty} \left( \sum_{j+k=n} c_{j,k} \right) z^n,$$

for  $f \in H^2(\mathbb{D}^2)$  with  $f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k$ .

In the proof of Theorem 4.2, we have seen that, for  $0 < \theta \leq 1/2$ , the composition operator  $C_{\lambda_\theta}$  is bounded from  $B^2(\mathbb{D})$  into  $H^2(\mathbb{D})$ ; we get hence the factorization:

$$C_{\Phi_\theta}: H^2(\mathbb{D}^2) \xrightarrow{J} B^2(\mathbb{D}) \xrightarrow{C_{\lambda_\theta}} H^2(\mathbb{D}) \xrightarrow{I} H^2(\mathbb{D}^2),$$

Now, the lens maps have a semi-group property:

$$(4.7) \quad \lambda_{\theta_1 \theta_2} = \lambda_{\theta_1} \lambda_{\theta_2},$$

giving  $C_{\lambda_{\theta_1 \theta_2}} = C_{\lambda_{\theta_1}} \circ C_{\lambda_{\theta_2}}$ .

For  $0 < \theta < 1/2$ , we therefore can write  $C_{\lambda_\theta} = C_{\lambda_{2\theta}} \circ C_{\lambda_{1/2}}$  (note that  $2\theta < 1$ , so  $C_{\lambda_{2\theta}}: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$  is bounded), and we get:

$$C_{\Phi_\theta} = I C_{\lambda_{2\theta}} C_{\lambda_{1/2}} J.$$

Consequently:

$$a_n(C_{\Phi_\theta}) \leq \|I\| \|J\| \|C_{\lambda_{1/2}}\|_{B^2 \rightarrow H^2} a_n(C_{\lambda_{2\theta}}).$$

Now, we know ([16], Theorem 2.1) that  $a_n(C_{\lambda_{2\theta}}) \lesssim e^{-b\sqrt{n}}$ , so we get that  $a_n(C_{\Phi_\theta}) \lesssim e^{-b\sqrt{n}}$ .  $\square$

**Remark.** In [2], we saw that for a truly 2-dimensional symbol  $\Phi$ , we have  $\beta_2^-(C_\Phi) > 0$ . Here the symbol  $\Phi_\theta$  is not truly 2-dimensional, but we nevertheless have  $\beta_2(C_{\Phi_\theta}) > 0$ . In fact, let  $E = \{f \in H^2(\mathbb{D}^2); \frac{\partial f}{\partial z_2} \equiv 0\}$ ;  $E$  is isometrically isomorphic to  $H^2(\mathbb{D})$  and the restriction of  $C_{\Phi_\theta}$  to  $E$  behaves as the 1-dimensional composition operator  $C_{\lambda_\theta}: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ ; hence ([19], Proposition 6.3):

$$e^{-b_0\sqrt{n}} \lesssim a_n(C_{\lambda_\theta}) = a_n(C_{\Phi_\theta|_E}) \leq a_n(C_{\Phi_\theta}),$$

and  $\beta_2^-(C_{\Phi_\theta}) \geq e^{-b_0} > 0$ .

## 5 Triangularly separated variables

In this section, we consider symbols of the form:

$$(5.1) \quad \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) z_2),$$

where  $\phi, \psi: \mathbb{D} \rightarrow \mathbb{D}$  are non-constant analytic maps.

Such maps  $\Phi$  are truly 2-dimensional.

More generally, if  $h \in H^\infty$ , with  $h(0) = 0$  and  $\|h\|_\infty \leq 1$ , has its powers  $h^k$ ,  $k \geq 0$ , orthogonal in  $H^2$  (for convenience, we shall say that  $h$  is a *Rudin function*), we can consider:

$$(5.2) \quad \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2))$$

For such  $h$  we can take for example an inner function vanishing at the origin, but there are other such functions, as shown by C. Bishop:



**Theorem** (Bishop [4]). *The function  $h$  is a Rudin function if and only if the pull-back measure  $\mu = \mu_h$  is radial and Jensen, i.e for every Borel set  $E$ :*

$$\mu(e^{i\theta} E) = \mu(E) \quad \text{and} \quad \int_{\mathbb{D}} \log(1/|z|) d\mu(z) < \infty.$$

*Conversely, for every probability measure  $\mu$  supported by  $\overline{\mathbb{D}}$ , which is radial and Jensen, there exists  $h$  in the unit ball of  $H^\infty$ , with  $h(0) = 0$ , such that  $\mu = \mu_h$ .*

If we take for  $\mu$  the Lebesgue measure of  $\mathbb{T}$ , we get an inner function. But, as remarked in [4], we can take for  $\mu$  the Lebesgue measure on the union  $\mathbb{T} \cup (1/2)\mathbb{T}$ , normalized in order that  $\mu(T) = \mu((1/2)\mathbb{T}) = 1/2$ . Then the corresponding  $h$  is not inner since  $|h| = 1/2$  on a subset of  $\mathbb{T}$  of positive measure. He also showed that  $h(z)/z$  may be a non-constant outer function. Also, P. Bourdon ([6]) showed that the powers of  $h$  are orthogonal if and only if its Nevanlinna counting function is almost everywhere constant on each circle centered on the origin.

## 5.1 General facts

We first observe that if  $f \in H^2(\mathbb{D}^2)$  and:

$$f(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} z_1^j z_2^k,$$

then we can write:

$$f(z_1, z_2) = \left( \sum_{k \geq 0} f_k(z_1) \right) z_2^k$$

with:

$$f_k(z_1) = \sum_{j \geq 0} c_{j,k} z_1^j,$$

and:

$$\|f\|_{H^2(\mathbb{D}^2)}^2 = \sum_{j,k \geq 0} |c_{j,k}|^2 = \sum_{k \geq 0} \|f_k\|_{H^2(\mathbb{D})}^2.$$

That means that we have an isometric isomorphism:

$$J: H^2(\mathbb{D}^2) \longrightarrow \bigoplus_{k=0}^{\infty} H^2(\mathbb{D}),$$

defined by  $Jf = (f_k)_{k \geq 0}$ .

Now, for symbols  $\Phi$  as in (5.1), we have:

$$(C_\Phi f)(z_1, z_2) = \sum_{j,k \geq 0} c_{j,k} [\phi(z_1)]^j [\psi(z_1)]^k z_2^k,$$

so that  $J C_\Phi J^{-1}$  appears as the operator  $\bigoplus_k M_{\psi^k} C_\phi$  on  $\bigoplus_k H^2(\mathbb{D})$ , where  $M_{\psi^k}$  is the multiplication operator by  $\psi^k$ :

$$[(M_{\psi^k} C_\phi) f_k](z_1) = [\psi(z_1)]^k [(f_k \circ \phi)(z_1)].$$

When  $\Phi$  is as in (5.2), we have:

$$(C_\Phi f)(z_1, z_2) = \sum_{j, k \geq 0} c_{j, k} [\phi(z_1)]^j [\psi(z_1)]^k [h(z_2)]^k,$$

with:

$$\|C_\Phi f\|^2 \leq \sum_{k=0}^{\infty} \|T_k f_k\|^2$$

and:

$$T_k = M_{\psi^k} C_\phi;$$

hence  $J C_\Phi J^{-1}$  appears as pointwise dominated by the operator  $T = \bigoplus_k T_k$  on  $\bigoplus_k H^2(\mathbb{D})$ . This implies a factorization  $C_\Phi = AT$  with  $\|A\| \leq 1$ , so that  $a_n(C_\Phi) \leq a_n(T)$  for all  $n \geq 1$ .

We recall the following elementary fact.

**Lemma 5.1.** *Let  $(H_k)_{k \geq 0}$  be a sequence of Hilbert spaces and  $T_k: H_k \rightarrow H_k$  be bounded operators. Let  $H = \bigoplus_{k=0}^{\infty} H_k$  and  $T: H \rightarrow H$  defined by  $Tx = (T_k x_k)_k$ . Then:*

- 1)  *$T$  is bounded on  $H$  if and only if  $\sup_k \|T_k\| < \infty$ ;*
- 2)  *$T$  is compact on  $H$  if and only if each  $T_k$  is compact and  $\|T_k\| \xrightarrow[k \rightarrow \infty]{} 0$ .*

Going back to the symbols of the form (5.1), we have  $\|M_{\psi^k}\| \leq \|\psi^k\|_\infty \leq 1$ , since  $\|\psi\|_\infty \leq 1$ ; hence  $\|M_{\psi^k} C_\phi\| \leq \|C_\phi\|$  and the operator  $(M_{\psi^k} C_\phi)_k$  is bounded on  $\bigoplus_k H^2(\mathbb{D})$ . Therefore  $C_\Phi$  is bounded on  $H^2(\mathbb{D}^2)$ .

For approximation numbers, we have the following two facts.

**Lemma 5.2.** *Let  $T_k: H_k \rightarrow H_k$  be bounded linear operators between Hilbert spaces  $H_k$ ,  $k \geq 0$ . Let  $H = \bigoplus_k H_k$  and  $T = (T_k)_k: H \rightarrow H$ , assumed to be compact. Then, for every  $n_1, \dots, n_K \geq 1$ , and  $0 \leq m_1 < \dots < m_K$ ,  $K \geq 1$ , we have:*

$$(5.3) \quad a_N(T) \geq \inf_{1 \leq k \leq K} a_{n_k}(T_{m_k}),$$

where  $N = n_1 + \dots + n_K$ .

*Proof.* We use the Bernstein numbers  $b_n$  (see (1.4)), which are equal to the approximation numbers (see (1.7)).

For  $k = 1, \dots, K$ , there is an  $n_k$ -dimensional subspace  $E_k$  of  $H_{m_k}$  such that:

$$b_{n_k}(T_{m_k}) \leq \|T_{m_k} x\|, \quad \text{for all } x \in S_{E_k}.$$

Then  $E = \bigoplus_{k=1}^K E_k$  is an  $N$ -dimensional subspace of  $H$  and for every  $x = (x_1, x_2, \dots) \in E$ , we have:

$$\begin{aligned} \|Tx\|^2 &= \sum_{k \leq K} \|T_{m_k} x_{m_k}\|^2 \geq \sum_{k \leq K} [b_{n_k}(T_{m_k})]^2 \|x_{m_k}\|^2 \\ &\geq \inf_{k \leq K} [b_{n_k}(T_{m_k})]^2 \sum_{k \leq K} \|x_{m_k}\|^2 = \inf_{k \leq K} [b_{n_k}(T_{m_k})]^2 \|x\|^2; \end{aligned}$$

hence  $b_N(T) \geq \inf_{k \leq K} b_{n_k}(T_{m_k})$ , and we get the announced result.  $\square$

**Lemma 5.3.** *Let  $T = \bigoplus_{k=0}^{\infty} T_k$  acting on a Hilbertian sum  $H = \bigoplus_{k=0}^{\infty} H_k$ . Let  $n_0, \dots, n_K$  be positive integers and  $N = n_0 + \dots + n_K - K$ . Then, the approximation numbers of  $T$  satisfy:*

$$(5.4) \quad a_N(T) \leq \max \left( \max_{0 \leq k \leq K} a_{n_k}(T_k), \sup_{k > K} \|T_k\| \right).$$

*Proof.* Denote by  $S$  the right-hand side of (5.4). Let  $R_k$ ,  $0 \leq k \leq K$  be operators on  $H_k$  of respective rank  $< n_k$  such that  $\|T_k - R_k\| = a_{n_k}(T_k)$  and let  $R = \bigoplus_{k=0}^K R_k$ . Then  $R$  is an operator of rank  $\leq n_0 + \dots + n_K - K - 1 < N$ . If  $f = \sum_{k=0}^{\infty} f_k \in H$ , we see that:

$$\begin{aligned} \|Tf - Rf\|^2 &= \sum_{k=0}^K \|T_k f_k - R_k f_k\|^2 + \sum_{k > K} \|T_k f_k\|^2 \\ &\leq \sum_{k=0}^K a_{n_k}(T_k)^2 \|f_k\|^2 + \sum_{k > K} \|T_k f_k\|^2 \leq S^2 \sum_{k=0}^{\infty} \|f_k\|^2 = S^2 \|f\|^2, \end{aligned}$$

hence the result.  $\square$

We give now two corollaries of Lemma 5.3.

**Example 1.** We first use lens maps. We get:

**Theorem 5.4.** *Let  $\lambda_\theta$  the lens map of parameter  $\theta$  and let  $\psi: \mathbb{D} \rightarrow \mathbb{D}$  such that  $\|\psi\|_\infty := c < 1$  and  $h$  a Rudin function. We consider:*

$$\Phi(z_1, z_2) = (\lambda_\theta(z_1), \psi(z_1) h(z_2)).$$

*Then, for some positive constant  $\beta$ , we have, for all  $N \geq 1$ :*

$$(5.5) \quad a_N(C_\Phi) \lesssim e^{-\beta N^{1/3}}.$$

*Proof.* Let  $T_k = M_{\psi^k} C_{\lambda_\theta}$ . We have  $\|T_k\| \leq c^k$ , so  $\sup_{k > K} \|T_k\| \leq c^K$ . On the other hand, we have  $a_n(T_k) \leq c^k a_n(C_{\lambda_\theta}) \leq a_n(C_{\lambda_\theta}) \lesssim e^{-\beta_\theta \sqrt{n}}$  ([16], Theorem 2.1). Taking  $n_0 = n_1 = \dots = n_K = K^2$  in Lemma 5.3, we get:

$$\max_{0 \leq k \leq K} a_{n_k}(T_k) \lesssim e^{-\beta_\theta K}.$$

Since  $n_0 + \dots + n_K - K \approx K^3$ , we obtain  $a_{K^3} \lesssim e^{-\beta_\theta K}$ , which gives the claimed result, by taking  $\beta = \max(\beta_\theta, \log(1/c))$ .  $\square$

**Example 2.** We consider the cusp map  $\chi$ . We have:

**Theorem 5.5.** *Let  $\chi$  be the cusp map,  $h$  a Rudin function, and  $\psi$  in the unit ball of  $H^\infty$ , with  $\|\psi\|_\infty := c < 1$ . Let:*

$$\Phi(z_1, z_2) = (\chi(z_1), \psi(z_1)h(z_2)).$$

Then, for positive constant  $\beta$ , we have, for all  $N \geq 1$ :

$$a_N(C_\Phi) \lesssim e^{-\beta\sqrt{N}/\sqrt{\log N}}.$$

*Proof.* Let  $T_k = M_{\psi^k}C_\chi$ . As above, we have  $\sup_{k>K} \|T_k\| \leq c^K$ . For the cusp map, we have  $a_n(C_\chi) \lesssim e^{-\alpha n/\log n}$  ([20], Theorem 4.3); hence  $a_n(T_k) \lesssim e^{-\alpha n/\log n}$ . We take  $n_0 = n_1 = \dots = n_K = K \lfloor \log K \rfloor$  (where  $\lfloor \log K \rfloor$  is the integer part of  $\log K$ ). Since  $n_0 + \dots + n_K \approx K^2 \lfloor \log K \rfloor$ , we get, for another  $\alpha > 0$ :

$$a_{K^2 \lfloor \log K \rfloor}(C_\Phi) \lesssim e^{-\alpha K},$$

which reads:  $a_N(C_\Phi) \lesssim e^{-\beta\sqrt{N}/\sqrt{\log N}}$ , as claimed.  $\square$

## 5.2 Lower bounds

In this subsection, we give lower bounds for approximation numbers of composition operators on  $H^2$  of the bidisk, attached to a symbol  $\Phi$  of the previous form  $\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1)h(z_2))$  where  $h$  is a Rudin function. The sharpness of those estimates will be discussed in the next subsection. We first need some lemmas in dimension one.

**Lemma 5.6.** *Let  $u, v: \mathbb{D} \rightarrow \mathbb{D}$  be two non-constant analytic self-maps and  $T = M_v C_u: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$  be the associated weighted composition operator. For  $0 < r < 1$ , we set  $A = u(r\overline{\mathbb{D}})$  and  $\Gamma = \exp(-1/\text{Cap}(A))$ . Then, for  $0 < \delta \leq \inf_{|z|=r} |v(z)|$ , we have:*

$$(5.6) \quad a_n(T) \gtrsim \sqrt{1-r} \delta \Gamma^n.$$

In this lemma,  $\text{Cap}(A)$  denotes the Green capacity of the compact subset  $A \subseteq \mathbb{D}$  (see [21], § 2.3 for the definition).

For the proof, we need the following result ([27], Theorem 7, p. 353).

**Theorem 5.7** (Widom). *Let  $A$  be a compact subset of  $\mathbb{D}$  and  $\mathcal{C}(A)$  be the space of continuous functions on  $A$  with its natural norm. Set:*

$$\tilde{d}_n(A) = \inf_E \left[ \sup_{f \in B_{H^\infty}} \text{dist}(f, E) \right],$$

where  $E$  runs over all  $(n-1)$ -dimensional subspaces of  $\mathcal{C}(A)$  and  $\text{dist}(f, E) = \inf_{h \in E} \|f - h\|_{\mathcal{C}(A)}$ . Then

$$(5.7) \quad \tilde{d}_n(A) \geq \alpha e^{-n/\text{Cap}(A)}$$

for some positive constant  $\alpha$ .

*Proof of Lemma 5.6.* We apply Theorem 5.7 to the compact set  $A = u(r\overline{\mathbb{D}})$ .

Let  $E$  be an  $(n-1)$ -dimensional subspace of  $H^2 = H^2(\mathbb{D})$ ; it can be viewed as a subspace of  $\mathcal{C}(A)$ , so, by Theorem 5.7, there exists  $f \in H^\infty \subseteq H^2$  with  $\|f\|_2 \leq \|f\|_\infty \leq 1$  such that:

$$\|f - h\|_{\mathcal{C}(A)} \geq \alpha \Gamma^n, \quad \forall h \in E.$$

Then:

$$\|v(f \circ u - h \circ u)\|_{\mathcal{C}(r\mathbb{T})} \geq \delta \|(f - h) \circ u\|_{\mathcal{C}(r\mathbb{T})} = \delta \|f - h\|_{\mathcal{C}(A)} \geq \alpha \delta \Gamma^n.$$

But:

$$\|v(f \circ u - h \circ u)\|_{\mathcal{C}(r\mathbb{T})} \leq \frac{1}{\sqrt{1-r^2}} \|v(f \circ u - h \circ u)\|_{H^2};$$

Hence:

$$\|Tf - Th\|_{H^2} \geq \alpha \sqrt{1-r^2} \delta \Gamma^n \geq \alpha \sqrt{1-r} \delta \Gamma^n.$$

Since  $h$  is an arbitrary function of  $E$ , we get ( $B_{H^2}$  being the unit ball of  $H^2$ ):

$$\inf_{\dim E < n} \left[ \sup_{f \in B_{H^2}} \text{dist}(Tf, T(E)) \right] \geq \alpha \sqrt{1-r} \delta \Gamma^n.$$

But the left-hand side is equal to the Kolmogorov number  $d_n(T)$  of  $T$  (see [21], Lemma 3.12), and, as recalled in (1.7), in Hilbert spaces, the Kolmogorov numbers are equal to the approximation numbers; hence we obtain:

$$(5.8) \quad a_n(T) \geq \alpha \sqrt{1-r} \delta \Gamma^n, \quad n = 1, 2, \dots,$$

as announced. □

The next lemma shows that some Blaschke products are far away from 0 on some circles centered at 0.

We consider a *strongly interpolating sequence*  $(z_j)_{j \geq 1}$  of  $\mathbb{D}$  in the sense that, if  $\varepsilon_j := 1 - |z_j|$ , then:

$$(5.9) \quad \varepsilon_{j+1} \leq \sigma \varepsilon_j$$

and so  $\varepsilon_j \leq \sigma^{j-1} \varepsilon_1$ , where  $0 < \sigma < 1$  is fixed. Equivalently, the sequence  $(|z_j|)_{j \geq 1}$  is interpolating. We consider the corresponding interpolating Blaschke product:

$$(5.10) \quad B(z) = \prod_{j=1}^{\infty} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - z_j z}.$$

The following lemma is probably well-known, but we could find no satisfactory reference (see yet [10] for related estimates) and provide a simple proof.

**Lemma 5.8.** *Let  $(z_j)_{j \geq 1}$  be a strongly interpolating sequence as in (5.9) and  $B$  the associated Blaschke product (5.10).*

*Then there exists a sequence  $r_l := 1 - \rho_l$  such that:*

$$(5.11) \quad C_1 \sigma^l \leq \rho_l \leq C_2 \sigma^l,$$

where  $C_1, C_2$  are positive constants, and for which:

$$(5.12) \quad |z| = r_l \implies |B(z)| \geq \delta,$$

where  $\delta > 0$  does not depend on  $l$ .

*Proof.* Let us denote by  $p_l, 1 \leq p_l \leq l$ , the biggest integer such that  $\varepsilon_{p_l} \geq \sigma^{l-1} \varepsilon_1$ .

We separate two cases.

**Case 1:**  $\varepsilon_{p_l} \geq 2 \sigma^{l-1} \varepsilon_1$ .

Then, we choose  $\rho_l = \alpha \sigma^{l-1} \varepsilon_1$  with  $\alpha$  fixed,  $1 < \alpha < 2$ . Since  $\rho(\xi, \zeta) \geq \rho(|\xi|, |\zeta|)$  for all  $\xi, \zeta \in \mathbb{D}$  (recall that  $\rho$  is the pseudo-hyperbolic distance on  $\mathbb{D}$ ), we have the following lower bound for  $|z| = r_l$ :

$$|B(z)| = \prod_{j=1}^{\infty} \rho(z, z_j) \geq \prod_{j=1}^{\infty} \rho(r_l, |z_j|) = \prod_{j \leq p_l} \rho(r_l, |z_j|) \times \prod_{j > p_l} \rho(r_l, |z_j|) := P_1 \times P_2,$$

and we estimate  $P_1$  and  $P_2$  separately.

We first observe that  $\frac{\rho_l}{\varepsilon_{p_l}} \leq \frac{\alpha \sigma^{l-1} \varepsilon_1}{2 \sigma^{l-1} \varepsilon_1} \leq \frac{\alpha}{2}$ , and then:

$$\frac{\rho_l}{\varepsilon_j} = \frac{\rho_l}{\varepsilon_{p_l}} \frac{\varepsilon_{p_l}}{\varepsilon_j} \leq \frac{\alpha}{2} \sigma^{p_l - j}.$$

The inequality  $\rho(1-u, 1-v) \geq \frac{|u-v|}{(u+v)}$  for  $0 < u, v \leq 1$  now gives us:

$$(5.13) \quad \rho(r_l, |z_j|) \geq \frac{\varepsilon_j - \rho_l}{\varepsilon_j + \rho_l} = \frac{1 - \rho_l/\varepsilon_j}{1 + \rho_l/\varepsilon_j} \geq \frac{1 - (\alpha/2) \sigma^{p_l - j}}{1 + (\alpha/2) \sigma^{p_l - j}}, \quad \text{for } j \leq p_l,$$

and:

$$(5.14) \quad P_1 \geq \prod_{k=0}^{\infty} \left( \frac{1 - (\alpha/2) \sigma^k}{1 + (\alpha/2) \sigma^k} \right).$$

Similarly:

$$\frac{\varepsilon_{p_l+1}}{\rho_l} \leq \frac{\sigma^{l-1} \varepsilon_1}{\alpha \sigma^{l-1} \varepsilon_1} \leq \frac{1}{\alpha}$$

and:

$$\frac{\varepsilon_j}{\rho_l} \leq \frac{1}{\alpha} \sigma^{j-p_l-1} \quad \text{for } j > p_l;$$

so that:

$$(5.15) \quad \rho(r_l, |z_j|) \geq \frac{\rho_l - \varepsilon_j}{\rho_l + \varepsilon_j} = \frac{1 - \varepsilon_j/\rho_l}{1 + \varepsilon_j/\rho_l} \geq \frac{1 - \alpha^{-1} \sigma^{j-p_l-1}}{1 + \alpha^{-1} \sigma^{j-p_l-1}}, \quad \text{for } j > p_l,$$

and

$$(5.16) \quad P_2 \geq \prod_{k=0}^{\infty} \left( \frac{1 - \alpha^{-1} \sigma^k}{1 + \alpha^{-1} \sigma^k} \right).$$

Finally, the condition of lower and upper bound for  $\rho_l$  is fulfilled by construction.

**Case 2:**  $\varepsilon_{p_l} \leq 2 \sigma^{l-1} \varepsilon_1$ .

Then, we choose  $\rho_l = a \varepsilon_{p_l}$  with  $\sigma < a < 1$  fixed. Computations exactly similar to those of Case 1 give us:

$$(5.17) \quad |B(z)| \geq \prod_{k=0}^{\infty} \left( \frac{1 - a \sigma^k}{1 + a \sigma^k} \right) \times \prod_{k=0}^{\infty} \left( \frac{1 - a^{-1} \sigma^k}{1 + a^{-1} \sigma^k} \right) =: \delta > 0, \quad \text{for } |z| = r_l.$$

Moreover, in this case:

$$a \sigma^{l-1} \varepsilon_1 \leq \rho_l \leq 2 a \sigma^{l-1} \varepsilon_1,$$

and the proof is ended.  $\square$

Now, we have the following estimation.

**Theorem 5.9.** *Let  $\phi, \psi: \mathbb{D} \rightarrow \mathbb{D}$  be two non-constant analytic self-maps and  $\Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2))$ , where  $h$  is inner.*

*Let  $(r_l)_{l \geq 1}$  be an increasing sequence of positive numbers with limit 1 such that:*

$$\inf_{|z|=r_l} |\psi(z)| \geq \delta_l > 0,$$

*with  $\delta_l \leq e^{-1/\text{Cap}(A_l)}$ , where  $A_l = \phi(r_l \mathbb{D})$ .*

*Then the approximation numbers  $a_N(C_\Phi)$ ,  $N \geq 1$ , of the composition operator  $C_\Phi: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$  satisfy:*

$$(5.18) \quad a_N(C_\Phi) \gtrsim \sup_{l \geq 1} \left[ \sqrt{1 - r_l} \exp \left( -8 \sqrt{N} \sqrt{\log(1/\delta_l)} \sqrt{\log(1/\Gamma_l)} \right) \right],$$

*where:*

$$(5.19) \quad \Gamma_l = e^{-1/\text{Cap}(A_l)}.$$

*Proof.* Since  $h$  is inner, the sequence  $(h^k)_{k \geq 0}$  is orthonormal in  $H^2$  and hence  $a_n(C_\Phi) = a_n(T)$  for all  $n \geq 1$ , where  $T = \bigoplus_{k=0}^{\infty} T_k$  and  $T_k = M_{\psi^k} C_\phi$ . Then Lemma 5.6 gives:

$$(5.20) \quad a_n(T_k) \gtrsim \sqrt{1 - r_l} \delta_l^k \Gamma_l^n$$

for all  $n \geq 1$  and all  $k \geq 0$ .

Let now:

$$(5.21) \quad p_l = \left\lceil \frac{\log(1/\delta_l)}{\log(1/\Gamma_l)} \right\rceil,$$

where  $[\cdot]$  stands for the integer part, and:

$$(5.22) \quad n_k = p_l k, \quad \text{for } k = 1, \dots, K.$$

By Lemma 5.2, applied with  $m_k = k$  (i.e. to  $H_1, \dots, H_K$ ), we have, if  $N = n_1 + \dots + n_K$ :

$$a_N(T) \geq \inf_{1 \leq k \leq K} \alpha \sqrt{1-r_l} \delta_l^k \Gamma_l^n = \alpha \sqrt{1-r_l} \delta_l^K \Gamma_l^{n_K}.$$

But, since  $p_l \leq \log(1/\delta_l)/\log(1/\Gamma_l)$ :

$$\delta_l^K \Gamma_l^{n_K} = \exp \left[ - (K \log(1/\delta_l) + p_l K \log(1/\Gamma_l)) \right] \geq \exp[-2K \log(1/\delta_l)].$$

Since:

$$N = p_l \frac{K(K+1)}{2} \geq p_l \frac{K^2}{4} \geq \frac{K^2}{16} \frac{\log(1/\delta_l)}{\log(1/\Gamma_l)},$$

we get:

$$\delta_l^K \Gamma_l^{n_K} \geq \exp \left[ -8 \sqrt{N} \sqrt{\log(1/\delta_l)} \sqrt{\log(1/\Gamma_l)} \right],$$

and the result ensues.  $\square$

**Example 1.** We take  $\phi = \lambda_\theta$ , a lens map, and  $\psi = B$ , a Blaschke product associated to a strongly regular sequence, as defined in (5.10); then we get:

**Theorem 5.10.** *Let  $\Phi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$  be defined by:*

$$\Phi(z_1, z_2) = (\lambda_\theta(z_1), c B(z_1) h(z_2)),$$

where  $B$  is a Blaschke product as in (5.10),  $0 < c < 1$ , and  $h$  is an arbitrary inner function, we have, for some positive constant  $b$ , for all  $N \geq 1$ :

$$(5.23) \quad a_N(C_\Phi) \gtrsim \exp(-b N^{1/3}) = \exp(-b \sqrt{N}/N^{1/6}).$$

In particular  $\beta_2(C_\Phi) = \beta_2^\pm(C_\Phi) = 1$ .

**Remark.** We saw in Theorem 5.4 that this is the exact size, since we have:  $a_N(C_\Phi) \lesssim e^{-\beta N^{1/3}}$ .

*Proof.* By Lemma 5.8, there is a sequence of numbers  $r_l \approx \sigma^l$  such that  $|B(z)| \geq \delta$  for  $|z| = r_l$ , where  $\delta$  is a positive constant (depending on  $\sigma$ ). Since  $\lambda_\theta(0) = 0$ , we have:

$$\text{diam}_\rho(A_l) \geq \lambda_\theta(r_l) \gtrsim 1 - (1 - r_l)^\theta;$$

hence, by [21], Theorem 3.13, we have:

$$\text{Cap}(A_l) \gtrsim \log \frac{1}{1-r_l} \gtrsim l,$$

or, equivalently:  $\Gamma_l \geq e^{-b/l}$ , some  $b > 0$ . Then (5.18) gives, for all  $l \geq 1$  (with another  $b$ ):

$$a_N(C_\Phi) \gtrsim \exp \left[ -b \left( l + \frac{\sqrt{N}}{\sqrt{l}} \right) \right].$$

Taking  $l = N^{1/3}$ , we get the result.  $\square$



**Example 2.** By taking the cusp instead of a lens map, we obtain a better result, close to the extremal one.

**Theorem 5.11.** *Let  $\Phi(z_1, z_2) = (\chi(z_1), cB(z_1)h(z_2))$ , where  $\chi$  is the cusp map,  $B$  a Blaschke product as in (5.10),  $0 < c < 1$ , and  $h$  an arbitrary inner function. Then, for all  $N \geq 1$ :*

$$a_N(C_\Phi) \gtrsim e^{-b\sqrt{N}/\sqrt{\log N}}.$$

In particular  $\beta_2(C_\Phi) = 1$ .

**Remark.** We saw in Theorem 5.5 that this is the exact size, since we have:  $a_N(C_\phi) \lesssim e^{-\beta\sqrt{N}/\log N}$ .

*Proof.* The proof is the same as that of Proposition 5.10, except that, for the cusp map, we have (note that  $\chi(0) = 0$ ):

$$\text{diam}_\rho(A_l) \geq \chi(r_l).$$

But when  $r$  goes to 1:

$$1 - \chi(r) \sim \frac{\pi(\sqrt{2}-1)}{2} \frac{1}{\log(1/(1-r))}$$

(see [20], Lemma 4.2). Hence, by [21], Theorem 3.13, again, we have:

$$\text{Cap}(A_l) \gtrsim \log(\log(1/(1-r_l))),$$

so  $\Gamma_l \geq e^{-b/\log l}$ . Then, (5.18) gives (with another  $b$ ):

$$a_N(C_\Phi) \gtrsim \exp\left[-b\left(l + \frac{\sqrt{N}}{\sqrt{\log l}}\right)\right].$$

In taking  $l = \sqrt{N/\log N}$ , we get the announced result.  $\square$

### 5.3 Upper bounds

All previous results point in the direction that, if  $\|\Phi\|_\infty = 1$ , then however small  $a_n(C_\Phi)$  is, it will always be larger than  $\alpha e^{-\beta\varepsilon_n\sqrt{n}}$  with  $\varepsilon_n \rightarrow 0^+$ , as this is the case in dimension one (with  $n$  instead of  $\sqrt{n}$ ). But Theorem 5.12 to follow shows that we cannot hope, in full generality, to get the same result in dimension  $d \geq 2$ , and that other phenomena await to be understood. Here is our main result. It shows that, even for a truly 2-dimensional symbol  $\Phi$ , we can have  $\|\Phi\|_\infty = 1$  and nevertheless  $\beta_2^+(C_\Phi) < 1$ , in contrast to the 1-dimensional case where (1.1) holds.

**Theorem 5.12.** *There exist a map  $\Phi: \mathbb{D}^2 \rightarrow \mathbb{D}^2$  such that:*

- 1) *the composition operator  $C_\Phi: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D}^2)$  is bounded and compact;*
- 2) *we have  $\|\Phi\|_\infty = 1$  and  $\Phi$  is truly 2-dimensional, so that  $\beta_2^-(C_\Phi) > 0$ ;*
- 3) *the singular numbers satisfy  $a_n(C_\Phi) \leq \alpha e^{-\beta\sqrt{n}}$  for some positive constants  $\alpha, \beta$ ; in particular  $\beta_2^+(C_\Phi) < 1$ .*

*Proof.* Let  $0 < \theta < 1$  be fixed, and  $\lambda_\theta$  be the corresponding lens map. We set:

$$\begin{cases} \phi &= \frac{1 + \lambda_\theta}{2} \\ w(z) &= \exp \left[ - \left( \frac{1+z}{1-z} \right)^\theta \right] \\ \psi &= w \circ \phi. \end{cases}$$

Note that  $\|\phi\|_\infty = 1$ .

Setting  $\delta = \cos(\theta\pi/2) > 0$ , we have for  $z \in \mathbb{D}$ :

$$(5.24) \quad |1 - \phi(z)| = \frac{1}{2} |1 - \lambda_\theta(z)| = \left| \frac{(1-z)^\theta}{(1-z)^\theta + (1+z)^\theta} \right| \leq \frac{|1-z|^\theta}{\delta}.$$

Indeed, the argument  $\alpha$  of  $(1 \pm z)^\theta$  satisfies  $|\alpha| \leq \theta\pi/2$  for  $z \in \mathbb{D}$ , and we get:

$$|(1-z)^\theta + (1+z)^\theta| \geq \Re[(1-z)^\theta + (1+z)^\theta] \geq \delta(|1+z|^\theta + |1-z|^\theta) \geq \delta.$$

We also see that  $\phi(\mathbb{D})$  touches the boundary  $\partial\mathbb{D}$  only at 1 in a non-tangential way, meaning that for some constant  $C > 1$ :

$$1 - |\phi(z)| \geq \frac{1}{C} |1 - \phi(z)|, \quad \forall z \in \mathbb{D}.$$

Now, we have the following two inequalities:

$$(5.25) \quad \Re z \geq 0 \implies |w(z)| \leq \exp \left( - \frac{\delta}{|1-z|^\theta} \right)$$

$$(5.26) \quad z \in \mathbb{D} \implies |\psi(z)| \leq \exp \left( - \frac{\delta^2}{|1-z|^{\theta^2}} \right).$$

Indeed, with  $S(z) = \left( \frac{1+z}{1-z} \right)^\theta$ , we have  $\Re S(z) \geq \delta |S(z)| \geq \delta |1-z|^{-\theta}$  when  $\Re z \geq 0$ , giving (5.25), and (5.24) and (5.25) imply, since  $\Re \phi(z) \geq 0$ :

$$|\psi(z)| = |w(\phi(z))| \leq \exp \left( - \frac{\delta}{|1-\phi(z)|^\theta} \right) \leq \exp \left( - \frac{\delta^2}{|1-z|^{\theta^2}} \right).$$

We now set:

$$(5.27) \quad \Phi(z_1, z_2) = (\phi(z_1), \psi(z_1) h(z_2)),$$

with  $h$  a Rudin function.

Observe that  $\phi \in A(\mathbb{D})$  and  $\psi = w \circ \phi \in A(\mathbb{D})$  as well ( $w \in A(\mathbb{D})$  with  $w(1) = 0$ ; this is due to the presence of the parameter  $\theta < 1$ ). hence if we take for  $h$  a finite Blaschke product, the two components of  $\Phi$  are in the bidisk algebra  $A(\mathbb{D}^2)$ .

We have  $\|\psi\|_\infty := \rho < 1$ . In fact, for  $\Re u \geq 0$ , we have:

$$\left| \frac{1+u}{1-u} \right| \geq 2^{-\theta} |1+u|^\theta \geq 2^{-\theta} (1 + \Re u)^\theta \geq 2^{-\theta},$$

hence:

$$\Re \left[ \left( \frac{1+u}{1-u} \right)^\theta \right] \geq \left( \cos \frac{\theta\pi}{2} \right) \left| \frac{1+u}{1-u} \right|^\theta \geq \left( \cos \frac{\theta\pi}{2} \right) 2^{-\theta} = \delta 2^{-\theta},$$

and  $\|w \circ \phi\|_\infty \leq e^{2^{-\theta}\delta}$ .

Now, 1) follows from the orthogonal model presented in Section 5.1, because  $\|\psi\|_\infty < 1$ .

The assertion 2) follows from [2], Theorem 3.1, since  $\|\phi\|_\infty = 1$ .

We now prove 3).

As observed,  $C_\Phi$  can be viewed as a direct sum  $T = \bigoplus_{k=0}^\infty T_k$  acting on a Hilbertian sum  $H = \bigoplus_{k=0}^\infty H_k$ , where  $T_k$  acts on a copy  $H_k$  of  $H^2(\mathbb{D})$  with:

$$T_k = M_{\psi^k} C_\phi.$$

We fix the positive integer  $n$ . The rest of the proof will consist of three lemmas.

**Lemma 5.13.** *We have  $\|T_k\| \leq 2\rho^{-k} \leq 2\rho^{-n}$  for  $k > n$ .*

*Proof.* Indeed, since  $\phi(0) = 1/2$ , we know that  $\|C_\phi\| \leq \sqrt{\frac{1+\phi(0)}{1-\phi(0)}} = \sqrt{3} \leq 2$ , so that  $\|T_k\| \leq \|\psi^k\|_\infty \|C_\phi\| \leq \rho^{-k} \times 2$ .  $\square$

**Lemma 5.14.** *Set  $b = a/\delta^2$  where  $a > 0$  is given by  $e^{-a} = 4C/\sqrt{16C^2+1}$  and  $C$  is as in (2.1). Let  $m_k$  be the smallest integer such that  $k\delta^2 2^{m_k\theta^2} \geq an$ ; namely:*

$$(5.28) \quad m_k = \left\lceil \frac{\log(bn/k)}{\theta^2 \log 2} \right\rceil + 1,$$

where  $\lceil \cdot \rceil$  stands for the integer part. Then, with  $a' = \min(\log 2, a)$ :

$$a_{nm_k+1}(T_k) \lesssim e^{-a'n}.$$

*Proof.* This follows from Theorem 2.3 applied with  $w = \psi^k$ ,  $R = k\delta^2$  and  $\theta$  changed into  $\theta^2$ . This is possible thanks to (5.26) and to Lemma 5.13. Moreover we have adjusted  $m_k$  so as to make the two terms in Theorem 2.3 of the same order.  $\square$

**Lemma 5.15.** *The dimension  $d := \sum_{k=0}^n n m_k$  satisfies, for some positive constant  $\alpha$ :*

$$d \leq \alpha n^2.$$

*Proof.* Indeed, it is well-known that:

$$\sum_{k=1}^n \log k = n \log n - n + O(\log n),$$

and, in view of (5.28), we have  $m_k \leq \alpha'_\theta \log(bn/k) \leq \alpha''_\theta(\log n - \log k)$ ; hence:

$$\sum_{k=1}^n m_k \leq \alpha''_\theta [n \log n - (n \log n - n + O(\log n))] = \alpha''_\theta n + O(\log n),$$

and we get  $d \leq \alpha''_\theta n^2 + O(n \log n) \leq \alpha_\theta n^2$ .

Alternatively, we could have used a Riemann sum for the function  $\log(1/x)$  on  $(0, 1]$ .  $\square$

Finally, putting things together and using as well Proposition 5.3 with  $K = n$  and  $n_k = nm_k + 1$  so that  $(\sum_{k=0}^n n_k) - n = (\sum_{k=0}^n n m_k) + 1 = d + 1$ , we get ignoring once more multiplicative constants:

$$a_{n^2}(T) \lesssim a_d(T) \leq \alpha e^{-\beta n}$$

with positive constants  $\alpha, \beta$ . This ends the proof of Theorem 5.12.  $\square$

## 6 Monge-Ampère capacity and applications

### 6.1 Definition

Let  $K$  be a compact subset of  $\mathbb{D}^m$  (in this section, for notational reasons, we denote the dimension by  $m$  instead of  $d$ ). The Monge-Ampère capacity of  $K$  has been defined by Bedford and Taylor ([3]; see also [13], § 5 or [11], Chapter 1) as:

$$\text{Cap}_m(K) = \sup \left\{ \int_K (dd^c u)^m; u \in PSH \text{ and } 0 \leq u \leq 1 \right\},$$

where  $PSH$  is the set of plurisubharmonic functions on  $\mathbb{D}^m$ ,  $dd^c = 2i\partial\bar{\partial}$ , and  $(dd^c)^m = dd^c \wedge \dots \wedge dd^c$  ( $m$  times). When  $u \in PSH \cap C^2(\mathbb{D}^m)$ , we have:

$$(dd^c u)^m = 4^m m! \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) dV(z),$$

where  $dV(z) = (i/2)^m dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_m \wedge d\bar{z}_m$  is the usual volume in  $\mathbb{C}^m$ . A more convenient formula (because  $\mathbb{D}^m$  is bounded and hyperconvex: see [11], p. 80, for the definition) is:

$$\text{Cap}_m(K) = \int_K (dd^c u_K^*)^m,$$

where  $u_K^*$  is called the *extremal function of  $K$*  and is the upper semi-continuous regularization of:

$$u_K = \sup\{v \in PSH; v \leq 0 \text{ and } v \leq -1 \text{ on } K\},$$

but we will not need that.

As in [28], we set:

$$(6.1) \quad \tau_m(K) = \frac{1}{(2\pi)^m} \text{Cap}_m(K) .$$

For  $m = 1$ ,  $\tau(K) := \tau_1(K)$  is equal to the Green capacity  $\text{Cap}(K)$  of  $K$  with respect to  $\mathbb{D}$ , with the definition used in [21] (see [13], Theorem 8.1, where a factor  $2\pi$  is introduced).

We further set:

$$(6.2) \quad \Gamma_m(K) = \exp \left[ - \left( \frac{m!}{\tau_m(K)} \right)^{1/m} \right] .$$

We proved in [21] that, for  $m = 1$ , and  $\varphi: \mathbb{D} \rightarrow r\mathbb{D}$ , with  $0 < r < 1$ , we have:

$$(6.3) \quad \beta_1(C_\varphi) = \Gamma_1(\overline{\varphi(\mathbb{D})}) .$$

The goal of this section is to see that Theorem 5.12 shows that this no longer holds for  $m = 2$ .

## 6.2 A seminal example

In one variable, our initial motivation had been the simple-minded example  $\varphi(z) = rz$ ,  $0 < r < 1$ , for which  $C_\varphi(z^n) = r^n z^n$ , implying  $a_n(C_\varphi) = r^{n-1}$  and  $\beta_1(C_\varphi) = r$ . If  $K = \overline{\varphi(\mathbb{D})} = \overline{D}(0, r)$ , we have  $\text{Cap}(K) = \frac{1}{\log 1/r}$  and  $\Gamma_1(K) = r$ , so that  $\beta_1(C_\varphi) = \Gamma_1(K)$ . Let us examine the multivariate example (where  $0 < r_j < 1$ ):

$$\Phi(z_1, z_2, \dots, z_m) = (r_1 z_1, r_2 z_2, \dots, r_m z_m) .$$

If  $K = \overline{\Phi(\mathbb{D}^m)}$ , we have  $K = \prod_{k=1}^m \overline{D}(0, r_k)$ , and hence ([5], Theorem 3):

$$(6.4) \quad \tau_m(K) = \prod_{k=1}^m \frac{1}{\log(1/r_k)} .$$

On the other hand,  $C_\Phi(z_1^{n_1} z_2^{n_2} \dots z_m^{n_m}) = r_1^{n_1} r_2^{n_2} \dots r_m^{n_m} z_1^{n_1} z_2^{n_2} \dots z_m^{n_m}$  so that the sequence  $(a_n)_n$  of approximation numbers of  $C_\Phi$  is the non-increasing rearrangement of the numbers  $r_1^{n_1} r_2^{n_2} \dots r_m^{n_m}$ . It is convenient to state the following simple lemma.

**Lemma 6.1.** *Let  $\lambda_1, \dots, \lambda_m$  be positive numbers. Let  $N_A$  be the number of  $m$ -tuples of non-negative integers  $(n_1, \dots, n_m)$  such that  $\sum_{k=1}^m \lambda_k n_k \leq A$ . Then, as  $A \rightarrow \infty$ :*

$$N_A \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m) m!} .$$

Indeed, just apply Karamata's tauberian theorem (see [12] p. 30) to the generalized Dirichlet series:

$$S(\varepsilon) := \prod_{k=1}^m \frac{1}{1 - e^{-\lambda_k \varepsilon}} = \sum_{n_1, \dots, n_m \geq 0} e^{-(\sum_{k=1}^m \lambda_k n_k) \varepsilon};$$

we have  $S(\varepsilon) \sim \frac{\varepsilon^{-m}}{(\lambda_1 \cdots \lambda_m)}$  as  $\varepsilon \rightarrow 0^+$ .

Let now  $N$  be a positive integer and  $\varepsilon = a_N$ . Setting  $\lambda_k = \log(1/r_k)$  and  $A = \log(1/\varepsilon)$ , we see that  $N$  is the number of  $m$ -tuples  $(n_1, \dots, n_m)$  of non-negative integers such that  $r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m} \geq \varepsilon$ , i.e. such that  $\sum_{k=1}^m \lambda_k n_k \leq A$ . This number  $N$  is hence nothing but the number  $N_A$  of the previous lemma, so that:

$$N \sim \frac{A^m}{(\lambda_1 \cdots \lambda_m) m!}.$$

Inverting this formula, we get:

$$a_N(C_\Phi) = \exp \left[ - (1 + o(1)) (m! \lambda_1 \lambda_2 \cdots \lambda_m N)^{1/m} \right]$$

and:

$$\beta_m(C_\Phi) = \exp \left[ - (m! \lambda_1 \lambda_2 \cdots \lambda_m)^{1/m} \right] = \Gamma_m(K),$$

in view of (6.2) and (6.4).

On the view of the simple-minded previous example, the extension of the spectral radius formula (6.3) to the multivariate case holds, and it is tempting to conjecture that this is a general phenomenon as in dimension one, all the more as the following extension of Widom's theorem was proved by Zakharyuta, based on the solution by S. Nivoche of Zakharyuta's conjecture ([23]); see also [28], Theorem 5.4. A compact subset  $K$  of  $\mathbb{D}^m$  is said to be *regular* if its extremal function  $u_K^*$  is continuous on  $\mathbb{D}^m$ .

**Theorem 6.2** ([28], Theorem 5.6). *Let  $K$  be a regular compact subset of  $\mathbb{D}^m$  and  $J: H^\infty(\mathbb{D}^m) \rightarrow \mathcal{C}(K)$  the canonical injection; then the Kolmogorov numbers  $d_n(J)$  satisfy:*

$$(6.5) \quad \lim_{n \rightarrow \infty} [d_n(J)]^{1/n^{1/m}} = \exp \left[ - \left( \frac{m!}{\tau_m(K)} \right)^{1/m} \right].$$

Note that the right side is nothing but  $\Gamma_m(K)$ .

We will see consequences of this result in a forthcoming paper ([22]).

### 6.3 Upper bound

For the upper bound, the situation behaves better, as stated in the following theorem.

**Theorem 6.3** ([28], Proposition 6.1). *Let  $K$  be a compact subset of  $\mathbb{D}^m$  with non-void interior. Then:*

$$(6.6) \quad \limsup_{n \rightarrow \infty} [d_n(J)]^{1/n^{1/m}} \leq \exp \left[ - \left( \frac{m!}{\tau_m(K)} \right)^{1/m} \right].$$

Note that  $(K, \mathbb{D}^m)$  is a condenser since  $K$  has non-void interior. We deduce the following upper bound.

**Theorem 6.4.** *Let  $\Phi$  be an analytic self-map of  $\mathbb{D}^m$  with  $\|\Phi\|_\infty = \rho < 1$ , thus inducing a compact composition operator on  $H^2(\mathbb{D}^m)$ . Then we have:*

$$\beta_m^+(C_\Phi) \leq \Gamma_m(\overline{\Phi(\mathbb{D}^m)}).$$

*Proof.* This proof provides in particular a simplification of that given in [21] in dimension  $m = 1$ .

Changing  $n$  into  $n^m$ , Theorem 6.3 means that for every  $\varepsilon > 0$ , there exists an  $(n^m - 1)$ -dimensional subspace  $V$  of  $\mathcal{C}(K)$  such that, for any  $g \in H^\infty(\mathbb{D}^m)$ , there exists  $h \in V$  such that:

$$(6.7) \quad \|g - h\|_{\mathcal{C}(K)} \leq C_\varepsilon (1 + \varepsilon)^n [\Gamma_m(K)]^n \|g\|_\infty.$$

Let  $l$  be an integer to be adjusted later, and  $f(z) = \sum_\alpha b_\alpha z^\alpha \in B_{H^2}$ , as well as  $g(z) = \sum_{|\alpha| \leq l} b_\alpha z^\alpha$ . We first note that (with  $M_m$  depending only on  $m$  and  $\rho$ , and since the number of  $\alpha$ 's such that  $|\alpha| \leq p$  is  $O(p^m)$ ):

$$\sum_{|\alpha| > l} \rho^{2|\alpha|} \leq M_m \sum_{p > l} p^m \rho^{2p} \leq M_m l^m \frac{\rho^{2l}}{(1 - \rho^2)^{m+1}}.$$

We next observe that, by the Cauchy-Schwarz and Parseval inequalities:

$$(6.8) \quad \|g\|_\infty \leq M_m l^{m/2},$$

and

$$(6.9) \quad |f(z) - g(z)| \leq M_m l^{m/2} \frac{|z|_\infty^l}{(1 - |z|_\infty^2)^{(m+1)/2}}, \quad \forall z \in \mathbb{D}^m.$$

where  $|z|_\infty := \max_{j \leq m} |z_j|$  if  $z = (z_1, \dots, z_m)$ .

The subspace  $F$  formed by functions  $v \circ \Phi$ , for  $v \in V$ , can be viewed as a subspace of  $L^\infty(\mathbb{T}^m) \subseteq L^2(\mathbb{T}^m)$  with respect to the Haar measure of  $\mathbb{T}^m$ , the distinguished boundary of  $\mathbb{D}^m$  (indeed, we can write  $(v \circ \Phi)^* = v \circ \Phi^*$ , where  $\Phi^*$  denotes the almost everywhere existing radial limits of  $\Phi(rz)$ , which belong to  $K$ ). Let finally  $E = P(F) \subseteq H^2(\mathbb{D}^m)$  where  $P: L^2(\mathbb{T}^m) \rightarrow H^2(\mathbb{T}^m) = H^2(\mathbb{D}^m)$  is the orthogonal projection. This is a subspace of  $H^2$  with dimension  $< n^m$ . Set temporarily  $\eta = C_\varepsilon (1 + \varepsilon)^n [\Gamma_m(K)]^n$ . It follows from (6.7) and (6.8) that, for some  $h \in V$ :

$$\|g - h\|_{\mathcal{C}(K)} \leq \eta \|g\|_\infty \leq \eta M_m l^{m/2}$$

and hence:

$$\|g \circ \Phi - h \circ \Phi\|_2 \leq \|g \circ \Phi - h \circ \Phi\|_\infty \leq \eta M_m l^{m/2},$$

implying by orthogonal projection:

$$\text{dist}(C_\Phi g, E) \leq \|g \circ \Phi - P(h \circ \Phi)\|_2 \leq \eta M_m l^{m/2}.$$

Now, since  $C_\Phi f(z) - C_\Phi g(z) = f(\Phi(z)) - g(\Phi(z))$ , (6.9) gives:

$$\|C_\Phi f - C_\Phi g\|_2 \leq \|C_\Phi f - C_\Phi g\|_\infty \leq M_m l^{m/2} \frac{\rho^l}{(1 - \rho^2)^{(m+1)/2}}$$

and hence:

$$\text{dist}(C_\Phi f, E) \leq M_m l^{m/2} \left( \frac{\rho^l}{(1 - \rho^2)^{(m+1)/2}} + C_\varepsilon (1 + \varepsilon)^n [\Gamma_m(K)]^n \right).$$

It ensues, since  $a_N(C_\Phi) = d_N(C_\Phi)$ , that:

$$[a_{n^m}(C_\Phi)]^{1/n} \leq (M_m l^{m/2})^{1/n} \left[ \frac{\rho^{l/n}}{(1 - \rho^2)^{(m+1)/2n}} + C_\varepsilon^{1/n} (1 + \varepsilon) \Gamma_m(K) \right].$$

Taking now for  $l$  the integer part of  $n \log n$ , and passing to the upper limit as  $n \rightarrow \infty$ , we obtain (since  $l/n \rightarrow \infty$  and  $(\log l)/n \rightarrow 0$ ):

$$\beta_m^+(C_\Phi) \leq (1 + \varepsilon) \Gamma_m(K),$$

and Theorem 6.4 follows.  $\square$

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