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Approximation numbers of composition operators on the Hardy space of the infinite polydisk

Daniel Li, Hervé Queffélec, L. Rodríguez-Piazza

March 14, 2017

Abstract. We study the composition operators of the Hardy space on $\mathbb{D}^\infty \cap \ell_1$, the ℓ_1 part of the infinite polydisk, and the behavior of their approximation numbers.

1 Introduction

Recently, in [2], we investigated approximation numbers $a_n(C_\varphi)$, $n \geq 1$, of composition operators C_φ , $C_\varphi(f) = f \circ \varphi$, on the Hardy or Bergman spaces $H^2(\Omega)$, $B^2(\Omega)$ over a bounded symmetric domain $\Omega \subseteq \mathbb{C}^d$. Assuming that $\varphi(\Omega)$ has non-empty interior, one of the main results of this study was the following theorem.

Theorem 1.1 ([2]). *Let $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$ be compact. Then:*

- 1) *we always have $a_n(C_\varphi) \geq c e^{-C n^{1/d}}$ where c, C are positive constants;*
- 2) *if Ω is a product of balls and if $\varphi(\Omega) \subseteq r \Omega$ for some $r < 1$, then:*

$$a_n(C_\varphi) \leq C e^{-c n^{1/d}}.$$

As a result, the minimal decay of approximation numbers is slower and slower as the dimension d increases, which might lead one to think that, in infinite-dimension, no compact composition operators can exist, since their approximation numbers will not tend to 0. After all, this is the case for the Hardy space of a half-plane, which supports no compact composition operator ([12], Theorem 3.1; in [9], it is moreover proved that $\|C_\varphi\|_e = \|C_\varphi\|$ as soon as C_φ is bounded; see also [15] for a necessary and sufficient condition for $H^2(\Omega)$ has compact composition operators, where Ω is a domain of \mathbb{C}). We will see that this is not quite the case here, even though the decay will be severely limited. In particular, we will never have a decay of the form $C e^{-c n^\delta}$ for some $c, C, \delta > 0$.

2 Framework and reminders

2.1 Hardy spaces on \mathbb{D}^∞

Let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle of the set of complex numbers. We consider \mathbb{T}^∞ and equip it with its Haar measure m . This is a compact Abelian group with dual $\mathbb{Z}^{(\infty)}$, the set of eventually zero sequences $\alpha = (\alpha_j)_{j \geq 1}$ of integers. We denote $L^2_{\mathbb{N}^{(\infty)}}(\mathbb{T}^\infty)$ the Hilbert subspace of $L^2(\mathbb{T}^\infty)$ formed by the functions f whose Fourier spectrum is contained in $\mathbb{N}^{(\infty)}$:

$$\widehat{f}(\alpha) := \int_{\mathbb{T}^\infty} f(z) \bar{z}^\alpha dm(z) = 0 \quad \text{if } \alpha \notin \mathbb{N}^{(\infty)}.$$

The set $E := \mathbb{N}^{(\infty)}$ is called the *narrow cone of Helson*, and we also denote $L^2_{\mathbb{N}^{(\infty)}}(\mathbb{T}^\infty) = L^2_E(\mathbb{T}^\infty)$. Any element of that subspace can be formally written as:

$$f = \sum_{\alpha \geq 0} c_\alpha e_\alpha \quad \text{with } c_\alpha = \widehat{f}(\alpha) \quad \text{and} \quad \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty.$$

Here, $(e_\alpha)_{\alpha \in \mathbb{Z}^{(\infty)}}$ is the canonical basis of $L^2(\mathbb{T}^\infty)$ formed by characters, and accordingly $(e_\alpha)_{\alpha \in \mathbb{N}^{(\infty)}}$ is the canonical basis of $L^2_E(\mathbb{T}^\infty)$.

Now we consider $\Omega_2 = \mathbb{D}^\infty \cap \ell_2$.

Any $f \sim \sum_{\alpha \geq 0} c_\alpha e_\alpha \in L^2_E(\mathbb{T}^\infty)$ defines an analytic function on the infinite-dimensional Reinhardt domain Ω_2 by the formula:

$$(2.1) \quad f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha$$

where the series is absolutely convergent for each $z = (z_j)_{j \geq 1} \in \Omega_2$, as the pointwise product of two square-summable sequences. Indeed, using an Euler type formula, we get for $z \in \Omega_2$:

$$\sum_{\alpha \geq 0} |z^\alpha|^2 = \prod_{j=1}^{\infty} (1 - |z_j|^2)^{-1} < \infty,$$

and hence, by the Cauchy-Schwarz inequality:

$$\sum_{\alpha \geq 0} |c_\alpha z^\alpha| \leq \left(\sum_{\alpha \geq 0} |c_\alpha|^2 \right)^{1/2} \left(\sum_{\alpha \geq 0} |z^\alpha|^2 \right)^{1/2} < \infty.$$

If $\alpha \in E$ and $z \in \Omega_2$, we have set, as usual, $z^\alpha = \prod_{j \geq 1} z_j^{\alpha_j}$.

This shows that $L^2_E(\mathbb{T}^\infty)$ can be identified with $H^2(\Omega_2)$, the Hardy-Hilbert space of analytic functions $f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha$ on Ω_2 with

$$\|f\|^2 := \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty.$$

This setting is customary in connection with Dirichlet series (see [7]).

In this paper, for a technical reason appearing below in the proof of Proposition 2.5, we will consider, instead of $\Omega_2 = \mathbb{D}^\infty \cap \ell_2$, the sub-domain:

$$\Omega = \mathbb{D}^\infty \cap \ell_1,$$

i.e. the *open* subset of ℓ^1 formed by the sequences:

$$z = (z_n)_{n \geq 1} \quad \text{such that} \quad |z_n| < 1, \forall n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} |z_n| < \infty,$$

and the restrictions to Ω of the functions $f \in H^2(\Omega_2)$. We denote $H^2(\Omega)$ the space of such restrictions.

Hence $f \in H^2(\Omega)$ if and only if:

$$f(z) = \sum_{\alpha \geq 0} c_\alpha z^\alpha \quad \text{with } z \in \Omega,$$

and $\|f\|^2 := \sum_{\alpha \geq 0} |c_\alpha|^2 < \infty$.

We now identify the space $L_E^2(\mathbb{T}^\infty)$ with the space $H^2(\Omega)$.

We more generally define Hardy spaces $H^p(\Omega)$, for $1 \leq p < \infty$, in the usual way:

$$H^p = H^p(\Omega) = \{f: \Omega \rightarrow \mathbb{C}; \|f\|_p < \infty\},$$

where f is analytic in Ω and $\|f\|_p = \sup_{0 < r < 1} M_p(r, f) = \lim_{r \rightarrow 1^-} M_p(r, f)$ with:

$$M_p(r, f) = \left(\int_{\mathbb{T}^\infty} |f(rz)|^p dm(z) \right)^{1/p}, \quad 0 < r < 1.$$

We have $\|f\| = \|f\|_2$. Moreover, H^q contractively embeds into H^p for $p < q$.

2.2 Singular numbers

We begin with a reminder of operator-theoretic facts. We recall that the approximation numbers $a_n(T) = a_n$ of an operator $T: H \rightarrow H$ (with H a Hilbert space) are defined by:

$$a_n = \inf_{\text{rank } R < n} \|T - R\|.$$

According to a 1957's result of Allahverdiev (see [3], page 155), we have $a_n = s_n$, the n -th singular number of T . We also recall a basic result due to H. Weyl and one obvious consequence:

Theorem 2.1. *Let $T: H \rightarrow H$ be a compact operator with eigenvalues (λ_n) rearranged in decreasing order and singular numbers (a_n) . Then:*

$$\prod_{j=1}^n |\lambda_j| \leq \prod_{j=1}^n a_j \quad \text{for all } n \geq 1.$$

As a consequence:

$$|\lambda_{2n}|^2 \leq a_1 a_n.$$

2.3 Spectra of projective tensor products

The following operator-theoretic result will play a basic role in the sequel. Let E_1, \dots, E_n be Banach spaces and let $E = \otimes_{i=1}^n E_i$ their *projective* tensor product (the only tensor product we shall use). If $T_i \in \mathcal{L}(E_i)$, we define as usual their projective tensor product $T = \otimes_{i=1}^n T_i \in \mathcal{L}(E)$ by its action on the atoms of E , namely:

$$T(\otimes_{i=1}^n x_i) = \otimes_{i=1}^n T_i(x_i).$$

Denote in general $\sigma(x)$ the spectrum of $x \in \mathcal{A}$ where \mathcal{A} is a unital Banach algebra. We recall ([13], chap.11, Theorem 11.23) the following result.

Lemma 2.2. *Let \mathcal{A} be a unital Banach algebra, and x_1, \dots, x_n be pairwise commuting elements of \mathcal{A} . Then:*

$$\sigma(x_1 \cdots x_n) \subseteq \prod_{i=1}^n \sigma(x_i).$$

Here, $\prod_{i=1}^n \sigma(x_i)$ is the product in the Minkowski sense, namely:

$$\prod_{i=1}^n \sigma(x_i) = \left\{ \prod_{i=1}^n \lambda_i : \lambda_i \in \sigma(x_i) \right\}.$$

As a consequence, we then have the following lemma due to Schechter, which we prove under a weakened form, sufficient here, and which is indeed already in [1] (we just add a few details because this is a central point in our estimates).

Lemma 2.3. *Let F be a Banach space, $T_1, \dots, T_n \in \mathcal{L}(F)$ and $T = \otimes_{i=1}^n T_i$. Then $\sigma(T) \subset \prod_{i=1}^n \sigma(T_i)$.*

Proof. To save notation, we assume $n = 2$. Let $x_1 = T_1 \otimes I_2$ and $x_2 = I_1 \otimes T_2$ where I_i is the identity of E_i . Clearly,

$$x_1 x_2 = x_2 x_1 = T_1 \otimes T_2 = T \quad \text{and} \quad \sigma(x_i) = \sigma(T_i)$$

where the spectrum of x_i is in the Banach algebra $\mathcal{L}(E)$ and that of T_i in $\mathcal{L}(E_i)$. Lemma 2.2 now gives:

$$\sigma(T) = \sigma(x_1 x_2) \subseteq \sigma(x_1) \sigma(x_2) = \sigma(T_1) \sigma(T_2),$$

hence the result. □

2.4 Schur maps and composition operators

We now pass to some general facts on composition operators C_φ , defined by $C_\varphi(f) = f \circ \varphi$, associated with a Schur map, namely a *non-constant* analytic self-map φ of Ω . We say that φ is a *symbol* for $H^2(\Omega)$ if C_φ is a bounded linear operator from $H^2(\Omega)$ into itself.

The differential $\varphi'(a)$ of φ at some point $a \in \Omega$ is a bounded linear map $\varphi'(a): \ell^1 \rightarrow \ell^1$.

Definition 2.4. The symbol φ is said to be truly infinite-dimensional if the differential $\varphi'(a)$ is an injective linear map from ℓ^1 into itself for at least one point $a \in \Omega$.

In finite dimension, this amounts to saying that $\varphi(\Omega)$ has non-void interior.

We have the following general result.

Proposition 2.5. Let $(\varphi_j)_{j \geq 1}$ be a sequence of analytic self-maps of \mathbb{D} such that $\sum_{j \geq 1} |\varphi_j(0)| < \infty$. Then, the mapping $\varphi: \Omega \rightarrow \mathbb{C}^\infty$ defined by the formula $\varphi(z) = (\varphi_j(z_j))_{j \geq 1}$ maps Ω to itself and is a symbol for $H^2(\Omega)$.

Proof. First, the Schwarz inequality:

$$|\varphi_j(z_j) - \varphi_j(0)| \leq 2|z_j|$$

shows that $\varphi(z) \in \Omega$ when $z \in \Omega$. To see that φ is moreover a symbol for $H^2(\Omega)$, we use the fact ([8]) that:

$$(2.2) \quad \|C_{\varphi_j}\| \leq \sqrt{\frac{1 + |\varphi_j(0)|}{1 - |\varphi_j(0)|}}.$$

Now, by the separation of variables and Fubini's theorem, we easily get:

$$(2.3) \quad \|C_\varphi\| \leq \prod_{j=1}^{\infty} \|C_{\varphi_j}\| < \infty.$$

As $\sum_{j \geq 1} |\varphi_j(0)| < \infty$, by hypothesis, the infinite product

$$\prod_{j \geq 1} \sqrt{\frac{1 + |\varphi_j(0)|}{1 - |\varphi_j(0)|}}$$

converges and, in view of (2.2) and (2.3), C_φ is bounded. \square

We also have the following useful fact.

Lemma 2.6. The automorphisms of Ω act transitively on Ω and define bounded composition operators on $H^2(\Omega)$.

Proof. Let $a = (a_j)_j \in \Omega$ and let $\Psi_a: \Omega \rightarrow \mathbb{C}^\infty$ be defined by:

$$\Psi_a(z) = (\Phi_{a_j}(z_j))_{j \geq 1}$$

where in general $\Phi_u: \mathbb{D} \rightarrow \mathbb{D}$ is defined by $\Phi_u(z) = (z - u)/(1 - \bar{u}z)$. The Schwarz lemma gives $|\Phi_{a_j}(z_j) + a_j| \leq 2|z_j|$, and shows that Ψ_a maps Ω to itself. Clearly, Ψ_a is an automorphism of Ω with inverse Ψ_{-a} and $\Psi_a(a) = 0$. The fact that the composition operator C_{Ψ_a} is bounded on $H^2(\Omega)$ is a consequence of Proposition 2.5. \square

3 Spectrum of compact composition operators

We begin with the following definition, following [10].

Definition 3.1. Let $\varphi: \Omega \rightarrow \Omega$ be a truly infinite-dimensional symbol. We say that φ is compact if $\overline{\varphi(\Omega)}$ is a compact subset of Ω .

We then have the following result.

Lemma 3.2. If $\varphi: \Omega \rightarrow \Omega$ is a compact mapping, then:

- 1) $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$ is bounded and moreover compact.
- 2) If $a \in \Omega$ a fixed point of φ , $\varphi'(a) \in \mathcal{L}(\ell^1)$ is a compact operator.

Proof. 1) follows from a H. Schwarz type criterion via an Ascoli-Montel type theorem: every sequence (f_n) of $H^2(\Omega)$ bounded in norm contains a subsequence which converges uniformly on compact subsets of Ω . Indeed, we have the following ([4], chap. 17, p. 274): if A is a locally bounded set of holomorphic functions on Ω , then A is locally equi-Lipschitz, namely every point $a \in \Omega$ has a neighbourhood $U \subset \Omega$ such that:

$$z, w \in U \quad \text{and} \quad f \in A \quad \implies \quad |f(z) - f(w)| \leq C_{A,U} \|z - w\|.$$

The Ascoli-Montel theorem easily follows from this. Then, if $f_n \in H^2(\Omega)$ converges weakly to 0, it converges uniformly to 0 on compact subsets of Ω ; in particular on $\overline{\varphi(\Omega)}$. This means that $\|C_\varphi(f_n)\|_\infty = \|f_n \circ \varphi\|_\infty \rightarrow 0$, implying $\|f_n \circ \varphi\|_2 \rightarrow 0$ and the compactness of C_φ .

Actually, C_φ is compact on every Hardy space $H^p(\Omega)$, $1 \leq p \leq \infty$. This observation will be useful later on.

For 2), we may indeed dispense ourselves with the invariance of a , and force $a = 0$ to be a fixed point of φ . Indeed, we can replace φ by $\psi = \Psi_b \circ \varphi \circ \Psi_a$ where $b = \varphi(a)$ is arbitrary, and use Lemma 2.6 as well as the ideal property of compact linear operators. We set $A = \varphi'(0)$. Expanding each coordinate φ_j of φ in a series of homogeneous polynomials, we may write (since $\varphi(0) = 0$):

$$\varphi(z) = \sum_{|\alpha|=1} c_\alpha z^\alpha + \sum_{s=2}^{\infty} \left(\sum_{|\alpha|=s} c_\alpha z^\alpha \right) = A(z) + \sum_{s=2}^{\infty} \left(\sum_{|\alpha|=s} c_\alpha z^\alpha \right),$$

where $c_\alpha = (c_{\alpha,j})_{j \geq 1} \in \mathbb{C}^\infty$. We clearly have (looking at the Fourier series of $\varphi(z e^{i\theta})$):

$$(3.1) \quad \|z\|_1 < 1 \quad \implies \quad z \in \Omega \quad \implies \quad A(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(z e^{i\theta}) e^{-i\theta} d\theta.$$

Since φ is compact, this clearly implies, with B the open unit ball of ℓ^1 , that $A(B)$ is totally bounded, proving the compactness of A . \square

The following extension of results of [11], then [1] and [6], which themselves extend a theorem of G. Königs ([14], p. 93) will play an essential role for lower bounds of approximation numbers.

Theorem 3.3. *Let $\varphi: \Omega \rightarrow \Omega$ be a compact symbol. Assume there is $a \in \Omega$ such that $\varphi(a) = a$ and that $\varphi'(a) \in \mathcal{L}(\ell^1)$ is injective. Then, the spectrum of $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$ is exactly formed by the numbers λ^α , $\alpha \in \mathbb{N}^{(\infty)}$, and $0, 1$, where $(\lambda_j)_{j \geq 1}$ denote the eigenvalues of $A := \varphi'(a)$ and:*

$$\lambda^\alpha = \prod_{j \geq 1} \lambda_j^{\alpha_j} \quad \text{if } \alpha = (\alpha_j)_{j \geq 1} \in \mathbb{N}^{(\infty)}.$$

Proof. This is proved in [1] for the unit ball B_E of an arbitrary Banach space E and for the space $H^\infty(B_E)$, in four steps which are the following:

1. If $\varphi(B_E)$ lies strictly inside B_E (namely if $\varphi(B_E) \subseteq rB_E$ for some $r < 1$), in particular when φ is compact, φ has a unique fixed point $a \in B_E$, according to a theorem of Earle and Hamilton.
2. The spectrum of C_φ contains the numbers λ where λ is an eigenvalue of $\varphi'(a)$ or $\lambda = 0, 1$.
3. It is then proved that the spectrum of C_φ contains the numbers λ^α and $0, 1$.
4. It is finally proved that spectrum of C_φ is contained in the numbers λ^α and $0, 1$.

Here, handling with the domain Ω , we see that:

1. True or not for Ω , the Earle-Hamilton theorem is not needed since we will force, by a change of the compact symbol φ in another compact symbol $\psi = \Psi_b \circ \varphi \circ \Psi_a$, the point 0 to be a fixed point. Moreover $A = \psi'(0)$ is injective if $\varphi'(a)$ is, since Ψ'_a and Ψ'_b are invertible.
2. The second step (non-surjectivity) is valid for any domain and for $H^2(\Omega)$, or $H^p(\Omega)$, in exactly the same way.
3. The third step consists of proving $\{\lambda^\alpha\} \subseteq \sigma(C_\varphi)$.
For that purpose, assume that $\lambda^\alpha = \prod_{l=1}^m \lambda_l \neq 0$ with λ_l an eigenvalue of $\varphi'(0)$ and with repetitions allowed. As we already mentioned, under the assumption of compactness of φ , C_φ is compact on $H^p(\Omega)$ as well, for any $p \geq 1$. We take here $p = 2m$. Step 2 provides us with non-zero functions $f_i \in H^p(\Omega)$ such that $f_i \circ \varphi = \lambda_i f_i$, $1 \leq i \leq m$, since for the compact operator $C_\varphi: H^p \rightarrow H^p$, non-surjectivity implies non-injectivity. Let $f = \prod_{1 \leq i \leq m} f_i$. Then, using the integral representation of the norm and the Hölder inequality, we see that $f \in H^2(\Omega)$, $f \neq 0$ and $f \circ \varphi = \lambda^\alpha f$, proving our claim.
4. The fourth step is valid as well, with a slight simplification: we have to show that, if $\mu \neq 1$ is not of the form λ^α , then $C_\varphi - \mu I$ is injective. Let $f \in H^2(\Omega)$ satisfying $f \circ \varphi = \mu f$ and let:

$$f(z) = \sum_{m=0}^{\infty} \frac{d^m f(0)}{m!} (z^m)$$

be the Taylor expansion of f about $z = 0$ (observe that Ω is a Reinhardt domain). As usual, $d^m f(0) =: L_m$ is an m -linear symmetric form on $F = \ell^1$ and the notation $L_m(z^m)$ means $L_m(z, z, \dots, z)$.

Observe that L_m can be isometrically identified with an element (denoted $\overline{L_m}$) of $\mathcal{L}(F^{\otimes n})$ defined by the formula:

$$\overline{L_m}(x_1 \otimes \cdots \otimes x_n) = L_m(x_1, \dots, x_m).$$

We will prove by induction that $L_n = 0$ for each n . For this, we can avoid the appeal to transposes of [1] as follows: if the result holds for L_m with $m < n$, one gets (comparing the terms in z^n in both members of $f \circ \varphi = \mu f$):

$$(3.2) \quad \mu A = A \circ B \quad \text{where} \quad A = \overline{L_n} \quad \text{and} \quad B = \varphi'(0)^{\otimes n}.$$

That is $A(B - \mu I) = 0$ where I is the identity map of $F^{\otimes n}$. Now, $B - \mu I$ is invertible in $\mathcal{L}(F)$ by Lemma 3.3, so that $A = A(B - \mu I)(B - \mu I)^{-1} = 0$.

The proof is complete. \square

The following theorem summarizes and exploits the preceding theorem. Possibly, some restrictions can be removed, and we could only assume the compactness of C_φ , not of φ itself. After all, in dimension one, there are symbols φ with $\|\varphi\|_\infty = 1$ for which $C_\varphi: H^2 \rightarrow H^2$ is compact.

Theorem 3.4. *Let $\varphi: \Omega \rightarrow \Omega$ be a truly infinite-dimensional compact mapping of Ω . Then:*

- 1) $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$ is bounded and even compact.
- 2) $A = \varphi'(0)$ is compact.
- 3) No $\delta > 0$ can exist such that $a_n(C_\varphi) \leq C e^{-c n^\delta}$ for all $n \geq 1$. More precisely, the numbers a_n satisfy:

$$(3.3) \quad \sum_{n \geq 1} \frac{1}{\log^p(1/a_n)} = \infty \quad \text{for all} \quad p < \infty.$$

Proof. The proof is based on the previous Theorem 3.3. Without loss of generality, we can assume that $\varphi(0) = 0$ and $\varphi'(0)$ is injective, by using a point a at which $\varphi'(a)$ is injective, and then the fact that automorphisms of Ω act transitively on Ω , act boundedly on $H^2(\Omega)$, and the ideal property of approximation numbers. More precisely, we pass to $\Psi = \Psi_b \circ \varphi \circ \Psi_a$ with $b = \varphi(a)$ and get:

$$\Psi(0) = 0 \quad \text{and} \quad \Psi'(b) = \Psi'_b(b) \varphi'(a) \Psi'_a(0)$$

injective, since $\Psi'_b(b)$ and $\Psi'_a(0)$ are, and Ψ_a and Ψ_b are automorphisms of Ω .

We now have the following simple but crucial lemma.

Lemma 3.5. *Whatever the choice of the numbers λ_j with $0 < |\lambda_j| < 1$, denoting by $(\delta_n)_{n \geq 1}$ the non-increasing rearrangement of the numbers λ^α , one has:*

$$\sum_{n \geq 1} \frac{1}{\log^p(1/\delta_n)} = \infty \quad \text{for all} \quad p < \infty.$$

Proof of the Lemma. For any positive integer p , we set:

$$q = 2p, \quad \log 1/|\lambda_j| = A_j,$$

and we use that:

$$\sum_{1 \leq j \leq q} \alpha_j A_j \leq \left(\sum_{1 \leq j \leq q} \alpha_j^2 \right) \left(\sum_{1 \leq j \leq q} A_j^2 \right) =: C_q \left(\sum_{1 \leq j \leq q} \alpha_j^2 \right) = C_q \|\alpha\|^2,$$

where $\|\cdot\|$ stands for the euclidean norm in \mathbb{R}^q . We then get:

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{\log^p(1/\delta_n)} &= \sum_{\alpha > 0} \frac{1}{\log^p(1/|\lambda^\alpha|)} \\ &\geq \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{\log^p(1/|\lambda_1^{\alpha_1}| \cdots 1/|\lambda_q^{\alpha_q}|)} \\ &= \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{(\alpha_1 A_1 + \cdots + \alpha_q A_q)^p} \\ &\geq C_q^{-p} \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{(\alpha_1^2 + \cdots + \alpha_q^2)^p} \\ &= C_q^{-p} \sum_{\alpha_j \geq 1, 1 \leq j \leq q} \frac{1}{\|\alpha\|^q} = \infty, \end{aligned}$$

because:

$$\int_{x \in \mathbb{R}^q, \|x\| \geq 1} \frac{1}{\|x\|^q} dx = c_q \int_1^\infty \frac{r^{q-1}}{r^q} dr = \infty.$$

This proves the lemma. \square

This can be transferred to the approximation numbers $a_n = a_n(C_\varphi)$ to end the proof of Theorem 3.4. Indeed, we know from Lemma 3.5 that the non-increasing rearrangement (δ_n) of the eigenvalues λ^α of C_φ satisfies

$$\sum_{n \geq 1} \frac{1}{\log^p(1/\delta_n)} = \infty.$$

Since a divergent series of non-negative and non-increasing numbers u_n satisfies $\sum u_{2n} = \infty$, we further see that:

$$\sum_{n \geq 1} \frac{1}{\log^p(1/\delta_{2n})} = \infty \quad \text{for all } p < \infty.$$

Moreover, by Theorem 2.1 we have:

$$(3.4) \quad \left(\frac{1}{2 \log 1/\delta_{2n}} \right)^p \leq \left(\frac{1}{\log 1/(a_1 a_n)} \right)^p.$$

Since $1/(\log 1/a_1 a_n) \sim 1/(\log 1/a_n)$, Lemma 3.5 then gives the result. This clearly prevents an inequality of the form $a_n \leq C e^{-c n^\delta}$ for some positive numbers c, C, δ and all $n \geq 1$. Indeed, this would imply:

$$\sum_{n \geq 1} \frac{1}{\log^p(1/a_n)} < \infty \quad \text{for } p > 1/\delta,$$

contradicting (3.3). □

Remarks. Let us briefly comment on the assumptions in Theorem 3.4.

1) We do not need the Earle-Hamilton theorem under our assumptions. The Schauder-Tychonoff theorem gives the existence (if not the uniqueness) of a fixed point for φ in Ω (bounded and convex).

2) The Earle-Hamilton theorem is in some sense more general (for analytic maps) since it remains valid when $\overline{\varphi(\Omega)}$ is only assumed to lie strictly inside Ω , i.e. when $\varphi(\Omega) \subseteq r\Omega$ for some $r < 1$. But this assumption does not ensure the compactness of C_φ as indicated by the simple example $\varphi(z) = rz$, $0 < r < 1$. The coordinate functions $z \mapsto z_n$ converge weakly to 0, while $\|C_\varphi(z_n)\|_{H^2(\Omega)} = r$.

3) The mere assumption that $\overline{\varphi(\Omega)}$ is compact is not sufficient either. Just take:

$$\varphi(z) = \left(\frac{1+z_1}{2}, 0, \dots, 0, \dots \right).$$

Since the composition operator C_{φ_1} associated with $\varphi_1(z) = \frac{1+z}{2}$ is notoriously non-compact on $H^2(\mathbb{D})$, neither is C_φ on $H^2(\Omega)$. Yet, $\overline{\varphi(\Omega)}$ is obviously compact in ℓ^1 .

4 Possible upper bounds

Recall that $\Omega = \mathbb{D}^\infty \cap \ell^1$.

4.1 A general example

Theorem 4.1. *Let $\varphi((z_j)_j) = (\lambda_j z_j)_j$ with $|\lambda_j| < 1$ for all j , so that $\varphi(\Omega) \subseteq \Omega$ and $\varphi'(0)$ is the diagonal operator with eigenvalues λ_j , $j \geq 1$, on the canonical basis of ℓ^1 . Let $p > 0$. Then:*

$$(\lambda_j)_j \in \ell^p \quad \implies \quad C_\varphi \in S_p.$$

In particular, there exist truly infinite-dimensional symbols on Ω such that the composition operator $C_\varphi: H^2(\Omega) \rightarrow H^2(\Omega)$ is in all Schatten classes S_p , $p > 0$.

Proof. Since C_φ is diagonal on the orthonormal basis $(z^\alpha)_\alpha$ of the Hilbert space $H^2(\Omega)$, with $C_\varphi(z^\alpha) = \varphi^\alpha$, its approximation numbers are the non-increasing rearrangement of the moduli of eigenvalues λ^α , so that an Euler product-type computation gives:

$$\sum_{n=1}^{\infty} a_n^p = \sum_{\alpha \in E} |\lambda^\alpha|^p = \sum_{\alpha_j \in \mathbb{N}} \prod_{j \geq 1} |\lambda_j|^{p\alpha_j} = \prod_{j=1}^{\infty} (1 - |\lambda_j|^p)^{-1} < \infty.$$

To obtain $C_\varphi \in \bigcap_{p>0} S_p$, just take $\lambda_n = e^{-n}$. This ends the proof. \square

4.2 A sharper upper bound

By making a more quantitative study, we can prove the following result.

Theorem 4.2. *For any $0 < \delta < 1$, there exists a compact composition operator on $H^2(\Omega)$, with a truly infinite-dimensional symbol, such that, for some positive constants c, C, b , we have:*

$$a_n(C_\varphi) \leq C \exp\left(-c e^{b(\log n)^\delta}\right).$$

Proof. Take the same operator C_φ as in Theorem 4.1, with $\lambda_n = e^{-A_n}$ where the positive numbers A_n have to be adjusted. Its approximation numbers a_N are then the non-increasing rearrangement of the sequence of numbers $(\varepsilon_n)_n := (\lambda^\alpha)_\alpha$. This suggests using a generating function argument, namely considering $\sum \varepsilon_n x^n$, but the rearrangement perturbs the picture. Accordingly, we follow a slightly different route. Fix an integer $N \geq 1$ and a real number $r > 0$. Observe that, following the proof of Theorem 4.1:

$$N a_N^r \leq \sum_{n=1}^N a_n^r \leq \sum_{n=1}^{\infty} a_n^r = \prod_{n=1}^{\infty} (1 - e^{-rA_n})^{-1}.$$

First, consider the simple example $A_n = n$. We get:

$$N a_N^r \leq \eta(e^{-r})$$

where η is the Dedekind eta function (see [5]) given by:

$$\eta(x) = \prod_{n=1}^{\infty} (1 - x^n)^{-1} = \sum_{n=0}^{\infty} p(n) x^n, \quad |x| < 1,$$

where $p(n)$ is the number of partitions of the integer n . It is well-known ([5], Ch. 7, p. 169) that $\eta(e^{-r}) \leq e^{D/r}$ with $D = \pi^2/6$, so that:

$$a_N \leq \exp\left(\frac{D}{r^2} - \frac{\log N}{r}\right).$$

Optimizing with $r = 2D/\log N$, we get:

$$a_N \leq \exp(-c \log^2 N),$$

with $c = 1/4D$. This is more precise than Theorem 4.1.

We now show that if A_n increases faster, we can achieve the decay of Theorem 4.2. As before, we get in general:

$$(4.1) \quad a_N \leq \inf_{x>1} (\exp [x(\log F(x^{-1}) - \log N)]),$$

where

$$F(r) = \prod_{n=1}^{\infty} (1 - e^{-rA_n})^{-1}.$$

We have:

$$\log F(r) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{e^{-rmA_n}}{m} \right) = \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{n=1}^{\infty} e^{-rmA_n} \right).$$

Now, take $A_n = e^{n^\alpha}$ where $\alpha > 0$ is to be chosen. We have:

$$\sum_{n=1}^{\infty} e^{-rm e^{n^\alpha}} \leq \int_0^{\infty} e^{-rm e^{t^\alpha}} dt =: I_m(r).$$

Standard estimates now give, for $r < 1$:

$$\begin{aligned} I_m(r) &= \int_1^{\infty} e^{-rmx} \frac{1}{\alpha} (\log x)^{\frac{1}{\alpha}-1} \frac{dx}{x} = \int_{rm}^{\infty} e^{-y} \frac{1}{\alpha} \left(\log \frac{y}{rm} \right)^{\frac{1}{\alpha}-1} \frac{dy}{y} \\ &\lesssim \left(\log \frac{1}{r} \right)^{\frac{1}{\alpha}-1} \int_{rm}^{\infty} e^{-y} \frac{dy}{y} \lesssim e^{-rm} \left(\log \frac{1}{r} \right)^{\frac{1}{\alpha}}, \end{aligned}$$

so that:

$$\log F(r) \lesssim (\log 1/r)^{\frac{1}{\alpha}} \sum_{m=1}^{\infty} m^{-1} e^{-rm} \lesssim (\log 1/r)^{\frac{1}{\alpha}+1}.$$

Going back to (4.1), we get, for some constant $C > 0$, and for $x = 1/r > 1$:

$$a_N \leq C \exp [C x ((\log x)^{\frac{1}{\alpha}+1} - \log N)].$$

Adjusting $x = x_N > 1$ so as to have $(\log x)^{\frac{1}{\alpha}+1} = \log N - 1$, that is:

$$x_N = \exp [(\log(N/e))^{\frac{\alpha}{\alpha+1}}],$$

we get $a_N \leq C e^{-c x_N}$, which is the claimed result with $\delta = \alpha/(\alpha + 1)$.

This δ can be taken arbitrarily in $(0, 1)$ by choosing α suitable, and we are done. \square

Remark. Of course, $\delta = 1$ is forbidden, because this would give $a_n \leq C e^{-c n^b}$, implying:

$$\sum_{n=1}^{\infty} \frac{1}{(\log 1/a_n)^p} \lesssim \sum_{n=1}^{\infty} n^{-bp} < \infty,$$

for large p , and contradicting Theorem 3.4.

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