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DAVID CHATAUR, MARTINTXO SARALEGI-ARANGUREN, AND DANIEL TANRÉ

Abstract. For having a Poincaré duality via a cap product between the intersection homology of a paracompact oriented pseudomanifold and the cohomology given by the dual complex, G. Friedman and J. E. McClure need a coefficient field or an additional hypothesis on the torsion. In this work, by using the classical geometric process of blowing-up, adapted to a simplicial setting, we build a cochain complex which gives a Poincaré duality via a cap product with intersection homology, for any commutative ring of coefficients. We prove also the topological invariance of the blown-up intersection cohomology with compact supports in the case of a paracompact pseudomanifold with no codimension one strata.

This work is written with general perversities, defined on each stratum and not only in function of the codimension of strata. It contains also a tame intersection homology, suitable for large perversities.

Contents

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Introduction

Intersection homology was defined by M. Goresky and R. MacPherson in [17], [18], with the existence of a Poincaré duality in the case of rational coefficients. If X is a compact, oriented, n-dimensional PL-pseudomanifold, Goresky and MacPherson establish in their first paper on intersection homology ([17, Theorem 1], see also [12], [15]) the existence of an intersection product, $\pitchfork\colon H_i^{\overline{p}}(X;\mathbb{Z}) \times H_j^{\overline{q}}(X;\mathbb{Z}) \to H^{\overline{r}}_{i+j-n}(X;\mathbb{Z})$, for perversities such that $\overline{p}+\overline{q} \leq \overline{r}$. Let $\overline{t}$ be the top perversity defined by $\overline{t}(i) = i-2$. By composing with an augmentation, $\varepsilon\colon H_0^{\overline{t}}(X;\mathbb{Z}) \to \mathbb{Z}$, the authors show ([17, Theorem 2]) that this correspondence gives a bilinear form,

$$H_i^{\overline{p}}(X;\mathbb{Z}) \times H_{n-i}^{\overline{t}-\overline{p}}(X;\mathbb{Z}) \to H_0^{\overline{t}}(X;\mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z},$$

which is non degenerate after tensorisation with $\mathbb{Q}$. As showed by Goresky and Siegel in [20], an extension of this result to $\mathbb{Z}$ cannot remain without an hypothesis on the torsion of the intersection homology of the links of the pseudomanifold (see [11] for an extension to homotopically stratified spaces).

Besides, mention the different approach of M. Banagl ([1]) who associates a CW-complex $I^{\overline{p}}X$ to certain stratified spaces. The rational homology of these spaces satisfies a generalized form of Poincaré duality and present some concrete advantages. Their homology being different from intersection homology, their study needs an ad'hoc approach and they are not considered in this work.

There exists also an approach of Poincaré duality of a manifold by mixing homology and cohomology with a cap product. This method was achieved with success in intersection homology and cohomology by G. Friedman et J.E. McClure ([16]) in the case of field coefficients, or with an hypothesis on the torsion of the intersection homology of the links ([9, Chapter 8]). Their intersection cohomology is defined as the homology of the linear dual of the intersection chain complex; we denote it by $H^*_{\overline{p}}(X;R)$ with R a commutative ring. In this context, the extension of such result to any commutative ring is not possible.

In this work, we continue with the paradigm of chain and cochain complexes. But, instead of taking the linear dual of the intersection chain complex, we consider a complex coming from a simplicial adaptation of the geometric blow-up which was already present in [2], [25]. For any commutative ring R, we define a cochain complex endowed with a cup product, $\widetilde{N}^*_{\overline{\bullet}}(X;R)$, whose homology in perversity $\overline{p}$ is denoted $\mathscr{H}^*_{\overline{p}}(X;R)$ and called *blown-up intersection cohomology* (or Thom-Whitney cohomology in some previous works, [3], [6], [4], [5]). A version with compact supports is introduced in Definition 2.2 and denoted $\mathscr{H}^*_{\overline{p},c}(X;R)$. In the case of Goresky and MacPherson perversities ([17]), our main result can be stated as follows.

Main Theorem. *Let R be a commutative ring and X an oriented, paracompact, n-dimensional pseudomanifold. Then, for any Goresky and MacPherson perversity, the cap product with the orientation class of X defines an isomorphism*

$$\mathcal{D}\colon \mathscr{H}^i_{\overline{p},c}(X;R) \xrightarrow{\cong} H^{\overline{p}}_{n-i}(X;R).$$

If we change of paradigm and consider the sheaf version of intersection homology ([18]), the blown-up cohomology appears as the Deligne sheaf defining intersection homology. We prove it explicitly with a direct approach in [7].

The complex $\widetilde{N}^*_\bullet(X;R)$ has several properties which facilitate its use. For instance, the complex $\widetilde{N}^*_{\overline{\bullet}}(X;R)$ is local in essence and it allows the determination of the admissibility of a cochain by considering individually each simplex of its support. We quote also that the operations *cup* and *cap* are defined from cochain complexes and not only in the derived category. The existence of cup_i products at the cochain level allowed in [6] an explicit determination of the rank of perversities in the definition of Steenrod intersection squares. As a consequence, we were able to give a positive answer ([6, Theorem B]) to a conjecture of M. Goresky et W. Pardon ([19]).

Actually, we prove the Main Theorem in the setting of general perversities introduced by MacPherson in [22], cf. Theorem B. These perversities are defined individually on each stratum and not only as a function of their codimension (cf. Definition 1.4). This allows a larger spectrum of the values taken by the perversities.

Without going too much into details at the level of this introduction, we may observe that, in the case of a perversity $\overline{p}$ such that $\overline{p} \leq \overline{t}$, each $\overline{p}$-allowable simplex as well as its boundary have a support which is not included in the singular part. As this property disappears if $\overline{p} \not\leq \overline{t}$, we introduce what we call *tame intersection homology* and denote $\mathfrak{H}^{\overline{p}}_*(X;R)$. The tame intersection homology keeps the behavior of intersection homology (see [4]) and is isomorphic to it when $\overline{p} \leq \overline{t}$. We denote $\mathfrak{H}^*_{\overline{p}}(X;R)$ the associated cohomology and $\mathfrak{H}^*_{\overline{p},c}(X;R)$ the variant with compact supports. In the case of a paracompact oriented pseudomanifold, Theorem B gives an isomorphism between the blown-up intersection cohomology $\mathscr{H}^*_{\overline{p},c}(X;R)$ and $\mathfrak{H}^{\overline{p}}_{n-*}(X;R)$ for any commutative ring R and any perversity $\overline{p}$. We complete this work with a proof of the topological invariance of the blown-up cohomology with compact supports in Theorem A.

Section 1 is a recall on intersection homology. To achieve the program above, we define and establish the main properties of the blown-up cohomology with compact supports, $\mathscr{H}^*_{\overline{p},c}(-)$, in Section 2: existence of a Mayer-Vietoris sequence (Proposition 2.12), cohomology of a cone (Proposition 2.14), cohomology of the product $X \times \mathbb{R}$ (Proposition 2.15) and comparison of $\mathscr{H}^*_{\overline{p},c}(-)$ and $\mathfrak{H}^*_{\overline{p},c}(-)$ (Proposition 2.20). In particular, we prove $\mathscr{H}^*_{\overline{p},c}(X;R) \cong \mathfrak{H}^*_{\overline{t}-\overline{p},c}(X;R)$ if R is a field and X a paracompact pseudomanifold. The topological invariance for a paracompact CS set with no codimension one strata and a Goresky and MacPherson perversity is established in Section 3 as Theorem A. Section 4 is concerned with the proof of Poincaré duality (Theorem B).

In all the text, R is a commutative ring (always supposed with unit) and we do not mention it explicitly in the proofs. The degree of an element a of a graded module is represented by $|a|$. For any topological space X, we denote by $\mathsf{c}X = X \times [0,1]/X \times \{0\}$ the cone on X and by $\mathring{\mathsf{c}}X = X \times [0,1[/X \times \{0\}$ the open cone on X.

1. Background on intersection homology

We recall the basics we need, sending the reader to [17], [9] or [3] for more details.

1.1. **Pseudomanifolds.**

Definition 1.1. A *filtered space of dimension* n, $(X, (X_i)_{0\leq i\leq n})$, is a Hausdorff space together with a filtration by closed subsets,

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X,$$

such that $X_n \backslash X_{n-1} \neq \emptyset$. The connected components S of $X_i \backslash X_{i-1}$ are the *strata* of X and we set $\dim S = i$, $\operatorname{codim} S = \dim X - \dim S$. The strata of $X_n \backslash X_{n-1}$ are called *regular*. The set of non-empty strata of X is denoted $\mathcal{S}_X$. The subspace X_{n-1} is called *the singular set.*

An open subset U of X is endowed with the *induced filtration,* defined by $U_i = U \cap X_i$. If M is a manifold, the *product filtration* is defined by $(M \times X)_i = M \times X_i$.

The CS sets introduced in [27] are a weaker version of pseudomanifolds that is sufficient for the topological invariance property.

Definition 1.2. A *CS set* of dimension n is a filtered space,

$$X_{-1} = \emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-2} \subseteq X_{n-1} \subsetneqq X_n = X,$$

such that, for each $i \in \{0, \ldots, n\}$, $X_i \backslash X_{i-1}$ is a topological manifold of dimension i or the empty set. Moreover each $x \in X_i \backslash X_{i-1}$ with $i \neq n$ admits

(i) an open neighborhood V of x in X, endowed with the induced filtration,
(ii) an open neighborhood U of x in $X_i \backslash X_{i-1}$,
(iii) a filtered compact space L of dimension $n-i-1$, whose cone $\mathring{\mathrm{c}}L$ is endowed with the conic filtration, $(\mathring{\mathrm{c}}L)_i = \mathring{\mathrm{c}}L_{i-1}$,
(iv) a homeomorphism, $\varphi \colon U \times \mathring{\mathrm{c}}L \to V$, such that
  (a) $\varphi(u, \mathtt{v}) = u$, for any $u \in U$, where $\mathtt{v}$ is the apex of $\mathring{\mathrm{c}}L$,
  (b) $\varphi(U \times \mathring{\mathrm{c}}L_j) = V \cap X_{i+j+1}$, for any $j \in \{0, \ldots, n-i-1\}$.

The filtered space L is called the *link* of x. The CS set is called *normal* if its links are connected.

We take over the original definition of pseudomanifold given by Goresky and MacPherson ([17]) but without the restriction on the existence of strata of codimension 1.

Definition 1.3. A *topological pseudomanifold of dimension* n (or a pseudomanifold) is a CS set of dimension n whose links of points $x \in X_i \backslash X_{i-1}$ are topological pseudomanifolds of dimension $(n-i-1)$ for all $i \in \{0, \ldots, n-1\}$. Any open subset of a pseudomanifold is a pseudomanifold for the induced structure.

1.2. **Perversities.** We begin with the perversities of [17] and continue with a more general notion of perversity, introduced in [22] and already present in [25], [26], [13], [14], [16].

Definition 1.4. A GM-*perversity* is a map $\overline{p} \colon \mathbb{N} \to \mathbb{Z}$ such that $\overline{p}(0) = \overline{p}(1) = \overline{p}(2) = 0$ and $\overline{p}(i) \leq \overline{p}(i+1) \leq \overline{p}(i) + 1$, for all $i \geq 2$. Among them, mention the null perversity $\overline{0}$ constant with value 0 and the *top perversity* defined by $\overline{t}(i) = i - 2$.

A *perversity on a filtered space,* $(X, (X_i)_{0\leq i\leq n})$, is an application, $\overline{p}_X \colon \mathcal{S}_X \to \mathbb{Z}$, defined on the set of strata of X and taking the value 0 on the regular strata. The pair $(X, \overline{p}_X)$ is called a *perverse space* and denoted $(X, \overline{p})$ if there is no ambiguity. (In the case

of a CS set or a pseudomanilfold we use the expressions *perverse CS set* and *perverse pseudomanifold.*)

If $\overline{p}$ and $\overline{q}$ are two perversities on X, we set $\overline{p} \leq \overline{q}$ if we have $\overline{p}(S) \leq \overline{q}(S)$, for all $S \in \mathcal{S}_X$. A GM-perversity induces a perversity on X by $\overline{p}(S) = \overline{p}(\operatorname{codim} S)$. For any perversity, $\overline{p}$, the perversity $D\overline{p} := \overline{t} - \overline{p}$ is called the *complementary perversity* of $\overline{p}$.

1.3. **Intersection Homology.** We specify the chain complex used for the determination of intersection homology, cf. [4].

Definition 1.5. Let X be a filtered space. A *filtered simplex* is a continuous map $\sigma\colon \Delta \to X$, from an euclidean simplex endowed with a decomposition $\Delta = \Delta_0 * \Delta_1 * \cdots * \Delta_n$, called *$\sigma$-decomposition of* Δ, such that

$$\sigma^{-1}X_i = \Delta_0 * \Delta_1 * \cdots * \Delta_i,$$

for all $i \in \{0, \ldots, n\}$. The sets Δ_i may be empty, with the convention $\emptyset * Y = Y$, for any space Y. The simplex σ is *regular* if $\Delta_n \neq \emptyset$. A chain is *regular* if it is a linear combination of regular simplices. For putting in evidence that the filtration on Δ is induced from the filtration of X by σ, we sometimes denote $\Delta = \Delta_\sigma$.

Definition 1.6. Let $(X, \overline{p})$ be a perverse space. The *perverse degree of* a filtered simplex $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ is the $(n+1)$-uple, $\|\sigma\| = (\|\sigma\|_0, \ldots, \|\sigma\|_n)$, where $\|\sigma\|_i = \dim \sigma^{-1}X_{n-i} = \dim(\Delta_0 * \cdots * \Delta_{n-i})$, with the convention $\dim \emptyset = -\infty$. For each stratum S of X, the *perverse degree of σ along S* is defined by

$$\|\sigma\|_S = \begin{cases} -\infty, & \text{if } S \cap \sigma(\Delta) = \emptyset, \\ \|\sigma\|_{\operatorname{codim} S}, & \text{otherwise.} \end{cases}$$

A *filtered simplex is $\overline{p}$-allowable* if

$$\|\sigma\|_S \leq \dim \Delta - \operatorname{codim} S + \overline{p}(S), \tag{1}$$

for each stratum S of X. A chain ξ is said *$\overline{p}$-allowable* if it is a linear combination of $\overline{p}$-allowable simplices, and of *$\overline{p}$-intersection* if ξ together with its boundary are $\overline{p}$-allowable. We denote by $C^{\overline{p}}_*(X; R)$ the complex of $\overline{p}$-intersection chains and by $H^{\overline{p}}_*(X; R)$ its homology, called *$\overline{p}$-intersection homology.*

In [4, Théorème B], we prove that $H^{\overline{p}}_*(X; R)$ is naturally isomorphic to the intersection homology of Goresky and MacPherson.

Lemma 1.7. [4, Lemme 7.5] *If the perversity $\overline{p}$ satisfies $\overline{p} \leq \overline{t}$, then any $\overline{p}$-allowable filtered simplex and its boundary are regular.*

Notice that the hypothesis of Lemma 1.7 is satisfied for any GM-perversity. On the contrary, if $\overline{p} \not\leq \overline{t}$, some $\overline{p}$-allowable filtered simplices can be included in the singular part. As such simplex cannot be considered in the definition of the blown-up intersection cohomology (see the definition of the cap product in Section 2), we adapt the definition of intersection homology to this situation as follows. First, we decompose the boundary of a filtered simplex $\Delta = \Delta_0 * \cdots * \Delta_n$ as

$$\partial\Delta = \partial_{\text{reg}}\Delta + \partial_{\text{sing}}\Delta,$$

where $\partial_{\mathrm{reg}}\Delta$ contains all the regular simplices. In particular, we have

$$\partial_{\mathrm{sing}}\Delta = \begin{cases} \partial\Delta & \text{if } \Delta_n = \emptyset, \\ (-1)^{|\Delta|+1}\Delta_0 * \cdots * \Delta_{n-1} & \text{if } \dim \Delta_n = 0, \\ 0 & \text{if } \dim \Delta_n > 0. \end{cases}$$

If $\sigma\colon \Delta \to X$ is a regular simplex, its boundary is decomposed in $\partial\sigma = \partial_{\mathrm{reg}}\sigma + \partial_{\mathrm{sing}}\sigma$.

Definition 1.8. Let $(X, \overline{p})$ be a perverse space. The chain complex $\mathfrak{C}^{\overline{p}}_*(X;R)$ is the R-module formed of the regular $\overline{p}$-allowable chains whose boundary by ∂_{reg} is $\overline{p}$-allowable. We call $(\mathfrak{C}^{\overline{p}}_*(X;R), \mathfrak{d} = \partial_{\mathrm{reg}})$ the tame $\overline{p}$-intersection complex and its homology, $\mathfrak{H}^{\overline{p}}_*(X;R)$, the *tame $\overline{p}$-intersection homology.*

Similar complexes have been already introduced by the second author in [26] and by G. Friedman in [10] and [9, Chapter 6]. In [4], we show that $\mathfrak{H}^{\overline{p}}_*(X;R)$ is isomorphic to them. We recall now the main properties of $\mathfrak{H}^{\overline{p}}_*(X;R)$ established in [4], see also [9, Chapter 6].

Theorem 1.9. [4, Propositions 7.10 and 7.15] *Let $(X, \overline{p})$ be a perverse space. The following properties are satisfied.*

(1) If $\overline{p} \leq \overline{t}$, the intersection homology coincides with the tame intersection homology,

$$H^{\overline{p}}_*(X;R) = \mathfrak{H}^{\overline{p}}_*(X;R).$$

(2) For any open cover $\mathcal{U} = \{U, V\}$ of X, there exists a Mayer-Vietoris exact sequence,

$$\ldots \to \mathfrak{H}^{\overline{p}}_i(U \cap V;R) \to \mathfrak{H}^{\overline{p}}_i(U;R) \oplus \mathfrak{H}^{\overline{p}}_i(V;R) \to \mathfrak{H}^{\overline{p}}_i(X;R) \to \mathfrak{H}^{\overline{p}}_{i-1}(U \cap V;R) \to \ldots$$

Proposition 1.10. [4, Corollaire 7.8] *Let $(X, \overline{p})$ be a perverse CS set. Then the inclusions $\iota_z\colon X \hookrightarrow \mathbb{R} \times X$, $x \mapsto (z, x)$ with $z \in \mathbb{R}$ fixed, and the projection $p_X\colon \mathbb{R} \times X \to X$, $(t, x) \mapsto x$, induce isomorphisms, $\mathfrak{H}^{\overline{p}}_k(\mathbb{R} \times X;R) \cong \mathfrak{H}^{\overline{p}}_k(X;R)$.*

Proposition 1.11. [4, Proposition 7.9] *Let X be a compact filtered space of dimension n. We endow the cone $\mathring{\mathtt{c}}X$ with a perversity $\overline{p}$ and with the the conic filtration, $(\mathring{\mathtt{c}}X)_i = \mathring{\mathtt{c}}X_{i-1}$. We denote also $\overline{p}$ the induced perversity on X. Then, the tame $\overline{p}$-intersection homology of the cone is determined by,*

$$\mathfrak{H}^{\overline{p}}_k(\mathring{\mathtt{c}}X;R) \cong \begin{cases} \mathfrak{H}^{\overline{p}}_k(X;R) & \text{if } k < n - \overline{p}(\mathtt{w}), \\ 0 & \text{if } k \geq n - \overline{p}(\mathtt{w}), \end{cases}$$

where the isomorphism is induced by $\iota_{\mathring{\mathtt{c}}X}\colon X \to \mathring{\mathtt{c}}X$, $x \mapsto [x, t]$ with $t \in]0, \infty[$.

2. Blown-up intersection cohomology with compact supports

In this section we recall the blown-up intersection cohomology of a perverse space and introduce its version with compact supports. We consider a filtered space X of dimension n and a commutative ring R.

2.1. **Definitions.** Let $N_*(\Delta)$ and $N^*(\Delta)$ be the simplicial chain and cochain complexes of an euclidean simplex Δ, with coefficients in R. For each simplex $F \in N_*(\Delta)$, we write $\mathbf{1}_F$ the element of $N^*(\Delta)$ taking the value 1 on F and 0 otherwise. Given a face F of Δ, we denote by $(F,0)$ the same face viewed as face of the cone $\mathtt{c}\Delta = \Delta * [\mathtt{v}]$ and by $(F,1)$ the face $\mathtt{c}F$ of $\mathtt{c}\Delta$. The apex is denoted $(\emptyset,1) = \mathtt{c}\emptyset = [\mathtt{v}]$. Cochains on the cone are denoted $\mathbf{1}_{(F,\varepsilon)}$ for $\varepsilon = 0$ or 1. For defining the blown-up intersection complex, we first set

$$\widetilde{N}^*(\Delta) = N^*(\mathtt{c}\Delta_0) \otimes \cdots \otimes N^*(\mathtt{c}\Delta_{n-1}) \otimes N^*(\Delta_n).$$

A basis of $\widetilde{N}^*(\Delta)$ is composed of the elements $\mathbf{1}_{(F,\varepsilon)} = \mathbf{1}_{(F_0,\varepsilon_0)} \otimes \cdots \otimes \mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \otimes \mathbf{1}_{F_n}$, where $\varepsilon_i \in \{0,1\}$ and F_i is a face of Δ_i for $i \in \{0,\dots,n\}$ or the empty set with $\varepsilon_i = 1$ if $i < n$. We set $|\mathbf{1}_{(F,\varepsilon)}|_{>s} = \sum_{i>s}(\dim F_i + \varepsilon_i)$, with the convention $\dim \emptyset = -1$.

Definition 2.1. Let $\ell \in \{1,\dots,n\}$ and $\mathbf{1}_{(F,\varepsilon)} \in \widetilde{N}^*(\Delta)$. The *$\ell$-perverse degree* of $\mathbf{1}_{(F,\varepsilon)} \in N^*(\Delta)$ is

$$\|\mathbf{1}_{(F,\varepsilon)}\|_\ell = \begin{cases} -\infty & \text{if} \quad \varepsilon_{n-\ell} = 1, \\ |\mathbf{1}_{(F,\varepsilon)}|_{>n-\ell} & \text{if} \quad \varepsilon_{n-\ell} = 0. \end{cases}$$

For a cochain $\omega = \sum_b \lambda_b\ \mathbf{1}_{(F_b,\varepsilon_b)} \in \widetilde{N}^*(\Delta)$ with $\lambda_b \neq 0$ for all b, the *ℓ-perverse degree* is

$$\|\omega\|_\ell = \max_b \|\mathbf{1}_{(F_b,\varepsilon_b)}\|_\ell.$$

By convention, we set $\|0\|_\ell = -\infty$.

Let $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ be a filtered simplex. We set $\widetilde{N}^*_\sigma = \widetilde{N}^*(\Delta)$. If $\delta_\ell\colon \Delta' \to \Delta$ is an inclusion of a face of codimension 1, we denote by $\partial_\ell\sigma$ the filtered simplex defined by $\partial_\ell\sigma = \sigma \circ \delta_\ell\colon \Delta' \to X$. If $\Delta = \Delta_0 * \cdots * \Delta_n$ is filtered, the induced filtration on Δ' is denoted $\Delta' = \Delta'_0 * \cdots * \Delta'_n$. The *blown-up intersection complex* of X is the cochain complex $\widetilde{N}^*(X)$ composed of the elements ω associating to each regular filtered simplex $\sigma\colon \Delta_0 * \cdots * \Delta_n \to X$ an element $\omega_\sigma \in \widetilde{N}^*_\sigma$ such that $\delta^*_\ell(\omega_\sigma) = \omega_{\partial_\ell\sigma}$, for any face operator $\delta_\ell\colon \Delta' \to \Delta$ with $\Delta'_n \neq \emptyset$. The differential $\delta\omega$ is defined by $(\delta\omega)_\sigma = \delta(\omega_\sigma)$. The *perverse degree of ω along a singular stratum S* equals

$$\|\omega\|_S = \sup\left\{\|\omega_\sigma\|_{\operatorname{codim} S} \mid \sigma\colon \Delta \to X \text{ regular such that } \sigma(\Delta) \cap S \neq \emptyset\right\}.$$

We denote $\|\omega\|$ the map which associates $\|\omega\|_S$ to any singular stratum S and 0 to any regular one. A *cochain $\omega \in \widetilde{N}^*(X)$ is $\overline{p}$-allowable* if $\|\omega\| \leq \overline{p}$ and of *$\overline{p}$-intersection* if ω and $\delta\omega$ are $\overline{p}$-allowable. We denote $\widetilde{N}^*_{\overline{p}}(X;R)$ the complex of $\overline{p}$-intersection cochains and $\mathscr{H}^*_{\overline{p}}(X;R)$ its homology called *blown-up intersection cohomology* of X for the perversity $\overline{p}$.

Definition 2.2. Let $(X,\overline{p})$ be a perverse space. A non-empty subspace K is a *support of the cochain* $\omega \in \widetilde{N}^*(X;R)$ if $\omega_\sigma = 0$, for any regular simplex σ such that $\sigma(\Delta) \cap K = \emptyset$. A cochain $\omega \in \widetilde{N}^*(X;R)$ is with *compact supports* if it has a compact support. We denote $\widetilde{N}^*_{\overline{p},c}(X;R)$ the complex of $\overline{p}$-intersection cochains with compact supports and $\mathscr{H}^*_{\overline{p},c}(X;R)$ its homology.

When the space X is compact, we clearly have $\mathscr{H}^*_{\overline{p},c}(X;R) \cong \mathscr{H}^*_{\overline{p}}(X;R)$. As in the classical case of a manifold (see [23, Appendix A]) the cohomology $\mathscr{H}^*_{\overline{p},c}(X;R)$ can be

obtained as a direct limit. To state it, we need to recall the notion of $\mathcal{U}$-small cochains in intersection cohomology.

Definition 2.3. Let $\mathcal{U}$ be an open cover of a space X. An $\mathcal{U}$-*small simplex* is a regular simplex $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$ such that there exists $U \in \mathcal{U}$ with $\operatorname{Im}\sigma \subset U$. The set of $\mathcal{U}$-small simplices is denoted $\mathtt{Simp}_{\mathcal{U}}$.

The *complex of blown-up $\mathcal{U}$-small cochains, with coefficients in R,* $\widetilde{N}^{*,\mathcal{U}}(X;R)$, is the cochain complex composed of the elements ω, associating to any $\mathcal{U}$-small simplex, $\sigma\colon \Delta = \Delta_0 * \cdots * \Delta_n \to X$, an element $\omega_\sigma \in \widetilde{N}^*(\Delta)$, such that $\delta_\ell^*(\omega_\sigma) = \omega_{\partial_\ell\sigma}$, for any face operator, $\delta_\ell\colon \Delta_0' * \cdots * \Delta_n' \to \Delta_0 * \cdots * \Delta_n$, with $\Delta_n' \neq \emptyset$. If $\overline{p}$ is a perversity on X, we denote $\widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(X;R)$ the cochain subcomplex of elements $\omega \in \widetilde{N}^{*,\mathcal{U}}(X;R)$ such that $\|\omega\| \leq \overline{p}$ and $\|\delta\omega\| \leq \overline{p}$.

The set of $\mathcal{U}$-small cochains admitting a compact support is denoted $\widetilde{N}^{*,\mathcal{U}}_c(X)$. Its subcomplex composed of the cochains of $\overline{p}$-intersection is designed by $\widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(X;R)$ of homology $\mathscr{H}^{*,\mathcal{U}}_{\overline{p},c}(X;R)$.

Proposition 2.4. [8, Theorem B] *Let $(X,\overline{p})$ be a perverse space and $\mathcal{U}$ an open cover of X. Then the restriction map, $\rho_{\mathcal{U}}\colon \widetilde{N}^*_{\overline{p}}(X;R) \to \widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(X;R)$, is a quasi-isomorphism.*

We establish the version with compact supports of Proposition 2.4.

Proposition 2.5. *Let $(X,\overline{p})$ be a perverse space and $\mathcal{U}$ an open cover of X. Then the restriction map, $\rho_{\mathcal{U},c}\colon \widetilde{N}^*_{\overline{p},c}(X;R) \to \widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(X;R)$, is a quasi-isomorphism.*

We postpone for a while the proof of this result. Recall that an open cover $\mathcal{V}$ of X is *finer* than the open cover $\mathcal{U}$ of X if any element $V \in \mathcal{V}$ is included in an element $U \in \mathcal{U}$. We denote $\mathcal{U} \preceq \mathcal{V}$ this relation. If $\mathcal{U} \preceq \mathcal{V}$, we have an inclusion $\mathtt{Simp}_{\mathcal{V}} \subset \mathtt{Simp}_{\mathcal{U}}$ and a natural map $I_X^{\mathcal{U},\mathcal{V}}\colon \widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(X;R) \to \widetilde{N}^{*,\mathcal{V}}_{\overline{p},c}(X;R)$. We consider the direct limit of these maps and set

$$\widetilde{\mathbb{N}}^*_{\overline{p},c}(X;R) = \varinjlim_{\mathcal{U}} \widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(X;R). \tag{2}$$

Proposition 2.5 implies immediatly the next characterisation of $\mathscr{H}^*_{\overline{p},c}(X;R)$.

Corollary 2.6. *Let $(X,\overline{p})$ be a perverse space. The canonical map from $\widetilde{N}^*_{\overline{p},c}(X;R)$ to the previous limit,*

$$\iota_c\colon \widetilde{N}^*_{\overline{p},c}(X;R) \xrightarrow{\simeq} \widetilde{\mathbb{N}}^*_{\overline{p},c}(X;R),$$

is a quasi-isomorphism.

Proof of Proposition 2.5. This is an adaptation of the proof of Proposition 2.4 made in [8]. For proving that the map $\rho_{\mathcal{U}}\colon \widetilde{N}^*_{\overline{p}}(X) \to \widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(X)$ induces an isomorphism in homology, we have built a cochain map, $\varphi_{\mathcal{U}}\colon \widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(X) \to \widetilde{N}^*_{\overline{p}}(X)$, and a homotopy $\Theta\colon \widetilde{N}^*_{\overline{p}}(X) \to \widetilde{N}^{*-1}_{\overline{p}}(X)$ such that $\rho_{\mathcal{U}} \circ \varphi_{\mathcal{U}} = \mathrm{id}$ and $\delta \circ \Theta + \Theta \circ \delta = \mathrm{id} - \varphi_{\mathcal{U}} \circ \rho_{\mathcal{U}}$. The maps $\varphi_{\mathcal{U}}$ and Θ are defined from an application $\widetilde{\mathrm{T}}\colon \widetilde{N}^*(X) \to \widetilde{N}^{*-1}(X)$. Here, it is thus sufficient to prove that the image by these maps of a cochain with compact supports is a cochain with the same support. This is direct for $\rho_{\mathcal{U}}$.

Concerning $\widetilde{\mathrm{T}}$, we consider $\omega \in \widetilde{N}^*_{\overline{p},c}(X)$. By definition, there exists a compact $L \subset X$ such that $\omega_\sigma = 0$ for any regular simplex $\sigma\colon \Delta \to X$ such that $\sigma(\Delta) \cap L = \emptyset$. By definition of $\widetilde{\mathrm{T}}$ (see [8, Proposition 9.9]), we have $(\widetilde{\mathrm{T}}(\omega))_\sigma = \widetilde{\mathrm{T}}_\Delta(\omega_{K(\Delta)})$, with

$$\omega_{K(\Delta)} = \sum_{\substack{F*G\subset K(\Delta)\\ |(F*G,\varepsilon)|=k}} \omega_{\sigma_{F*G}}(F*G,\varepsilon)\,\mathbf{1}_{(F*G,\varepsilon)},$$

where the simplex $\sigma_{F*G}$is a restriction of σ. Therefore, the image of σ_{F*G} is included in the image of σ and we have $\omega_{K(\Delta)} = 0$ as required. □

2.2. **Cup and cap products.** We have already defined a cup product in [3] and cup_i products in [6] on the blown-up intersection cochains in the case of filtered face sets with GM-perversities. In [8], a definition of a cup product has been made in the general case considered here; we recall this definition.

Definition 2.7. Two ordered simplices $F = [a_0, \dots, a_k]$ and $G = [b_0, \dots, b_\ell]$ of an euclidean simplex Δ are *compatible* if $a_k = b_0$. In this case, we set

$$F \cup G = [a_0, \dots, a_k, b_1, \dots, b_\ell] \in N_*(\Delta).$$

The *cup product on* $N^*(\Delta)$ is defined on the dual basis by

$$\mathbf{1}_F \cup \mathbf{1}_G = (-1)^{|F|\,|G|}\mathbf{1}_{F\cup G},$$

if F and G are compatible and 0 otherwise. If $\Delta = \Delta_0 * \cdots * \Delta_n$ is a regular euclidean simplex, this product is extended to $\widetilde{N}^*(\Delta)$ with the classical rule of commutation of graded objects, as follows:
if $\omega_0 \otimes \cdots \otimes \omega_n$ and $\eta_0 \otimes \cdots \otimes \eta_n$ are elements of $N^*(\mathsf{c}\Delta_0) \otimes \cdots \otimes N^*(\Delta_n)$, we set

$$(\omega_0 \otimes \cdots \otimes \omega_n) \cup (\eta_0 \otimes \cdots \otimes \eta_n) = (-1)^{\sum_{i>j} |\omega_i|\,|\eta_j|}(\omega_0 \cup \eta_0) \otimes \cdots \otimes (\omega_n \cup \eta_n). \tag{3}$$

Recall the main property of this cup product.

Proposition 2.8. [8, Proposition 4.2] *Let X be a filtered space endowed with two perversities $\overline{p}$ and $\overline{q}$. The previous cup product gives an associative product,*

$$- \cup -\colon \widetilde{N}^k_{\overline{p}}(X;R) \otimes \widetilde{N}^\ell_{\overline{q}}(X;R) \to \widetilde{N}^{k+\ell}_{\overline{p}+\overline{q}}(X;R), \tag{4}$$

defined by $(\omega \cup \eta)_\sigma = \omega_\sigma \cup \eta_\sigma$, for any $(\omega, \eta) \in \widetilde{N}^*_{\overline{p}}(X;R) \times \widetilde{N}^*_{\overline{q}}(X;R)$ and any regular filtered simplex $\sigma\colon \Delta \to X$. Moreover, this morphism induces a graded commutative product, called intersection cup product,

$$- \cup -\colon \mathscr{H}^k_{\overline{p}}(X;R) \otimes \mathscr{H}^\ell_{\overline{q}}(X;R) \to \mathscr{H}^{k+\ell}_{\overline{p}+\overline{q}}(X;R). \tag{5}$$

We recall the intersection cap product studied in [8] and [5]. Let $\Delta = [e_0, \dots, e_r, \dots, e_m]$ be an euclidean simplex. The (classical) cap product $- \cap \Delta\colon N^*(\Delta) \to N_{m-*}(\Delta)$ is defined by

$$\mathbf{1}_F \cap \Delta = \begin{cases} [e_r, \dots, e_m] & \text{if } F = [e_0, \dots, e_r], \text{ for any } r \in \{0, \dots, m\},\\ 0 & \text{otherwise.}\end{cases}$$

If $\Delta = \Delta_0 * \cdots * \Delta_n$ is a regular filtered simplex, we set $\widetilde{\Delta} = \mathring{\mathsf{c}}\Delta_0 \times \cdots \times \mathring{\mathsf{c}}\Delta_{n-1} \times \Delta_n$. The previous cap product is extended with the rule of permutation of graded objects as follows. If $\mathbf{1}_{(F,\varepsilon)} = \mathbf{1}_{(F_0,\varepsilon_0)} \otimes \cdots \otimes \mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \otimes \mathbf{1}_{F_n} \in \widetilde{N}^*(\Delta)$, we define

$$\begin{aligned} \mathbf{1}_{(F,\varepsilon)} \widetilde{\cap}\, \widetilde{\Delta} &= (-1)^{\nu(F,\varepsilon,\Delta)} (\mathbf{1}_{(F_0,\varepsilon_0)} \cap \mathsf{c}\Delta_0) \otimes \cdots \otimes (\mathbf{1}_{(F_{n-1},\varepsilon_{n-1})} \cap \mathsf{c}\Delta_{n-1}) \otimes (\mathbf{1}_{F_n} \cap \Delta_n), \\ &\in N_*(\mathsf{c}\Delta_0) \otimes \cdots \otimes N_*(\mathsf{c}\Delta_{n-1}) \otimes N_*(\Delta_n), \end{aligned} \tag{6}$$

where $\nu(F,\varepsilon,\Delta) = \sum_{j=0}^{n-1} (\dim \Delta_j + 1) \left(\sum_{i=j+1}^{n} |(F_i,\varepsilon_i)|\right)$, with the convention $\varepsilon_n = 0$.

An intersection cap product on $\widetilde{\Delta}$ must take values in the chain complex $N_*(\Delta)$. For that, we construct a morphism $\mu_\Delta \colon N_*(\mathsf{c}\Delta_0) \otimes \cdots \otimes N_*(\mathsf{c}\Delta_{n-1}) \otimes N_*(\Delta_n) \to N_*(\Delta)$, by its value on $(F,\varepsilon) = (F_0,\varepsilon_0) \otimes \cdots \otimes (F_{n-1},\varepsilon_{n-1}) \otimes F_n$. Let ℓ be the smallest integer j such that $\varepsilon_j = 0$. We set

$$\mu_\Delta(F,\varepsilon) = \begin{cases} F_0 * \cdots * F_\ell & \text{if } \dim(F,\varepsilon) = \dim(F_0 * \cdots * F_\ell), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

This application is a chain map (cf. [8, Lemma 6.4]) and we define the *local intersection cap product*

$$- \cap \Delta \colon \widetilde{N}^*(\Delta) \to N_{m-*}(\Delta) \text{ as } \omega \cap \Delta = \mu_\Delta(\omega \,\widetilde{\cap}\, \widetilde{\Delta}).$$

This expression may be carried on to filtered simplices of a filtered space X.

Definition 2.9. Let $\omega \in \widetilde{N}^*(X;R)$ and $\sigma \colon \Delta_\sigma \to X$ be a filtered simplex. We set

$$\omega \cap \sigma = \begin{cases} \sigma_*(\omega_\sigma \cap \Delta_\sigma) & \text{if } \sigma \text{ is regular,} \\ 0 & \text{otherwise.} \end{cases}$$

With a linear extension, the *intersection cap product* is defined as a map

$$- \cap - \colon \widetilde{N}^k(X;R) \otimes C_m(X;R) \to C_{m-k}(X;R).$$

As proved in [8], the cap product respects the tame intersection chains.

Proposition 2.10. [8, Propositions 6.5 and 6.6] *Let X be a filtered space endowed with two perversities $\overline{p}$ and $\overline{q}$. The cap product defines a homomorphism*

$$- \cap - \colon \widetilde{N}^k_{\overline{p}}(X;R) \otimes \mathfrak{C}^{\overline{q}}_m(X;R) \to \mathfrak{C}^{\overline{p}+\overline{q}}_{m-k}(X;R)$$

satisfying the following properties.

(i) *This is a chain map:* $\mathfrak{d}(\omega \cap \xi) = (\delta\omega) \cap \xi + (-1)^{|\omega|} \omega \cap (\mathfrak{d}\xi)$.
(ii) *The cap and the cup products are compatible:* $(\omega \cup \eta) \cap \xi = (-1)^{|\omega|\,|\eta|} \eta \cap (\omega \cap \xi)$.

2.3. **Properties of the blown-up cohomology with compact supports.** In this section, we establish the properties allowing the use of Proposition 2.19 for the proof of Theorem A and Theorem B. In particular, we construct a Mayer-Vietoris exact sequence, compute the intersection cohomology of a cone and of a product with $\mathbb{R}$, in the case of compact supports.

Cochains with compact supports on an open subset. Let U be an open subset of a filtered space X and $\mathcal{V}$ an open cover of U. To any $\omega \in \widetilde{N}_c^{*,\mathcal{V}}(U;R)$ of compact support $K \subset U$, we associate the open cover $\mathcal{U} = \mathcal{V} \cup \{X \backslash K\}$ of X. Let $\sigma\colon \Delta \to X$, $\sigma \in \mathtt{Simp}_{\mathcal{U}}$, be a regular simplex. We define:

$$\eta_\sigma = \begin{cases} \omega_\sigma & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{V}}, \\ 0 & \text{if} \quad \operatorname{Im}\sigma \cap K = \emptyset. \end{cases} \tag{8}$$

There is no ambiguity in this construction since K is a support of ω.

Let $\delta_\ell\colon \Delta' \to \Delta$ be a regular face of codimension 1. The conditions $\sigma \in \mathtt{Simp}_{\mathcal{V}}$ and $\operatorname{Im}\sigma \cap K = \emptyset$ imply $\partial_\ell\sigma \in \mathtt{Simp}_{\mathcal{V}}$ and $\operatorname{Im}\partial_\ell\sigma \cap K = \emptyset$. Therefore, the compatibility of ω with the face operators gives $\delta_\ell^*\eta_\sigma = \eta_{\partial_\ell\sigma}$. Moreover, as $K \subset U \subset X$ is a compact support of η, we get $\eta \in \widetilde{N}_c^{*,\mathcal{U}}(X;R)$. Let $\boldsymbol{\eta}$ be the class of η in $\widetilde{\mathbb{N}}_{\overline{p},c}^*(X;R)$, see (2). The association $\omega \mapsto \boldsymbol{\eta}$ defines an application

$$I_{U,X}^{\mathcal{V}}\colon \widetilde{N}_c^{*,\mathcal{V}}(U;R) \to \widetilde{\mathbb{N}}_c^*(X;R),$$

compatible with the differentials since $(\delta\omega)_\sigma = \delta(\omega_\sigma)$.

Let $\overline{p}$ be a perversity on X. We endow the open subset $U \subset X$ with the induced perversity also denoted $\overline{p}$. Let $\omega \in \widetilde{N}_{\overline{p},c}^{*,\mathcal{V}}(U)$ of compact support K. For any regular simplex $\sigma \in \mathtt{Simp}_{\mathcal{U}}$ and any $\ell \in \{1,\dots,n\}$, we have the inequality $\|\eta_\sigma\|_\ell \leq \|\omega_\sigma\|_\ell$ from which we deduce a cochain map,

$$I_{U,X}^{\mathcal{V}}\colon \widetilde{N}_{\overline{p},c}^{*,\mathcal{V}}(U;R) \to \widetilde{\mathbb{N}}_{\overline{p},c}^*(X;R).$$

Proposition 2.11. *Let $(X,\overline{p})$ be a perverse space and U an open subspace, endowed with the induced perversity. The maps $I_{U,X}^{\mathcal{V}}$ defined above induce an injective application of cochain complexes,*

$$\boldsymbol{I}_{U,X}\colon \widetilde{\mathbb{N}}_{\overline{p},c}^*(U;R) \to \widetilde{\mathbb{N}}_{\overline{p},c}^*(X;R).$$

Proof. Let $\mathcal{V} \preceq \mathcal{V}'$ be two open covers of U. The open covers $\mathcal{U} = \mathcal{V} \cup \{X\backslash K\}$ and $\mathcal{U}' = \mathcal{V}' \cup \{X\backslash K\}$ of X satisfy $\mathcal{U} \preceq \mathcal{U}'$. Thus we have a commutative diagram,

$$\begin{array}{ccc} \widetilde{N}_{\overline{p},c}^{*,\mathcal{V}}(U) & \xrightarrow{I_{U,X}^{\mathcal{V}}} & \widetilde{\mathbb{N}}_{\overline{p},c}^*(X). \\ {\scriptstyle I_U^{\mathcal{V},\mathcal{V}'}}\downarrow & \nearrow_{I_{U,X}^{\mathcal{V}'}} & \\ \widetilde{N}_{\overline{p},c}^{*,\mathcal{V}'}(U) & & \end{array}$$

The map $\boldsymbol{I}_{U,X}$ is then obtained by a passage to the limit. For proving the injectivity, we consider $\boldsymbol{\omega} \in \widetilde{\mathbb{N}}_{\overline{p},c}^*(U)$ such that $\boldsymbol{I}_{U,X}(\boldsymbol{\omega}) = 0$. The class $\boldsymbol{\omega}$ is the limit of elements $\omega \in \widetilde{N}_{\overline{p},c}^{*,\mathcal{V}}(U)$, where $\mathcal{V}$ is an open cover of U. Let $\eta \in \widetilde{N}_{\overline{p},c}^{*,\mathcal{U}}(X)$ be the element associated to ω in (8). From $\boldsymbol{I}_{U,X}(\boldsymbol{\omega}) = 0$, we get the existence of an open cover $\mathcal{W}$ of X finer than $\mathcal{U}$ and such that $\eta_\sigma = 0$ for any $\sigma \in \mathtt{Simp}_{\mathcal{W}}$. Thus the open cover $\mathcal{W} \cap \mathcal{V}$ of U formed of the intersections of elements of $\mathcal{V}$ and $\mathcal{W}$ verifies by definition $\eta_\sigma = 0$ for any $\sigma \in \mathtt{Simp}_{\mathcal{W}\cap\mathcal{V}} = \mathtt{Simp}_{\mathcal{W}} \cap \mathtt{Simp}_{\mathcal{V}}$. It follows $\boldsymbol{\omega} = 0$. □

Mayer-Vietoris exact sequence with compact supports.

Proposition 2.12. *Let $(X, \overline{p})$ be a locally compact and paracompact perverse space. The induced perversities on the open subsets of X are also denoted $\overline{p}$. If $X = U_1 \cup U_2$ is an open cover of X, then the sequence,*

$$0 \longrightarrow \widetilde{\mathbb{N}}^*_{\overline{p},c}(U_1 \cap U_2; R) \xrightarrow{(\boldsymbol{I}_1, \boldsymbol{I}_2)} \widetilde{\mathbb{N}}^*_{\overline{p},c}(U_1; R) \oplus \widetilde{\mathbb{N}}^*_{\overline{p},c}(U_2; R) \xrightarrow{\boldsymbol{I}_3 - \boldsymbol{I}_4} \widetilde{\mathbb{N}}^*_{\overline{p},c}(X; R) \longrightarrow 0,$$

whose applications $\boldsymbol{I}_\bullet$ are defined in Proposition 2.11, is exact.

Before giving the proof, we recall the following result from [8].

Lemma 2.13. [8, Lemma 10.2] *Let $(X, \overline{p})$ be a perverse space. Each application $g\colon X \to R$ defines a 0-cochain $\tilde{g} \in \widetilde{N}^0_{\overline{0}}(X)$. Moreover the association $g \mapsto \tilde{g}$ is R-linear.*

The cochain $\tilde{g}$ is defined as follows. Let $\sigma\colon \Delta_0 * \cdots * \Delta_n \to X$ be a regular filtered simplex and $b = (b_0, \ldots, b_n) \in \mathtt{c}\Delta_0 \times \cdots \times \mathtt{c}\Delta_{n-1} \times \Delta_n$. We set $i_0 = \min\{i \mid b_i \in \Delta_i\}$ and $\tilde{g}_\sigma(b) = g(\sigma(b_{i_0}))$.

Proof of Proposition 2.12. The injectivity of $(\boldsymbol{I}_1, \boldsymbol{I}_2)$ is a consequence of Proposition 2.11. The rest of the proof goes along the next steps.

- *The map $(\boldsymbol{I}_3 - \boldsymbol{I}_4) \circ (\boldsymbol{I}_1, \boldsymbol{I}_2)$ is constant with value 0.*

Let $\boldsymbol{\omega} \in \widetilde{\mathbb{N}}^*_{\overline{p},c}(U_1 \cap U_2)$. Consider an open cover $\mathcal{V}$ of $U_1 \cap U_2$ and a cochain $\omega \in \widetilde{N}^{*,\mathcal{V}}_{\overline{p},c}(U_1 \cap U_2)$ with compact support $K \subset U_1 \cap U_2 \subset X$, representing $\boldsymbol{\omega}$. We set $\boldsymbol{\eta}_1 = \boldsymbol{I}_3(\boldsymbol{I}_1(\omega))$, $\boldsymbol{\eta}_2 = \boldsymbol{I}_4(\boldsymbol{I}_2(\omega))$ and choose representing elements of $\boldsymbol{\eta}_i$, for $i = 1, 2$,

$$\eta_i \in \widetilde{N}^{*,\mathcal{V}\cup\{U_i\backslash K\}\cup\{X\backslash K\}}_{\overline{p},c}(X) = \widetilde{N}^{*,\mathcal{V}\cup\{X\backslash K\}}_{\overline{p},c}(X).$$

From the definition of the applications $\boldsymbol{I}_\bullet$, we have for $i = 1, 2$ and $\sigma \in \mathtt{Simp}_{\mathcal{V}\cup\{X\backslash K\}}$,

$$(\eta_i)_\sigma = \begin{cases} \omega_\sigma & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{V}}, \\ 0 & \text{if} \quad \operatorname{Im}\sigma \cap K = \emptyset. \end{cases}$$

This implies $(\eta_1)_\sigma = (\eta_2)_\sigma$ and $\boldsymbol{I}_3 \circ \boldsymbol{I}_1 = \boldsymbol{I}_4 \circ \boldsymbol{I}_2$.

- *The kernel of $\boldsymbol{I}_3 - \boldsymbol{I}_4$ is included in the image of $(\boldsymbol{I}_1, \boldsymbol{I}_2)$.*

Let $\boldsymbol{\omega}_i \in \widetilde{\mathbb{N}}^*_{\overline{p},c}(U_i)$, for $i = 1, 2$, such that $\boldsymbol{I}_3(\boldsymbol{\omega}_1) = \boldsymbol{I}_4(\boldsymbol{\omega}_2)$. Consider an open cover $\mathcal{V}_i$ of U_i and a cochain $\omega_i \in \widetilde{N}^{*,\mathcal{V}_i}_{\overline{p},c}(U_i)$ with compact support $K_i \subset U_i \subset X$, representing $\boldsymbol{\omega}_i$ for $i = 1, 2$. With the local compacity, there exist an open subset W and a compact subset F such that $K_1 \cap K_2 \subset W \subset F \subset U_1 \cap U_2$. Set $\boldsymbol{I}_3(\boldsymbol{\omega}_1) = \boldsymbol{\eta}_1$ and $\boldsymbol{I}_4(\boldsymbol{\omega}_2) = \boldsymbol{\eta}_2$. From the definition of the applications $\boldsymbol{I}_\bullet$, we get $\eta_i \in \widetilde{N}^{*,\mathcal{V}_i\cup\{X\backslash K_i\}}_{\overline{p},c}(X)$ and

$$\begin{aligned} (\eta_i)_\sigma &= (\omega_i)_\sigma \quad \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{V}_i}, && (9) \\ &= 0 \quad \text{si} \quad \operatorname{Im}\sigma \cap K_i = \emptyset. && (10) \end{aligned}$$

The equality $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2$ implies the existence of an open cover $\mathcal{U}$ of X such that $\mathcal{V}_i \cup \{X\backslash K_i\} \preceq \mathcal{U}$ for $i = 1, 2$ and

$$(\eta_1)_\sigma = (\eta_2)_\sigma \quad \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{U}}. \tag{11}$$

Without loss of generality, we may suppoose $\{X\backslash K_1, X\backslash K_2, W\} \preceq \mathcal{U}$. In particular, for any $U \in \mathcal{U}$, we have

$$U \cap K_1 = \emptyset \text{ or } U \cap K_2 = \emptyset \text{ or } U \subset W. \tag{12}$$

Thus the open cover $\mathcal{W} = \{U \cap U_1 \cap U_2 \mid U \in \mathcal{U}\}$ of $U_1 \cap U_2$ can be decomposed in $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$ with $\mathcal{W}_i = \{U \in \mathcal{W} \mid U \cap K_i = \emptyset\}$ for $i = 1, 2$ and $\mathcal{W}_3 = \{U \in \mathcal{W} \mid U \subset W\}$. For any regular simplex $\sigma \in \mathtt{Simp}_{\mathcal{W}}$, we set

$$\omega_\sigma = \begin{cases} (\eta_1)_\sigma = (\eta_2)_\sigma & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}_3}, \\ 0 & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}_1} \cup \mathtt{Simp}_{\mathcal{W}_2}. \end{cases} \tag{13}$$

The following paragraphs establish the validity of that definition.

- $(\eta_1)_\sigma = (\eta_2)_\sigma$ if $\sigma \in \mathtt{Simp}_{\mathcal{W}_3} \subset \mathtt{Simp}_{\mathcal{U}}$, cf. (11).
- $(\eta_i)_\sigma = 0$ if $\sigma \in \mathtt{Simp}_{\mathcal{W}_i} \cap \mathtt{Simp}_{\mathcal{W}_3}$, for $i = 1, 2$, cf. (10).
- $\delta_\ell \omega_\sigma = \omega_{\partial_\ell \sigma}$ for any face operator because η_1 satisfies this property and $\sigma \in \mathtt{Simp}_{\mathcal{W}_i}$ implies $\partial_\ell \sigma \in \mathtt{Simp}_{\mathcal{W}_i}$ for $i = 1, 2, 3$.
- $\|\omega_\sigma\|_\ell \leq \|(\eta_1)_\sigma\|_\ell$ and $\|\delta\omega_\sigma\|_\ell \leq \|(\delta\eta_1)_\sigma\|_\ell$ for any $\sigma \in \mathtt{Simp}_{\mathcal{W}_3}$ and $\ell \in \{1, \dots, n\}$.
- For any $\sigma \in \mathtt{Simp}_{\mathcal{W}}$, the property $\operatorname{Im}\sigma \cap F = \emptyset$ implies $\omega_\sigma = 0$ because $\sigma \in \mathtt{Simp}_{\mathcal{W}_1} \cup \mathtt{Simp}_{\mathcal{W}_2}$. (Note that F is a compact support of ω.)

We have constructed a cochain $\boldsymbol{\omega} \in \widetilde{\mathbb{N}}^*_{\overline{p},c}(U_1 \cap U_2)$ and we are reduced to prove $\boldsymbol{I}_i(\boldsymbol{\omega}) = \boldsymbol{\omega}_i$. We do it for $i = 1$, the second case being similar. Set $\boldsymbol{\gamma}_1 = \boldsymbol{I}_1(\boldsymbol{\omega})$.

We set $\mathcal{W}' = \{U \cap U_1 \backslash F \mid U \in \mathcal{U}\}$ and denote $\mathcal{H} = \mathcal{W} \cup \mathcal{W}'$ the open cover of U_1. This cover is a refinement of $\mathcal{W} \cup \{U_1 \backslash F\}$ and it is sufficient to prove $(\gamma_1)_\sigma = (\omega_1)_\sigma$ for any $\sigma \in \mathcal{H}$. The cover $\mathcal{H}$ being also a refinement of $\mathcal{U}$ and therefore of $\{X \backslash K_1, X \backslash K_2, W\}$ it is sufficient to consider the three following cases.

– If $\operatorname{Im}\sigma \cap K_1 = \emptyset$, by using the fact that F is a compact support of ω, we have,

$$(\gamma_1)_\sigma =_{(8)} \begin{cases} \omega_\sigma & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}} \\ 0 & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}'} \end{cases} = \begin{cases} \omega_\sigma & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}_1} \\ 0 & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}'} \end{cases} =_{(13)} 0 = (\omega_1)_\sigma,$$

– If $\operatorname{Im}\sigma \cap K_2 = \emptyset$ and $\operatorname{Im}\sigma \cap K_1 \neq \emptyset$, we have

$$(\gamma_1)_\sigma =_{(8)} \begin{cases} \omega_\sigma & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}} \\ 0 & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}'} \end{cases} = \begin{cases} \omega_\sigma & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}_2} \\ 0 & \text{if} \quad \sigma \in \mathtt{Simp}_{\mathcal{W}'} \end{cases} =_{(13)} 0.$$

As $\operatorname{Im}\sigma \cap K_1 \neq \emptyset$, we get $\sigma \in \mathtt{Simp}_{\mathcal{V}_1}$ and

$$(\omega_1)_\sigma =_{(9)} (\eta_1)_\sigma =_{(11)} (\eta_2)_\sigma =_{(10)} 0.$$

– If $\operatorname{Im}\sigma \subset U \subset W$ and $\operatorname{Im}\sigma \cap K_1 \neq \emptyset$, we have $U \in \mathcal{W}_3$ and $U \in \mathcal{V}_1$. This implies

$$(\gamma_1)_\sigma =_{(8)} \omega_\sigma =_{(13)} (\eta_1)_\sigma =_{(9)} (\omega_1)_\sigma.$$

• *The map $\boldsymbol{I}_3 - \boldsymbol{I}_4$ is onto.* Let $\boldsymbol{\omega} \in \widetilde{\mathbb{N}}^*_{\overline{p},c}(X)$. Consider an open cover $\mathcal{U}$ of X and a cochain $\omega \in \widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(X)$ with compact support $K \subset X$, representing $\boldsymbol{\omega}$.

From the paracompacity of X, we get two fonctions, $g_i \colon X \to \{0, 1\}$, $i = 1, 2$, satisfying $\operatorname{Supp} g_i \subset U_i$ and $g_1 + g_2 = 1$. We denote $\tilde{g}_i \in \widetilde{N}^0_{\overline{0}}(X)$ the associated 0-cochain, defined in Lemma 2.13. There exist also two relatively compact open subsets W_i such that $\operatorname{Supp} g_i \cap K \subset W_i \subset \overline{W}_i \subset U_i$. We fix $i = 1$. We already know $\tilde{g}_1 \cup \omega \in \widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(X)$. We define

$$\mathcal{A} = \{V \cap U_1 \backslash K \mid V \in \mathcal{U}\},\ \mathcal{B} = \{V \cap U_1 \backslash \operatorname{Supp} g_1 \mid V \in \mathcal{U}\},\ \mathcal{C} = \{V \cap W_1 \mid V \in \mathcal{U}\},$$

and consider the open cover $\mathcal{V}_1 = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ of U_1. By restriction, we have $\tilde{g}_1 \cup \omega \in \widetilde{N}^{*,\mathcal{V}_1}_{\overline{p}}(U_1)$. If $\sigma \in \mathtt{Simp}_{\mathcal{V}_1}$ is such that $\operatorname{Im}\sigma \cap \overline{W}_1 = \emptyset$, then we have $\operatorname{Im}\sigma \cap \operatorname{Supp} g_1 = \emptyset$

or $\operatorname{Im}\sigma \cap K = \emptyset$. In each case, we may write $(\tilde{g}_1 \cup \omega)_\sigma = 0$. Therefore $\overline{W}_1$ is a compact support of $\tilde{g}_1 \cup \omega$ and we get $\tilde{g}_1 \cup \omega \in \widetilde{N}^{*,\mathcal{V}_1}_{\overline{p},c}(U_1)$.

We use the same process for $i = 2$ with an open cover $\mathcal{V}_2$ of U_2 and the cochain $\tilde{g}_2 \cup \omega \in \widetilde{N}^{*,\mathcal{V}_2}_{\overline{p},c}(U_2)$. By choosing an open cover $\mathcal{X}$ of X finer than $\mathcal{V}_1 \cup \{X \backslash \overline{W}_1\}$ and than $\mathcal{V}_2 \cup \{X \backslash \overline{W}_2\}$, we see that $\boldsymbol{I}_3 - \boldsymbol{I}_4$ sends the class associated to $(\tilde{g}_1 \cup \omega, -\tilde{g}_2 \cup \omega)$ on $\boldsymbol{\omega}$. This proves the surjectivity of $\boldsymbol{I}_3 - \boldsymbol{I}_4$. □

Cohomology with compact supports of a cone.

Proposition 2.14. *Let X be a compact filtered space. The cone $\mathring{\mathtt{c}}X$ is endowed with the conic filtration and with a perversity $\overline{p}$. We denote also $\overline{p}$ the induced perversity on X. Then the following properties are satisfied for any commutative ring R.*

(a) *For any $k \geq \overline{p}(\mathtt{w}) + 2$, there exists an isomorphism,*

$$\mathscr{H}^{k-1}_{\overline{p}}(X; R) \xrightarrow{\cong} \mathscr{H}^k_{\overline{p},c}(\mathring{\mathtt{c}}X; R).$$

(b) *For any $k \leq \overline{p}(\mathtt{w}) + 1$, we have $\mathscr{H}^k_{\overline{p},c}(\mathring{\mathtt{c}}X; R) = 0$.*

Proof. Recall $\mathring{\mathtt{c}}X = (X \times [0, \infty[)/(X \times \{0\})$ and denote $\mathring{\mathtt{c}}_1 X = (X \times [0, 1[)/(X \times \{0\})$. The pair $\mathcal{U} = \{\mathring{\mathtt{c}}_1 X, X \times]0, \infty[\}$ is an open cover of $\mathring{\mathtt{c}}X$. The proof follows three steps.

• *Construction of an exact sequence.* We consider the short exact sequence

$$0 \longrightarrow \widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(\mathring{\mathtt{c}}X) \longrightarrow \widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X) \longrightarrow \frac{\widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X)}{\widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(\mathring{\mathtt{c}}X)} \longrightarrow 0. \tag{14}$$

For the study of the homology of the right-hand term, we introduce

$$\mathcal{G}^* = \left\{ \omega \in \widetilde{N}^*_{\overline{p}}(X \times]0, \infty[) \mid \exists a > 0 \text{ such that } \omega_\sigma = 0 \text{ if } \operatorname{Im}\sigma \cap (X \times]0, a[) = \emptyset \right\}.$$

The reduction $\widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X) \to \widetilde{N}^*_{\overline{p}}(X \times]0, \infty[)$ induces a cochain map

$$\varphi \colon \frac{\widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X)}{\widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(\mathring{\mathtt{c}}X)} \longrightarrow \frac{\widetilde{N}^*_{\overline{p}}(X \times]0, \infty[)}{\mathcal{G}^*}.$$

First we show that φ is an isomorphism. It is one-to-one because any cochain $\omega \in \widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X)$ such that $\varphi(\omega) = 0$ owns as compact support the closed cone $\mathtt{c}_a X = X \times [0, a]/(X \times \{0\})$. For proving the surjectivity, we define an application $g \colon \mathring{\mathtt{c}}X \to \{0, 1\}$ by $g([x, t]) = 0$ if $t \leq 1$ and $g([x, t]) = 1$ otherwise. We denote $\tilde{g} \in \widetilde{N}^0_{\overline{0}}(\mathring{\mathtt{c}}X)$ the 0-cochain associated to g by Lemma 2.13. Let $\omega \in \widetilde{N}^*_{\overline{p}}(X \times]0, \infty[)$ be a cochain. For any regular simplex $\sigma \colon \Delta \to \mathring{\mathtt{c}}X$, we set

$$\eta_\sigma = \begin{cases} 0 & \text{if} \quad \operatorname{Im}\sigma \subset \mathring{\mathtt{c}}_1 X, \\ \tilde{g}_\sigma \cup \omega_\sigma & \text{if} \quad \operatorname{Im}\sigma \subset X \times]0, \infty[. \end{cases} \tag{15}$$

If $\operatorname{Im}\sigma \subset X \times]0, 1[$, we have $\tilde{g}_\sigma = 0$ by construction of g and $\tilde{g}$. Therefore $\eta \in \widetilde{N}^{*,\mathcal{U}}(\mathring{\mathtt{c}}X)$ is well defined. From $\tilde{g} \cup \omega \in \widetilde{N}^*_{\overline{p}}(X \times]0, \infty[)$, we deduce $\eta \in \widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X)$. We are reduced to show $\omega - \varphi(\eta) \in \mathcal{G}^*$. For that, we choose $a = 2$ and consider $\sigma \colon \Delta \to \mathring{\mathtt{c}}X$ such that $\operatorname{Im}\sigma \cap]0, 2[= \emptyset$. By construction of g and $\tilde{g}$, we have $\tilde{g}_\sigma = 1$, thus $\eta_\sigma = \tilde{g}_\sigma \cup \omega_\sigma = \omega_\sigma$. The bijectivity of φ is now established.

• *Proof of (a).* The homeomorphism $\mathbb{R} \cong]0,\infty[$ and Lemma 2.16 imply the acyclicity of $\mathcal{G}^*$. Thus, the right-hand term in the short exact sequence (14) has for cohomology $\mathscr{H}^*_{\overline{p}}(X)$. From [8, Theorem B], we know that the complex $\widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X)$ has for cohomology $\mathscr{H}^*_{\overline{p}}(\mathring{\mathtt{c}}X)$ which has been determined in [8, Theorem E]. Thus, if $k \geq \overline{p}(\mathtt{w}) + 2$, the exact sequence associated to (14) can be reduced to exact sequences of the form

$$0 \to \mathscr{H}^{k-1}_{\overline{p}}(X) \xrightarrow{\delta_1} \mathscr{H}^k_{\overline{p},c}(\mathring{\mathtt{c}}X) \to 0,$$

where δ_1 is the connecting map determined by (15).

• *Proof of (b).* We first observe the commutativity of the diagram

$$\begin{array}{ccc} \widetilde{N}^*_{\overline{p}}(X\times]0,\infty[) & \longrightarrow & \dfrac{\widetilde{N}^*_{\overline{p}}(X\times]0,\infty[)}{\mathcal{G}^*} \\ \uparrow & & \cong \uparrow \varphi \\ \widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X) & \longrightarrow & \dfrac{\widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X)}{\widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(\mathring{\mathtt{c}}X)}. \end{array}$$

The top map is a quasi-isomorphism in all degrees, φ is an isomorphism and the left-hand vertical map is a quasi-isomorphism if $* \leq \overline{p}(\mathtt{w})$. Therefore, by using the determination $H^{\overline{p}(\mathtt{w})+1,\mathcal{U}}_{\mathrm{TW},\overline{p}}(\mathring{\mathtt{c}}X) = 0$ (see [8, Theorem E]), the map

$$\mathscr{H}^{k,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X) \to H^k\left(\frac{\widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{\mathtt{c}}X)}{\widetilde{N}^{*,\mathcal{U}}_{\overline{p},c}(\mathring{\mathtt{c}}X)}\right)$$

is an isomorphism for any $k \leq \overline{p}(\mathtt{w}) + 1$. The result follows. □

Cohomology with compact supports of the product with $\mathbb{R}$.

Proposition 2.15. *Let $(X,\overline{p})$ be a locally compact and paracompact perverse space. We denote also $\overline{p}$ the perversity induced on $X \times \mathbb{R}$. Then, for any $k > 0$, there exists an isomorphism,*

$$\mathscr{H}^k_{\overline{p},c}(X;R) \cong \mathscr{H}^{k+1}_{\overline{p},c}(X \times \mathbb{R};R).$$

Proof. With the notations of Lemma 2.16, we have a short exact sequence with an acyclic middle term,

$$0 \longrightarrow \mathcal{L}^* \cap \mathcal{R}^* \cap \mathcal{K}^* \longrightarrow (\mathcal{L}^* \cap \mathcal{K}^*) \oplus (\mathcal{R}^* \cap \mathcal{K}^*) \xrightarrow{\Phi} \mathcal{K}^* \longrightarrow 0. \tag{16}$$

Let $g\colon X \times \mathbb{R} \to \{0,1\}$ be the function defined by $g(x,t) = 0$ if $t \leq 1$ and $g(x,t) = 1$ otherwise. Let $\tilde{g}$ be the associated cochain (see Lemma 2.13). To any $\omega \in \mathcal{K}^*$, we associate $(\tilde{g} \cup \omega, (1-\tilde{g}) \cup \omega) \in (\mathcal{L}^* \cap \mathcal{K}^*) \oplus (\mathcal{R}^* \cap \mathcal{K}^*)$. This gives the surjectivity of Φ and determines the connecting map of the associated long exact sequence,

$$[\omega] \mapsto \delta(\tilde{g} \cup \omega). \tag{17}$$

• *The complex $\mathcal{K}^*$ is quasi-isomorphic to $\widetilde{N}^*_{\overline{p},c}(X)$.* Denote $I_0\colon X \to X \times \mathbb{R}$ and $\mathrm{pr}\colon X \times \mathbb{R} \to X$ the maps defined by $I_0(x) = (x,0)$ and $\mathrm{pr}(x,t) = x$. They induce cochain maps,

$I_0^*\colon \widetilde{N}^*_{\overline{p}}(X\times\mathbb{R}) \to \widetilde{N}^*_{\overline{p}}(X)$ and $\mathrm{pr}^*\colon \widetilde{N}^*_{\overline{p}}(X) \to \widetilde{N}^*_{\overline{p}}(X\times\mathbb{R})$, (see [8, Proposition 3.6]) which verify $I_0^*(\mathcal{K}^*) \subset \widetilde{N}^*_{\overline{p},c}(X)$ and $\mathrm{pr}^*(\widetilde{N}^*_{\overline{p},c}(X)) \subset \mathcal{K}^*$. Thus we get

$$\mathcal{K}^* \xrightarrow{I_0^*} \widetilde{N}^*_{\overline{p},c}(X) \xrightarrow{\mathrm{pr}^*} \mathcal{K}^*.$$

The map $I_0^* \circ \mathrm{pr}^* = (\mathrm{pr}\circ I_0)^*$ is the identity map. The application $\mathrm{pr}^* \circ I_0^* = (I_0 \circ \mathrm{pr})^*$ is homotopic to the identity map on $\widetilde{N}^*_{\overline{p}}(X\times\mathbb{R})$, see [8, Theorem D]. From (18) applied to $\mathcal{K}^*$, we deduce that $\mathrm{pr}^* \circ I_0^*$ is homotopic to the identity map on $\mathcal{K}^*$.

• *The complex $\mathcal{K}^* \cap \mathcal{L}^* \cap \mathcal{R}^*$ is quasi-isomorphic to $\widetilde{N}^*_{\overline{p},c}(X\times\mathbb{R})$.* We observe first that $\widetilde{N}^*_{\overline{p},c}(X\times\mathbb{R}) \subset \mathcal{K}^* \cap \mathcal{L}^* \cap \mathcal{R}^*$ and denote by ι the corresponding canonical injection. Let $\omega \in \mathcal{K}^* \cap \mathcal{L}^* \cap \mathcal{R}^*$. There exist $a > 0$ and a compact $K \subset X$ such that, if $\sigma\colon \Delta \to X\times\mathbb{R}$ satisfies *one* of the following conditions, then we have $\omega_\sigma = 0$,

(*i*) $\mathrm{Im}\,\sigma \subset X\times]-\infty, a]$ or (*ii*) $\mathrm{Im}\,\sigma \subset X\times[-a,\infty[$ or (*iii*) $(\mathrm{Im}\,\sigma)\cap(K\times\mathbb{R}) = \emptyset$.

Choose an open subset W of $X\times\mathbb{R}$ such that $K\times[-a,a] \subset W \subset \overline{W}$ and $\overline{W}$ compact. Set $\mathcal{U}_\omega = \{X\times]a,\infty[,\ X\times]-\infty,-a[,\ (X\backslash K)\times\mathbb{R}\}$. From the properties of ω, we deduce $\omega \in \widetilde{N}^{*,\mathcal{U}_\omega}_{\overline{p},c}(X\times\mathbb{R})$ with support $\overline{W}$. We have constructed a cochain map ψ which gives a commutative diagram with the quasi-isomorphism ι_c of Corollary 2.6,

$$\begin{array}{ccc} \mathcal{K}^* \cap \mathcal{L}^* \cap \mathcal{R}^* & \xrightarrow{\psi} & \widetilde{\mathbb{N}}^*_{\overline{p},c}(X\times\mathbb{R}) \\ & \nwarrow^{\iota} & \simeq \uparrow \iota_c \\ & & \widetilde{N}^*_{\overline{p},c}(X\times\mathbb{R}). \end{array}$$

So, we get the injectivity of the homomorphism ι^* induced by ι. We establish now its surjectivity. Let $\omega \in \mathcal{K}^* \cap \mathcal{L}^* \cap \mathcal{R}^*$ of associated open cover $\mathcal{U}_\omega$ and such that $\delta\omega = 0$. Let $\sigma\colon \Delta \to X\times\mathbb{R}$ be a regular $\mathcal{U}_\omega$-small simplex such that $(\mathrm{Im}\,\sigma)\cap\overline{W} = \emptyset$. It follows: $\mathrm{Im}\,\sigma \subset (X\times]a,\infty[) \cup (X\times]-\infty,-a[) \cup ((X\backslash K)\times\mathbb{R})$ and $\omega_\sigma = 0$ by hypothesis on ω. Thus, from Proposition 2.5, we get $\rho_{\mathcal{U}_\omega}(\omega) \in \widetilde{N}^{*,\mathcal{U}_\omega}_{\overline{p},c}(X\times\mathbb{R})$ and from the proof of Proposition 2.5, we deduce also $(\varphi_{\mathcal{U}_\omega}\circ\rho_{\mathcal{U}_\omega})(\omega) \in \widetilde{N}^*_{\overline{p},c}(X\times\mathbb{R}) \subset \mathcal{K}^* \cap \mathcal{L}^* \cap \mathcal{R}^*$ and $\omega - (\varphi_{\mathcal{U}_\omega}\circ\rho_{\mathcal{U}_\omega})(\omega) = \delta\Theta(\omega)$ with $\Theta(\omega) \in \widetilde{N}^*_{\overline{p},c}(X\times\mathbb{R}) \subset \mathcal{K}^* \cap \mathcal{L}^* \cap \mathcal{R}^*$. This proves the surjectivity of ι^*. □

Lemma 2.16. *Let $(X,\overline{p})$ be a perverse space. We consider the following subcomplexes of $\widetilde{N}^*_{\overline{p}}(X\times\mathbb{R};R)$,*

$$\begin{aligned} \mathcal{L}^* &= \{\omega \mid \exists a>0 \text{ such that } \omega_\sigma = 0 \text{ if } \mathrm{Im}\,\sigma \cap (X\times]a,\infty[) = \emptyset\}, \\ \mathcal{R}^* &= \{\omega \mid \exists b>0 \text{ such that } \omega_\sigma = 0 \text{ if } \mathrm{Im}\,\sigma \cap (X\times]-\infty,-b[) = \emptyset\}, \\ \mathcal{K}^* &= \{\omega \mid \exists K \text{ compact, such that } K\subset X \text{ and } \omega_\sigma = 0 \text{ if } \mathrm{Im}\,\sigma \cap (K\times\mathbb{R}) = \emptyset\}. \end{aligned}$$

Then the complexes $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{L}^* \cap \mathcal{K}^*$ et $\mathcal{R}^* \cap \mathcal{K}^*$ are acyclic.

Proof. • *The complex $\mathcal{L}^*$ is acyclic.* Let $\omega \in \mathcal{L}^k$, $\delta\omega = 0$. Denote a the positive number associated to ω, $I_0\colon X \to X\times\mathbb{R}$ the map defined by $I_0(x) = (x,0)$ and $\mathrm{pr}\colon X\times\mathbb{R} \to X$ the canonical projection. Let $\Delta\otimes[0,1]$ be the simplicial complex whose simplices are the

joins $F * G$ with $F \subset \Delta \times \{0\}$ and $G \subset \Delta \times \{1\}$. [8, Proposition 11.3] gives a homotopy $\Theta\colon \widetilde{N}^*(\Delta \otimes [0,1]) \to \widetilde{N}^{*-1}(\Delta)$ such that

$$\Theta \circ \delta + \delta \circ \Theta = (I_0 \circ \mathrm{pr})^* - \mathrm{id}. \tag{18}$$

We are going to prove $\delta(\Theta(\omega)) = -\omega$ and $\Theta(\omega) \in \mathcal{L}_*$.

For any regular simplex $\sigma\colon \Delta \to X \times \mathbb{R}$, we have $(I_0 \circ \mathrm{pr})^*(\omega_\sigma) = \omega_{I_0 \circ \mathrm{pr} \circ \sigma} = 0$ because $\mathrm{Im}\,(I_0 \circ \mathrm{pr} \circ \sigma) \cap (X \times]a, \infty[) = 0$. Thus (18) becomes $\delta(\Theta(\omega)) = -\omega$.

Let $\sigma\colon \Delta \to X \times \mathbb{R}$, $\sigma(x) = (\sigma_1(x), \sigma_2(x))$, such that $\mathrm{Im}\,\sigma \subset X \times]-\infty, a]$. At σ, we associate $\hat{\sigma}\colon \Delta \otimes [0,1] \to X \times \mathbb{R}$ defined by $\hat{\sigma}(x,t) = (\sigma_1(x), t\sigma_2(x))$. The expression of $\Theta(\omega)_\sigma$ given in the proof of [8, Proposition 11.3] depends on elements of the form $\omega_{\hat{\sigma} \circ \iota_{F*G}}$ with $\mathrm{Im}\,(\hat{\sigma} \circ \iota_{F*G}) \subset X \times]-\infty, a]$. This implies $(\Theta(\omega))_\sigma = 0$ and $\Theta(\omega) \in \mathcal{L}^*$.

• *The complex $\mathcal{L}^* \cap \mathcal{K}^*$ is acyclic.* Let $\omega \in \mathcal{L}^* \cap \mathcal{K}^*$ with $\delta\omega = 0$. Denote a the positive number and K the compact subset of X associated to ω. With the previous notations, we observe that the condition $(\mathrm{Im}\,\sigma) \cap (K \times \mathbb{R}) = \emptyset$ is equivalent to $(\mathrm{pr}(\mathrm{Im}\,\sigma)) \cap K = \emptyset$. The previous map $\hat{\sigma}$ satisfies $\mathrm{pr}(\mathrm{Im}\,\sigma) = \mathrm{pr}(\mathrm{Im}\,\hat{\sigma})$. Thus if σ is such that $\mathrm{pr}(\mathrm{Im}\,\sigma) = \emptyset$ implies $\omega_\sigma = 0$, we have also $\omega_{\hat{\sigma} \circ \iota_{F*G}} = 0$. We deduce $\Theta(\omega) \in \mathcal{L}^* \cap \mathcal{K}^*$, with the same number a and the same compact subset K.

• The proofs *of acyclicity of $\mathcal{R}^*$ and $\mathcal{R}^* \cap \mathcal{K}^*$* are similar. □

Corollary 2.17. *Let X be a compact filtered space. The cone $\mathring{c}X$ is endowed with the conic filtration and with a perversity $\overline{p}$. We denote also $\overline{p}$ the induced perversity on $X \times]0, \infty[$. Then, for any $k \geq \overline{p}(\mathtt{w}) + 2$, the canonical injection $I\colon X \times]0, \infty[\hookrightarrow \mathring{c}X$ induces an isomorphism*

$$\mathscr{H}^k_{\overline{p},c}(X \times]0, \infty[; R) \xrightarrow{\cong} \mathscr{H}^k_{\overline{p},c}(\mathring{c}X; R).$$

Proof. In (15), to any $\omega \in \widetilde{N}_{\overline{p}}(X \times]0, \infty[)$, we associate a cochain $\eta \in \widetilde{N}^{*,\mathcal{U}}_{\overline{p}}(\mathring{c}X)$. By precomposing with $\mathrm{pr}^*\colon \mathscr{H}^*_{\overline{p}}(X) \to \mathscr{H}^*_{\overline{p}}(X \times]0, \infty[)$, we get the connecting map of (14),

$$\delta_1\colon \mathscr{H}^*_{\overline{p}}(X) \to \mathscr{H}^{*+1}_{\overline{p},c}(\mathring{c}X), \tag{19}$$

defined by

$$[\gamma] \mapsto \begin{cases} [\delta(\tilde{g} \cup \mathrm{pr}^*(\gamma))] & \text{if } \mathrm{Im}\,\sigma \subset X \times]0, \infty[, \\ 0 & \text{if } \mathrm{Im}\,\sigma \subset \mathring{c}_1 X. \end{cases}$$

We observe that δ_1 is an isomorphism if $* \geq \overline{p}(\mathtt{w}) + 1$. In the proof of Proposition 2.15, with the same 0-cochain $\tilde{g}$, we have specified in (17) the connecting map of (16). By precomposing with $\mathrm{pr}^*\colon \mathscr{H}^*_{\overline{p}}(X) \to H^*(\mathcal{K}^*)$, we get an isomorphism

$$\delta_2\colon \mathscr{H}^*_{\overline{p}}(X) \to H^{*+1}(\mathcal{L}^* \cap \mathcal{R}^* \cap \mathcal{K}^*), [\gamma] \mapsto [\delta(\tilde{g} \cup \mathrm{pr}^*(\gamma))]. \tag{20}$$

In the degrees of the statement, with X compact, these two connecting maps are isomorphisms and give the following commutative diagram.

$$\begin{array}{ccc} \mathscr{H}^*_{\overline{p}}(X) & \xrightarrow{\quad\delta_1\quad} & \mathscr{H}^{*+1}_{\overline{p},c}(\mathring{c}X) \\ {\scriptstyle \delta_2}\downarrow & & \downarrow{\scriptstyle I^*} \\ H^{*+1}(\mathcal{L}^* \cap \mathcal{R}^* \cap \mathcal{K}^*) & \xleftarrow{\ \iota\ } & \mathscr{H}^{*+1}_{\overline{p},c}(X \times]0, \infty[). \end{array}$$

Thus the homomorphism I^* is an isomorphism. □

2.4. **Intersection cohomologies with compact supports.** In this section, we compare the blown-up intersection cohomology with an intersection cohomology defined by the dual complex of intersection chains, in the case of compact supports. This second cohomology has already been introduced in [16] for the study of a duality in intersection homology via cap products. We call it intersection cohomology with compact supports. We prove the existence of an isomorphism between these two intersection cohomologies with compact supports under an hypothesis on the torsion, that we precise in the next definition. Mention also the existence of examples for which the two cohomologies differ.

Definition 2.18. Let R be a commutative ring and $\overline{p}$ a perversity on a CS set X. The CS set X is *locally $(\overline{p}, R)$-torsion free* if, for each singular stratum S of link L_S, one has

$$\mathfrak{T}^{\overline{p}}_{\overline{q}(S)}(L_S; R) = 0,$$

where $\overline{q}(S) = \operatorname{codim} S - 2 - \overline{p}(S)$ and $\mathfrak{T}^{\overline{p}}_j(L_S; R)$ is the torsion R-submodule of $\mathfrak{H}^{\overline{p}}_j(L_S; R)$.

Note that the previous condition is always fulfilled if R is a field. Also, in the case of a GM-perversity, that is the torsion subgroup of $H^{\overline{p}}_j(L_S; R)$ which is involved. The existence of an isomorphism between the two cohomologies is based on the next result (cf. [21], [26]). A proof of it can be found in [9, Section 5.1].

Proposition 2.19. *Let $\mathcal{F}_X$ be the category whose objects are homeomorphic in a filtered way to open subsets of a fixed CS set X and whose arrows are the inclusions and homeomorphisms respecting the filtration. Let $\mathcal{A}b_*$ be the category of graded abelian groups. We consider two functors $F^*, G^* \colon \mathcal{F}_X \to \mathcal{A}b$ and a natural transformation $\Phi \colon F^* \to G^*$, such that the following properties are satisfied.*

(i) *The functors F^* and G^* have Mayer-Vietoris exact sequences and the natural transformation Φ induces a commutative diagram between these sequences.*
(ii) *If $\{U_\alpha\}$ is an increasing sequence of open subsets of X and $\Phi \colon F_*(U_\alpha) \to G_*(U_\alpha)$ is an isomorphism for each α, then $\Phi \colon F_*(\cup_\alpha U_\alpha) \to G_*(\cup_\alpha U_\alpha)$ is an isomorphism.*
(iii) *Let L be a compact filtered space such that $\mathbb{R}^i \times \mathring{c}L$ is homeomorphic, in a filtered way, to an open subset of X. If $\Phi \colon F^*(\mathbb{R}^i \times (\mathring{c}L \backslash \{\mathtt{v}\})) \to G^*(\mathbb{R}^i \times (\mathring{c}L \backslash \{\mathtt{v}\}))$ is an isomorphism, then so is $\Phi \colon F^*(\mathbb{R}^i \times \mathring{c}L) \to G^*(\mathbb{R}^i \times \mathring{c}L)$.*
(iv) *If U is an open subset of X, included in only one stratum and homeomorphic to an euclidean space, then $\Phi \colon F^*(U) \to G^*(U)$ is an isomorphism.*

Then $\Phi \colon F^*(X) \to G^*(X)$ is an isomorphism.

If $(X, \overline{p})$ is a perverse space, we set $\mathfrak{C}^*_{\overline{p}}(X; R) = \hom(\mathfrak{C}^{\overline{p}}_*(X; R), R)$ where $\mathfrak{C}^{\overline{p}}_*(X; R)$ is introduced in Definition 1.6. The homology of $\mathfrak{C}^*_{\overline{p}}(X; R)$ is denoted $\mathfrak{H}^*_{\overline{p}}(X; R)$ (or $\mathfrak{H}^*_{\overline{p}}(X)$ if there is no ambiguity) and called *$\overline{p}$-intersection cohomology.* A cochain map $\chi \colon \widetilde{N}^*_{\overline{p}}(X; R) \to \mathfrak{C}^*_{D\overline{p}}(X; R)$ can be defined as follows, see [8, Proposition 13.4]. If $\omega \in \widetilde{N}^*(X; R)$ and if $\sigma \colon \Delta_\sigma = \Delta_0 * \cdots * \Delta_n \to X$ is a filtered simplex, we set:

$$\chi(\omega)(\sigma) = \begin{cases} \omega_\sigma(\widetilde{\Delta}_\sigma) & \text{if } \sigma \text{ is regular,} \\ 0 & \text{otherwise.} \end{cases}$$

For a field of coefficients and GM-perversities, we showed in [3] that the map χ induces an isomorphism in homology. In [8, Theorem F], this result is extended to the cases of perversities defined at the level of each stratum, with coefficients in a Dedekind ring and for any paracompact, separable, locally $(D\overline{p}, R)$-free CS set. More precisely, under the previous hypotheses, we prove

$$\mathscr{H}^*_{\overline{p}}(X;R) \cong \mathfrak{H}^*_{D\overline{p}}(X;R). \tag{21}$$

The next result is the adaptation of (21) to cohomologies with compact supports, with the definition ([16]) $\mathfrak{H}^*_{\overline{q},c}(X;R) = \lim_{K\,\mathrm{compact}} \mathfrak{H}^*_{\overline{q}}(X, X\backslash K;R)$.

Proposition 2.20. *Let $(X,\overline{p})$ be a paracompact perverse CS set and R a Dedekind ring. Denote $\overline{q} = D\overline{p}$. We suppose that one of the following hypotheses is satisfied.*

(1) The ring R is a field.

(2) The CS set X is a locally $(\overline{q},R)$-torsion free pseudomanifold.

Then there is an isomorphism

$$\mathscr{H}^*_{\overline{p},c}(X;R) \cong \mathfrak{H}^*_{\overline{q},c}(X;R) \tag{22}$$

In the case of a GM-perversity $\overline{p}$, the conclusion of Proposition 2.20 can be stated as

$$\mathscr{H}^*_{\overline{p},c}(X;R) \cong \lim_{K\,\mathrm{compact}} H^*_{\overline{q}}(X, X\backslash K;R).$$

Proof of Proposition 2.20. This proof is an adaptation of that of [8, Theorem F]. Let U be an open subset of X, $\omega \in \widetilde{N}^*_{\overline{p},c}(U)$ with compact support K and σ a regular filtered simplex. From the construction of χ, we observe that $\chi(\omega)(\sigma) \in \mathfrak{C}^*_{\overline{q}}(U, U\backslash K)$. This gives a morphism $\chi_U\colon \widetilde{N}^*_{\overline{p},c}(U) \to \lim_{K\,\mathrm{compact}} \mathfrak{C}^*_{\overline{q}}(U, U\backslash K)$ which induces a natural transformation

$$\chi^*_U\colon \mathscr{H}^*_{\overline{p},c}(U) \to \lim_{K\,\mathrm{compact}} \mathfrak{C}^*_{\overline{q}}(U, U\backslash K).$$

For proving that $\chi^* = \chi^*_X$ is an isomorphism, we use Proposition 2.19 whose hypotheses are satisfied thanks to Proposition 2.15, Corollary 2.17 and [9, Chapter 7]. □

3. Topological invariance. Theorem A

In this section, we prove the topological invariance of $\mathscr{H}^*_{\overline{p},c}(-)$ in the case of GM-perversities and paracompact CS sets with no codimension one strata. We first establish some additional properties of the blown-up cohomology with compact supports. Later, for the proof of the topological invariance, we introduce a method developed by King in [21] and taken over with details and examples in [9, Section 5.5].

From Proposition 2.11, we deduce the existence of a short exact sequence defining the relative blown-up cohomology with compact supports, in the case of an open subset $U \subset X$ of a perverse space $(X,\overline{p})$,

$$0 \longrightarrow \widetilde{\mathbb{N}}_{\overline{p},c}(U;R) \xrightarrow{\boldsymbol{I}_{U,X}} \widetilde{\mathbb{N}}_{\overline{p},c}(X;R) \xrightarrow{\boldsymbol{R}_{U,X}} \widetilde{\mathbb{N}}_{\overline{p},c}(X,U;R) \longrightarrow 0. \tag{23}$$

The associated long exact sequence, Proposition 2.14 and Proposition 2.15 involve the next determination.

Corollary 3.1. *Let X be an n-dimensional compact filtered space and $\overline{p}$ be a GM-perversity. The cone $\mathring{\mathtt{c}}X$ is endowed with the induced filtration. Then we have:*

$$\mathscr{H}^j_{\overline{p},c}(\mathring{\mathtt{c}}X, \mathring{\mathtt{c}}X\backslash\{\mathtt{w}\}; R) = \begin{cases} 0 & \text{if} \quad j \geq \overline{p}(n+1)+1, \\ \mathscr{H}^j_{\overline{p},c}(X;R) & \text{if} \quad j < \overline{p}(n+1)+1. \end{cases} \tag{24}$$

The next result is an excision property.

Corollary 3.2. *Let $\overline{p}$ be a GM-perversity. Let X be a paracompact, locally compact filtered space, F a closed subset of X and U an open subset of X such that $F \subset U$. Then the canonical inclusion $(X\backslash F, U\backslash F) \hookrightarrow (X, U)$ induces an isomorphism,*

$$\mathscr{H}^*_{\overline{p},c}(X\backslash F, U\backslash F; R) \cong \mathscr{H}^*_{\overline{p},c}(X, U; R).$$

Proof. From the open covers $\{U, X\backslash F\}$ of X and $\{U, U\backslash F\}$ of U, we obtain a commutative diagram between the associated Mayer-Vietoris exact sequences (Proposition 2.12).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathbb{N}}^*_{\overline{p},c}(U \cap (U\backslash F)) & \longrightarrow & \widetilde{\mathbb{N}}^*_{\overline{p},c}(U) \oplus \widetilde{\mathbb{N}}^*_{\overline{p},c}(U\backslash F) & \longrightarrow & \widetilde{\mathbb{N}}^*_{\overline{p},c}(U) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\mathbb{N}}^*_{\overline{p},c}(U \cap (X\backslash F)) & \longrightarrow & \widetilde{\mathbb{N}}^*_{\overline{p},c}(U) \oplus \widetilde{\mathbb{N}}^*_{\overline{p},c}(X\backslash F) & \longrightarrow & \widetilde{\mathbb{N}}^*_{\overline{p},c}(X) & \longrightarrow & 0. \end{array}$$

The Ker-Coker exact sequence gives an isomorphisme $\widetilde{\mathbb{N}}^*_{\overline{p},c}(X\backslash F, U\backslash F) \cong \widetilde{\mathbb{N}}^*_{\overline{p},c}(X, U)$. □

We need also the blown-up cohomology with compact supports of a product with a sphere.

Corollary 3.3. *Let $\overline{p}$ be a GM-perversity and X a locally compact and paracompact filtered space. We denote S^ℓ the sphere of $\mathbb{R}^{\ell+1}$ and endow the product $S^\ell \times X$ with the product filtration $(S^\ell \times X)_i = S^\ell \times X_i$. Then, the projection $p_X \colon S^\ell \times X \to X$, $(z, x) \mapsto x$, induces isomorphisms*

$$\mathscr{H}^j_{\overline{p},c}(S^\ell \times X; R) \cong \mathscr{H}^j_{\overline{p},c}(X; R) \oplus \mathscr{H}^{j-\ell}_{\overline{p},c}(X; R).$$

Proof. Let $\{N, S\}$ be the two poles of the sphere S^ℓ. We do an induction in the Mayer-Vietoris exact sequence with $U_1 = X \times (S^\ell\backslash\{N\})$, $U_2 = X \times (S^\ell\backslash\{S\})$ and $U_1 \cap U_2 = X \times \mathbb{R} \times S^{\ell-1}$. Propositions 2.12 and 2.15 conclude the proof. □

We briefly recall King's construction. First, we say that two points x_0, x_1 of a topological space are *equivalent* if there exists a homeomorphism $h \colon (U_0, x_0) \to (U_1, x_1)$ between two neighbourhoods of x_0 and x_1. We denote this relation by $\sim$.

Let X be a CS set. We observe that the equivalence classes of $\sim$ are union of strata. We denote X^*_i the union of the equivalence classes formed of strata of dimension less than or equal to i. Let X^* be the space X endowed with this new filtration. As X^* is a CS set whose filtration does not depend on the initial filtration on X (see [9, Section 2.8]), we have an intrinsic CS set associated to X. The identity map as continuous application $\nu \colon X \to X^*$ is called *intrinsic aggregation* of X. In the next result we compare the blown-up intersection cohomology with compact supports of X and X^*.

Proposition 3.4. *Let $\overline{p}$ be a GM-perversity and X a paracompact CS set with no codimension one strata. We consider a stratum S of X and a conic chart (U,φ) of $x \in S$. If the intrinsic aggregation induces an isomorphism*

$$\nu_*\colon \mathscr{H}^*_{\overline{p},c}(U\backslash S;R) \xrightarrow{\cong} \mathscr{H}^*_{\overline{p},c}((U\backslash S)^*;R),$$

then it induces also an isomorphism

$$\nu_*\colon \mathscr{H}^*_{\overline{p},c}(U;R) \xrightarrow{\cong} \mathscr{H}^*_{\overline{p},c}(U^*;R).$$

Proof. We may suppose $U = \mathbb{R}^k \times \mathring{\mathtt{c}}W$, where W is a compact filtered space and $S \cap U = \mathbb{R}^k \times \{\mathtt{w}\}$. From [21, Lemma 2 and Proposition 1], we deduce the existence of a homeomorphism of filtered spaces,

$$h\colon (\mathbb{R}^k \times \mathring{\mathtt{c}}W)^* \xrightarrow{\cong} \mathbb{R}^m \times \mathring{\mathtt{c}}L, \tag{25}$$

where L is a (possibly empty) compact filtered space and $m \geq k$. Moreover h satisfies,

$$h(\mathbb{R}^k \times \{\mathtt{w}\}) \subset \mathbb{R}^m \times \{\mathtt{v}\} \text{ and } h^{-1}(\mathbb{R}^m \times \{\mathtt{v}\}) = \mathbb{R}^k \times \mathring{\mathtt{c}}A, \tag{26}$$

where A is an $(m-k-1)$-sphere, $\mathtt{v}$ and $\mathtt{w}$ are the respective apexes of $\mathring{\mathtt{c}}L$ and $\mathring{\mathtt{c}}W$. With these notations, the hypothesis and the conclusion of the statement become,

$$h\colon \mathscr{H}^*_{\overline{p},c}(\mathbb{R}^k \times \mathring{\mathtt{c}}W\backslash(\mathbb{R}^k \times \{\mathtt{w}\})) \xrightarrow{\cong} \mathscr{H}^*_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L\backslash h(\mathbb{R}^k \times \{\mathtt{w}\})) \tag{27}$$

and

$$h\colon \mathscr{H}^*_{\overline{p},c}(\mathbb{R}^k \times \mathring{\mathtt{c}}W) \xrightarrow{\cong} \mathscr{H}^*_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L). \tag{28}$$

Set $s = \dim W$ and $t = \dim L$. The isomorphism h of (25) implies $k + s = m + t$, and $s \geq t$ since $m \geq k$.

• The result is direct if $s = -1$ and we may suppose $s \geq 0$ and $\mathbb{R}^k \times \{\mathtt{w}\}$ a singular stratum. In fact, since X has no strata of codimension 1, we have $s \geq 1$.

• If $t = -1$, then $L = \emptyset$ and $\dim A = m-k-1 = s$. We have a series of isomorphisms,

$$\begin{aligned}
\mathscr{H}^j_{\overline{p},c}(\mathbb{R}^k \times \mathring{\mathtt{c}}W) &\cong \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^k \times \mathring{\mathtt{c}}A) \cong_{(1)} \mathscr{H}^{j-k}_{\overline{p},c}(\mathring{\mathtt{c}}A) \\
&\cong_{(2)} \begin{cases} \mathscr{H}^{j-k-1}_{\overline{p},c}(A) = H^{j-k-1}(A) & \text{if } j-k-1 \geq \overline{p}(s+1)+1, \\ 0 & \text{if } j-k-1 \leq \overline{p}(s+1), \end{cases} \\
&\cong \begin{cases} R & \text{if } j = s+k+1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{29}$$

The isomorphisms $\cong_{(1)}$ and $\cong_{(2)}$ arrive from Proposition 2.15 and Proposition 2.14 respectively. The last isomorphism is a consequence of

$$0 < \overline{p}(s+1)+1 \leq \overline{t}(s+1)+1 = s = \dim A.$$

• We suppose now $t \geq 0$ and $s \geq 1$ and split the proof in two cases.
First case: suppose $j \leq \overline{p}(s+1)+1+k$. The same properties than above imply the two following series of isomorphisms.

$$\mathscr{H}^j_{\overline{p},c}(\mathbb{R}^k \times \mathring{\mathtt{c}}W) \cong \mathscr{H}^{j-k}_{\overline{p},c}(\mathring{\mathtt{c}}W) \cong \begin{cases} \mathscr{H}^{j-k-1}_{\overline{p},c}(W) & \text{if } j-k-1 \geq \overline{p}(s+1)+1, \\ 0 & \text{if } j-k-1 \leq \overline{p}(s+1), \end{cases} \tag{30}$$

and

$$\mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L) \cong \mathscr{H}^{j-m}_{\overline{p},c}(\mathring{\mathtt{c}}L) \cong \begin{cases} \mathscr{H}^{j-m-1}_{\overline{p},c}(L) & \text{if } j-m-1 \geq \overline{p}(t+1)+1, \\ 0 & \text{if } j-m-1 \leq \overline{p}(t+1). \end{cases} \tag{31}$$

By using (30) and (31), the isomorphism $\mathscr{H}^j_{\overline{p},c}(\mathbb{R}^k \times \mathring{\mathtt{c}}W) \cong \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L) \cong 0$ is a consequence of the next equalities whose the first one results from Definition 1.4,

$$\overline{p}(s+1)+k+1 \leq \overline{p}(t+1)+s-t+k+1 \leq \overline{p}(t+1)+m+1.$$

Second case: suppose $j > \overline{p}(s+1)+1+k$. We repeat the arguments used above in two series of isomorphisms, together with additional properties detailed below. First, we get

$$\begin{aligned} \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash h(\mathbb{R}^k \times \{\mathtt{w}\})) &\cong \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^k \times \mathring{\mathtt{c}}W \backslash \mathbb{R}^k \times \{\mathtt{w}\}) \cong \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^k \times (\mathring{\mathtt{c}}W \backslash \{\mathtt{w}\})) \\ &\cong \mathscr{H}^{j-k-1}_{\overline{p},c}(W), \end{aligned} \tag{32}$$

where the first isomorphism is the hypothesis (27). Denote $B \times \{\mathtt{v}\} = h(\mathbb{R}^m \times \{\mathtt{w}\})$. We have also isomorphisms between the next relative cohomologies.

$$\begin{aligned} &\mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash h(\mathbb{R}^k \times \{\mathtt{w}\}), \mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash (\mathbb{R}^m \times \{\mathtt{v}\})) \quad \cong_{(1)} \\ &\mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash (B \times \{\mathtt{v}\}), \mathbb{R}^m \times (\mathring{\mathtt{c}}L \backslash \{\mathtt{v}\})) \cong \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \backslash B \times (\mathring{\mathtt{c}}L, \mathring{\mathtt{c}}L \backslash \{\mathtt{v}\})) \quad \cong \\ &\mathscr{H}^j_{\overline{p},c}(\mathbb{R}^{k+1} \times A \times (\mathring{\mathtt{c}}L, \mathring{\mathtt{c}}L \backslash \{\mathtt{v}\})) \cong \mathscr{H}^{j-k-1}_{\overline{p},c}(A \times (\mathring{\mathtt{c}}L, \mathring{\mathtt{c}}L \backslash \{\mathtt{v}\})) \quad \cong_{(2)} \\ &\mathscr{H}^{j-k-1}_{\overline{p},c}(\mathring{\mathtt{c}}L, \mathring{\mathtt{c}}L \backslash \{\mathtt{v}\}) \oplus \mathscr{H}^{j-k-1-\dim A}_{\overline{p},c}(\mathring{\mathtt{c}}L, \mathring{\mathtt{c}}L \backslash \{\mathtt{v}\}), \end{aligned} \tag{33}$$

where $\cong_{(1)}$ comes from the excision of $B \times (\mathring{\mathtt{c}}L \backslash \{\mathtt{v}\})$ (see Corollary 3.2) and $\cong_{(2)}$ from Corollary 3.3. In (33), we observe from Corollary 3.1 that the hypothesis $j > \overline{p}(s+1)+1+k$ implies $\mathscr{H}^{j-k-1}_{\overline{p},c}(\mathring{\mathtt{c}}L, \mathring{\mathtt{c}}L \backslash \{\mathtt{v}\}) = 0$. Moreover, we have $j-k-1-\dim A = j-m$. Thus, with the restriction on j imposed in this second case, the previous isomorphisms imply

$$\mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash h(\mathbb{R}^k \times \{\mathtt{w}\}), \mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash \mathbb{R}^m \times \{\mathtt{v}\}) \cong \mathscr{H}^{j-m}_{\overline{p},c}(\mathring{\mathtt{c}}L, \mathring{\mathtt{c}}L \backslash \{\mathtt{v}\}). \tag{34}$$

In the next diagram, left-hand arrows are a part of the long exact sequence of a pair and the horizontal isomorphisms come successively from (34), (32) and Proposition 2.15.

$$\begin{array}{ccc} \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash h(\mathbb{R}^k \times \{\mathtt{w}\}, \mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash \mathbb{R}^m \times \{\mathtt{v}\}) & \xrightarrow{\cong} & \mathscr{H}^{j-m}_{\overline{p},c}(\mathring{\mathtt{c}}L, \mathring{\mathtt{c}}L \backslash \{\mathtt{v}\}) \\ \downarrow & & \\ \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash h(\mathbb{R}^k \times \{\mathtt{w}\}) & \xrightarrow{\cong} & \mathscr{H}^{j-k-1}_{\overline{p},c}(W) \\ \downarrow & & \\ \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathtt{c}}L \backslash \mathbb{R}^m \times \{\mathtt{v}\}) & \xrightarrow{\cong} & \mathscr{H}^{j-m}_{\overline{p},c}(\mathring{\mathtt{c}}L \backslash \{\mathtt{v}\}) \end{array}$$

From this diagram and the long exact sequence associated to $(\mathring{\mathtt{c}}L, \mathring{\mathtt{c}}L \backslash \{\mathtt{v}\})$, we deduce

$$\mathscr{H}^{j-k-1}_{\overline{p},c}(W) \cong \mathscr{H}^{j-m}_{\overline{p},c}(\mathring{\mathtt{c}}L). \tag{35}$$

By using (35), the computation of the cohomology of a cone and the cohomology of a product with $\mathbb{R}$, we get

$$\mathscr{H}^j_{\overline{p},c}(\mathbb{R}^m \times \mathring{\mathrm{c}}L) \cong \mathscr{H}^{j-m}_{\overline{p},c}(\mathring{\mathrm{c}}L) \cong \mathscr{H}^{j-k-1}_{\overline{p},c}(W) \cong_{(1)} \mathscr{H}^{j-k}_{\overline{p},c}(\mathring{\mathrm{c}}W) \cong \mathscr{H}^j_{\overline{p},c}(\mathbb{R}^k \times \mathring{\mathrm{c}}W), \quad (36)$$

where $\cong_{(1)}$ is a consequence of the condition $j-k \geq \overline{p}(s+1)+2$ imposed in this second case. □

Notice that the hypothesis *"with no codimension one strata"* is used in (29) where we assume that the sphere A is of dimension $s > 0$.

The invariance property is deduced from Proposition 2.19 applied to the natural transformation $\Phi_U \colon \mathscr{H}^*_{\overline{p},c}(U) \to \mathscr{H}^*_{\overline{p},c}(U^*)$. All the ingredients being established, the proof goes as in Proposition 2.20 and we may leave it to the reader.

Theorem A. *Let $\overline{p}$ be a GM-perversity. For any n-dimensional paracompact CS set X, with no codimension one strata, the intrinsic aggregation $\nu \colon X \mapsto X^*$ induces an isomorphism*

$$\mathscr{H}_{\overline{p},c}(X;R) \cong \mathscr{H}_{\overline{p},c}(X^*;R).$$

4. Poincaré duality. Theorem B

4.1. **Intersection homology and Poincaré duality.** In this paragraph, X is an oriented (Definition 4.2) paracompact pseudomanifold and R is a commutative ring. We recall some known examples with the purpose of highlighting the conditions of existence of a Poincaré duality in intersection homology. First, Goresky and MacPherson display a bilinear form, $H^{\overline{p}}_i(X;\mathbb{Z}) \times H^{\overline{t}-\overline{p}}_{n-i}(X;\mathbb{Z}) \to \mathbb{Z}$, which becomes non degenerate after tensorisation by the rationals, cf. [17]. By denoting $T^{\overline{p}}_i(-)$ the torsion subgroup of $H^{\overline{p}}_i(-)$, M. Goresky et P. Siegel show in [20, Theorem 4.4] that the previous bilinear form generates a non degenerate bilinear form,

$$T^{\overline{p}}_i(X) \times T^{\overline{t}-\overline{p}}_{n-i-1}(X) \to \mathbb{Q}/\mathbb{Z},$$

under the hypothesis of locally $(\overline{p},\mathbb{Z})$-torsion free. Without this additional hypothesis, the property disappears. If we take as pseudomanifold X the suspension of $\mathbb{R}P^3$ endowed with the perversity $\overline{p}$ taking the value 1 on the two apexes of the suspension, we see that $H^{\overline{p}}_2(X;\mathbb{Z}) = 0$ and $H^{\overline{p}}_1(X;\mathbb{Z}) = \mathbb{Z}_2$.

We are interested now in the existence of a Poincaré duality given by a cap product between intersection homology and cohomology groups. We choose in this paragraph the intersection cohomology $H^*_{\overline{p}}(X;R)$ given by $C^*_{\overline{p}}(X;R) = \hom(C^{\overline{p}}_*(X;R),R)$. Even if we avoid the previous phenomenon of torsion by *choosing a field* R, some restrictions appear on the domain of values taken by the perversities.

Example 4.1. Consider a compact oriented manifold M of dimension $n-1$. We filter its suspension $X = \Sigma M$ by $X_0 = \{N,S\} = \cdots = X_{n-1} \subset X_n = X$. We choose a perversity $\overline{p}$ such that $\overline{p}(\{N\}) = \overline{p}(\{S\}) = p$. The $\overline{p}$-intersection homology of X is determined for

instance in [9, Section 4.4] as

$$H_i^{\overline{p}}(X;R) = \begin{cases} H_i(M;R) & \text{if } i < n-p-1, \\ 0 & \text{if } i = n-p-1 \text{ et } i \neq 0, \\ \tilde{H}_{i-1}(M;R) & \text{if } i > n-p-1 \text{ et } i \neq 0, \\ R & \text{if } 0 = i \geq n-p-1. \end{cases} \tag{37}$$

Even in the case of field coefficients R, we observe the lack of duality if the perversity $\overline{p}$ does not lie between $\overline{0}$ and $\overline{t}$. For instance, with the space $X = \Sigma M$, we have:

- $H_0^{\overline{p}}(X) = R$ and $H^n_{D\overline{p}}(X) = 0$ if $p > n-2 = \overline{t}(n)$,
- $H_n^{\overline{p}}(X) = 0$ and $H^0_{D\overline{p}}(X) = R$ if $p < 0$.

In Theorem B, to overcome the restriction $\overline{p} \in [\overline{0}, \overline{t}]$, we use the tame intersection homology recalled in Definition 1.8.

4.2. **Orientation of a pseudomanifold.** We recall the definition and properties of the orientation of pseudomanifolds, cf. [17] et [16].

Definition 4.2. An *R-orientation* of a pseudomanifold X of dimension n is an R-orientation of the manifold $X^n = X \backslash X_{n-1}$. For any $x \in X^n$, we denote the associated local orientation class by $\mathsf{o}_x \in H_n(X^n, X^n \backslash \{x\}; R) = \mathfrak{H}_n^{\overline{0}}(X, X\backslash\{x\}; R)$

Theorem 4.3 ([16])**.** *Let X be a pseudomanifold of dimension n, endowed with an R-orientation.*

(1) If X is normal, the sheaf generated by $U \to \mathfrak{H}_n^{\overline{0}}(X, X\backslash\overline{U}; R)$ is constant and there exists a unique global section s such that $s(x) = \mathsf{o}_x$ for any $x \in X^n$. Moreover for any $x \in X$, $\mathfrak{H}_i^{\overline{0}}(X, X\backslash\{x\}; R) = 0$ if $i \neq n$ and $\mathfrak{H}_n^{\overline{0}}(X, X\backslash\{x\}; R)$ is the free R-module generated by $s(x)$. Henceforth we denote $\mathsf{o}_x = s(x)$ for any $x \in X$.

(2) If X is not normal, we denote $\Pi\colon \hat{X} \to X$ the normalisation constructed by G. Padilla in [24] *and we endow $\hat{X}$ with the R-orientation induced by the homeomorphism $\Pi\colon \hat{X}\backslash\hat{X}_{n-1} \cong X\backslash X_{n-1}$. Then, we have $\mathfrak{H}_i^{\overline{0}}(X, X\backslash\{x\}; R) = 0$ si $i \neq n$ and $\mathfrak{H}_n^{\overline{0}}(X, X\backslash\{x\}; R)$ is the free R-module generated by $\{\Pi_*(\mathsf{o}_y) \mid y \in \Pi^{-1}(x)\}$. We denote $\mathsf{o}_x = \sum_{y\in\Pi^{-1}(x)} \Pi_*(\mathsf{o}_y)$.*

(3) For any compact $K \subset X$, there exists a unique element $\Gamma_K^X \in \mathfrak{H}_n^{\overline{0}}(X, X\backslash K; R)$ whose restriction equals o_x for any $x \in K$. The class Γ_K^X is called the fundamental class of X over K. *If there is no ambiguity, we denote $\Gamma_K = \Gamma_K^X$.*

Remark 4.4. Let $U \subset V \subset X$ be two open subsets of a pseudomanifold X. If $K \subset U$ is a compact subset, the canonical inclusion $U \hookrightarrow V$ induces a homomorphism, $I_*\colon \mathfrak{H}_n^{\overline{0}}(U, U\backslash K; R) \to \mathfrak{H}_n^{\overline{0}}(V, V\backslash K; R)$. By construction, the fundamental class satisfies

$$I_*(\Gamma_K^U) = \Gamma_K^V. \tag{38}$$

4.3. **The main theorem.** In this section, we prove that the cap product with the fundamental class of a pseudomanifold is the isomorphism of Poincaré duality.

Proposition 4.5. *Let R be a commutative ring and X an oriented pseudomanifold of dimension n, endowed with a perversity $\overline{p}$. The cap product with the fundamental class of X defines a homomorphism,*

$$\mathcal{D}\colon \mathscr{HC}_{\overline{p},c}^k(X;R) \to \mathfrak{H}_{n-k}^{\overline{p}}(X;R). \tag{39}$$

Proof. Let $\omega \in \widetilde{N}^k_{\overline{p},c}(X)$ be a cocycle with compact support $K \subset X$. We choose a representing element $\gamma_K \in \mathfrak{C}^{\overline{0}}_n(X, X\backslash K)$ of the fundamental class $\Gamma^X_K \in \mathfrak{H}^{\overline{0}}_n(X, X\backslash K)$. The differential of the chain $\omega \cap \gamma_K$ equals

$$\mathfrak{d}(\omega \cap \gamma_K) = (\delta\omega) \cap \gamma_K + (-1)^k \omega \cap (\mathfrak{d}\gamma_K) = (-1)^k \omega \cap (\mathfrak{d}\gamma_K).$$

The chain γ_K being a relative cycle, its differential satisfies $\mathfrak{d}\gamma_K \in \mathfrak{C}^{\overline{0}}_{n-1}(X\backslash K)$. The subset K being a support of ω, we have $\omega \cap \mathfrak{d}\gamma_K = 0$ and thus $\omega \cap \gamma_K$ is a cycle in $\mathfrak{C}^{\overline{p}}_{n-k}(X)$. Denote $[\omega \cap \gamma_K] \in \mathfrak{H}^{\overline{p}}_{n-k}(X)$ the associated tame intersection homology class. We have to prove that this class does not depend on the choices done in its construction.

- *The class $[\omega \cap \gamma_K]$ does not depend on the choice of the representing element γ_K of Γ_K.* This is a consequence of the two following observations.
  - If we replace γ_K by $\gamma_K + \mu$ with $\mu \in \mathfrak{C}^{\overline{0}}_n(X\backslash K)$, we have, by definition of the cap product, $\omega \cap (\gamma_K + \mu) = \omega \cap \gamma_K + \omega \cap \mu = \omega \cap \gamma_K$.
  - If we replace γ_K by $\gamma_K + \mathfrak{d}\mu$ with $\mu \in \mathfrak{C}^{\overline{0}}_{n+1}(X, X\backslash K)$, the same argument implies $[\omega \cap (\gamma_K + \mathfrak{d}\mu)] = [\omega \cap \gamma_K]$.
- *The class $[\omega \cap \gamma_K]$ does not depend on the choice of the support K of ω.* If K and L are two supports of ω, we may suppose $L \subset K$. Therefore, the cycle $\gamma_K \in \mathfrak{C}^{\overline{0}}_n(X, X\backslash K)$ is also a cycle in $\mathfrak{C}^{\overline{0}}_n(X, X\backslash L)$. By uniqueness of the fundamental class over a compact, the classes in $\mathfrak{H}^{\overline{0}}_n(X, X\backslash L)$ associated to γ_L and γ_K are equal. Therefore, there exist $\alpha \in \mathfrak{C}^{\overline{0}}_{n+1}(X, X\backslash L)$ and $\beta \in \mathfrak{C}^{\overline{0}}_n(X\backslash L)$ such that $\gamma_K - \gamma_L = \mathfrak{d}\alpha + \beta$. Then we may deduce,

$$[\omega \cap \gamma_K] - [\omega \cap \gamma_L] = [\omega \cap \mathfrak{d}\alpha] + [\omega \cap \beta] = 0.$$

- *The class $[\omega \cap \gamma_K]$ does not depend on the choice of the cocycle ω in its associated cohomology class.* This is a consequence of the Leibniz formula $[(\omega + \delta\eta) \cap \gamma_K] = [\omega \cap \gamma_K \pm \mathfrak{d}(\eta \cap \gamma_K) \pm \eta \cap \mathfrak{d}\gamma_K] = [\omega \cap \gamma_K]$.

From Proposition 2.10, we get the homomorphism $\mathcal{D}$ of the statement. □

Theorem B. *Let R be a commutative ring and X an oriented paracompact pseudomanifold of dimension n, endowed with a perversity $\overline{p}$. Then, the cap product with the fundamental class of X induces an isomorphism between the blown-up intersection cohomology with compact supports and the tame intersection homology,*

$$\mathcal{D} \colon \mathscr{H}^k_{\overline{p},c}(X; R) \xrightarrow{\cong} \mathfrak{H}^{\overline{p}}_{n-k}(X; R).$$

By using Proposition 2.20, Theorem B gives also the duality theorem established by Friedman and McClure in [16], see also [9].

Corollary 4.6. *Let R be a Dedekind ring and X an oriented paracompact pseudomanifold of dimension n, endowed with a perversity $\overline{p}$. If X is locally $(D\overline{p}, R)$-torsion free, the cap product with the fundamental class induces an isomorphism,*

$$\mathcal{D} \colon \mathfrak{H}^k_{D\overline{p},c}(X; R) \xrightarrow{\cong} \mathfrak{H}^{\overline{p}}_{n-k}(X; R).$$

In the case of a compact pseudomanifold, we retrieve the first result in this direction, proved by Goresky and MacPherson in [17].

Corollary 4.7. *Let X be an oriented compact pseudomanifold of dimension n, with no strata of codimension 1. For any GM-perversity $\overline{p}$, there exists an isomorphism,*

$$\mathcal{D}\colon H^k_{D\overline{p}}(X;\mathbb{Q}) \xrightarrow{\cong} H^{\overline{p}}_{n-k}(X;\mathbb{Q}).$$

Proof of Theorem B. As any open subset $U \subset X$ is a pseudomanifold, we may consider the associated homomorphism defined in Proposition 4.5, $\mathcal{D}_U\colon \mathscr{H}^k_{\overline{p},c}(U) \to \mathfrak{H}^{\overline{p}}_{n-k}(U)$. If $U \subset V \subset X$ are two open subsets of X, the equality (38) gives the commutativity of the next diagram,

$$\begin{array}{ccc} \mathscr{H}^k_{\overline{p},c}(V) & \xrightarrow{\mathcal{D}_V} & \mathfrak{H}^{\overline{p}}_{n-k}(V) \\ {\scriptstyle I^*}\uparrow & & \uparrow{\scriptstyle I_*} \\ \mathscr{H}^k_{\overline{p},c}(U) & \xrightarrow{\mathcal{D}_U} & \mathfrak{H}^{\overline{p}}_{n-k}(U) \end{array} \tag{40}$$

where I_* et I^* are induced by the canonical inclusion $U \hookrightarrow V$ (see Proposition 2.11).

The morphisms $\mathcal{D}_U$ give a natural transformation between the functors $\mathscr{H}^k_{\overline{p},c}(-)$ and $\mathfrak{H}^{\overline{p}}_{n-k}(-)$ and we apply Proposition 2.19 after having checked its hypotheses.

• Condition (ii) is direct and condition (iv) is the classical Poincaré duality theorem of manifolds.

• By applying (40) to $V = \mathbb{R}^i \times \mathring{\mathtt{c}}L$ et $U = \mathbb{R}^i \times \mathring{\mathtt{c}}L\backslash\{\mathtt{v}\} \cong \mathbb{R}^i \times L\times]0,\infty[$, the condition (iii) comes from the properties of the blown-up cohomology with compact supports established in Proposition 2.14 and Corollary 2.17 together with the properties of tame intersection homology recalled in the Propositions 1.10 and 1.11.

• We consider now condition (i). The two theories, $\mathscr{H}^*_{\overline{p},c}(-)$ and $\mathfrak{H}^{\overline{p}}_{n-*}(-)$, have Mayer-Vietoris exact sequences, cf. Proposition 2.12 and Theorem 1.9. It is thus sufficient to prove that the map $\mathcal{D}$ induces a commutative diagram between these two sequences. This problem is reduced to two cases:

- a square with an open subset U of a pseudomanifold X and that is exactly the situation of (40),
- a square containing the connecting maps of the two sequences and that we detail now. We consider the following diagram where $X = U_1 \cup U_2$ and the maps δ_c, δ_h are the connecting maps.

$$\begin{array}{ccc} \mathscr{H}^k_{\overline{p},c}(X) & \xrightarrow{\mathcal{D}_X} & \mathfrak{H}^{\overline{p}}_{n-k}(X) \\ {\scriptstyle \delta_c}\downarrow & & \downarrow{\scriptstyle \delta_h} \\ \mathscr{H}^{k+1}_{\overline{p},c}(U_1\cap U_2) & \xrightarrow{\mathcal{D}_{U_1\cap U_2}} & \mathfrak{H}^{\overline{p}}_{n-k-1}(U_1\cap U_2). \end{array} \tag{41}$$

Let $\omega \in \widetilde{N}^{k,\mathfrak{U}}_{\overline{p},c}(X)$ be a cocycle of compact support K. For $i = 1, 2$, let $g_i\colon X \to \{0,1\}$ be a partition of unity with $\operatorname{Supp} g_i \subset U_i$ and $\tilde{g}_i \in \widetilde{N}^0_{\overline{0}}(X)$ the associated 0-cochain defined in Lemma 2.13. The connecting map δ_c is constructed as follows in Proposition 2.12: *we choose relatively compact open subsets, W_1, W_1', W_2, W_2' such that* $\operatorname{Supp} g_i \cap K \subset W_i' \subset \overline{W}_i' \subset W_i \subset \overline{W}_i \subset U_i$, *and we define a cochain $\tilde{g}_i \cup \omega \in \widetilde{N}^{k,\mathfrak{U}}_{\overline{p},c}(X)$ with compact support $\overline{W}_i'$, pour $i = 1, 2$. The open subset $W = W_1 \cap W_2$ and the compact $F = \overline{W}_1 \cap \overline{W}_2$*

satisfy

$$\operatorname{Supp} g_1 \cap \operatorname{Supp} g_2 \cap K \subset W \subset F \subset U_1 \cap U_2.$$

We define also an open cover $\mathcal{W}$ *of* $U_1 \cap U_2$ *and we set* $\delta_c([\omega]) = [\delta\tilde{g}_1 \cup \omega]$ *where* $\delta\tilde{g}_1 \cup \omega \in \widetilde{N}^{*,\mathcal{W}}_{\overline{p},c}(U_1 \cap U_2)$ *is a cochain of compact support* F*. By composing with the duality map, we get,*

$$(\mathcal{D}_{U_1\cap U_2} \circ \delta_c)([\omega]) = [(\delta\tilde{g}_1 \cup \omega) \cap \gamma_F], \tag{42}$$

with $\gamma_F \in \mathfrak{C}^{\overline{0}}_n(X, X\backslash F)$ a representing element of the fundamental class of X over the compact F. The compact $L = K \cup \overline{W}_1 \cup \overline{W}_2$ is also a compact support of ω. From the open cover $\{U_1\backslash\overline{W}_1, U_2\backslash\overline{W}_2, U_1 \cap U_2\}$ of X, by using properties of the subdivision process in intersection homology (cf. [4, Proposition 7.10]), we decompose a representing element $\gamma_L \in \mathfrak{C}^{\overline{0}}_n(X, X\backslash L)$ of the fundamental class of X over the compact L as

$$\gamma_L = \alpha_1 + \alpha_2 + \alpha_{12} + (\mathfrak{d}\mathfrak{T}^s + \mathfrak{T}^s\mathfrak{d})(\gamma_L), \tag{43}$$

where s is an integer, $\alpha_i \in \mathfrak{C}^{\overline{0}}_n(U_i\backslash\overline{W}_i)$, $i = 1, 2$ and $\alpha_{12} \in \mathfrak{C}^{\overline{0}}_n(U_1 \cap U_2)$. The chain γ_L is a relative cycle and by construction we have $\mathfrak{T}^s\mathfrak{d}(\gamma_L) \in \mathfrak{C}_n(X\backslash L)$. The chains α_1, α_2 having a support in $X\backslash F$, we get $[\gamma_L] = [\alpha_{12}] \in \mathfrak{H}^{\overline{0}}_n(X, X\backslash F)$. With Remark 4.4 and $F \subset U_1 \cap U_2$, we can choose $[\alpha_{12}] \in \mathfrak{H}^{\overline{0}}_n(U_1 \cap U_2, U_1 \cap U_2\backslash F)$ as fundamental class of $U_1 \cap U_2$ over F. Then, the equality (42) becomes,

$$\begin{aligned}
\mathcal{D}_{U_1\cap U_2}(\delta_c([\omega])) &= [(\delta\tilde{g}_1 \cup \omega) \cap \alpha_{12}] = [(\delta(\tilde{g}_1 \cup \omega)) \cap \alpha_{12}] \\
&= -(-1)^{|\omega|}[(\tilde{g}_1 \cup \omega) \cap \mathfrak{d}\alpha_{12}] \\
&=_{(1)} -(-1)^{|\omega|}[(\tilde{g}_1 \cup \omega) \cap \mathfrak{d}(\alpha_1 + \alpha_{12})] \\
&=_{(2)} (-1)^{|\omega|}[(\tilde{g}_1 \cup \omega) \cap \mathfrak{d}\alpha_2] \qquad (44) \\
&=_{(3)} (-1)^{|\omega|}[\omega \cap \mathfrak{d}\alpha_2], \qquad (45)
\end{aligned}$$

where

- the equality (1) is a consequence of the fact that $\tilde{g}_1 \cup \omega$ has for support $\overline{W}_1$ and $\alpha_1 \in \mathfrak{C}^{\overline{0}}_n(U_1\backslash\overline{W}_1)$,
- the equality (2) comes from $\mathfrak{d}\alpha_1 + \mathfrak{d}\alpha_{12} = \mathfrak{d}\gamma_L - \mathfrak{d}\alpha_2 - \mathfrak{d}\mathfrak{T}^s\mathfrak{d}(\gamma_L)$ and $(\tilde{g}_1 \cup \omega) \cap \mathfrak{d}\gamma_L = (\tilde{g}_1 \cup \omega) \cap \mathfrak{d}\mathfrak{T}^s\mathfrak{d}(\gamma_L) = 0$, because $\tilde{g}_1 \cup \omega$ has for support $\overline{W}_1$ and $\mathfrak{d}\gamma_L \in \mathfrak{C}^{\overline{0}}_{n-1}(X\backslash L)$,
- the equality (3) happens from $\tilde{g}_1 + \tilde{g}_2 = 1$, from the fact that $\tilde{g}_2 \cup \omega$ admits $\overline{W}_2$ as support and from $\alpha_2 \in \mathfrak{C}^{\overline{0}}_n(U_2\backslash\overline{W}_2)$.

We proceed now to the determination of $(\delta_h \circ \mathcal{D}_X)([\omega])$. As the duality map does not depend on the choice of the support of ω, we have, with the notations of (43),

$$\mathcal{D}_X([\omega]) = [\omega \cap \gamma_L] = [\omega \cap \alpha_2 + \omega \cap (\alpha_1 + \alpha_{12})],$$

with $\omega \cap \alpha_2 \in \mathfrak{C}^{\overline{p}}_*(U_2)$ and $\omega \cap (\alpha_1 + \alpha_{12}) \in \mathfrak{C}^{\overline{p}}_*(U_1)$. It follows

$$\delta_h(\mathcal{D}_X([\omega]) = [\mathfrak{d}(\omega \cap \alpha_2)] = (-1)^{|\omega|}[\omega \cap \mathfrak{d}\alpha_2]. \tag{46}$$

The result is now a consequence of the equalities (45) and (46). □

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