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# Approximation numbers of composition operators on $H^p$

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza\*

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**Abstract.** *We give estimates for the approximation numbers of composition operators on the  $H^p$  spaces,  $1 \leq p < \infty$ .*

**Mathematics Subject Classification 2010.** Primary: 47B33 – Secondary: 30H10 ; 30H20 ; 30J10 ; 47B06

**Key-words.** approximation numbers; Blaschke product ; composition operator; Hardy space; interpolation sequence ; pseudo-hyperbolic metric

## 1 Introduction

Recently, the study of approximation numbers of composition operators on  $H^2$  has been initiated (see [10], [11], [8], [18], [12]), and (upper and lower) estimates have been given. However, most of the techniques used there are specifically Hilbertian (in particular Weyl's inequality; see [10]). Here, we consider the case of composition operators on  $H^p$  for  $1 \leq p < \infty$ . We focus essentially on lower estimates, because the upper ones are similar, with similar proofs, as in the Hilbertian case. We give in Theorem 2.4 a minoration involving the uniform separation constant of finite sequences in the unit disk and the interpolation constant of their images by the symbol. We finish with some upper estimates.

### 1.1 Preliminary

Recall that if  $X$  and  $Y$  are two Banach spaces of analytic functions on the unit disk  $\mathbb{D}$ , and  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  is an analytic self-map of  $\mathbb{D}$ , one says that  $\varphi$  induces a *composition operator*  $C_\varphi: X \rightarrow Y$  if  $f \circ \varphi \in Y$  for every  $f \in X$ ;  $\varphi$  is then called the *symbol* of the composition operator. One also says that  $\varphi$  is a symbol for  $X$  and  $Y$  if it induces a composition operator  $C_\varphi: X \rightarrow Y$ .

For every  $a \in \mathbb{D}$ , we denote by  $e_a \in (H^p)^*$  the evaluation map at  $a$ , namely:

$$(1.1) \quad e_a(f) = f(a), \quad f \in H^p.$$

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We know that ([22], p. 253):

$$(1.2) \quad \|e_a\| = \left( \frac{1}{1 - |a|^2} \right)^{1/p}$$

and the mapping equation

$$(1.3) \quad C_\varphi^*(e_a) = e_{\varphi(a)}$$

still holds.

Throughout this section we denote by  $\|\cdot\|$ , without any subscript, the norm in the dual space  $(H^p)^*$ .

Let us stress that this dual norm of  $(H^p)^*$  is, for  $1 < p < \infty$ , equivalent, but not equal, to the norm  $\|\cdot\|_q$  of  $H^q$ , and the equivalence constant tends to infinity when  $p$  goes to 1 or to  $\infty$ .

As usual, the notation  $A \lesssim B$  means that there is a constant  $c$  such that  $A \leq cB$  and  $A \approx B$  means that  $A \lesssim B$  and  $B \lesssim A$ .

## 1.2 Singular numbers

For an operator  $T: X \rightarrow Y$  between Banach spaces  $X$  and  $Y$ , its *approximation numbers* are defined, for  $n \geq 1$ , as:

$$(1.4) \quad a_n(T) = \inf_{\text{rank } R < n} \|T - R\|.$$

One has  $\|T\| = a_1(T) \geq a_2(T) \geq \dots \geq a_n(T) \geq a_{n+1}(T) \geq \dots$ , and (assuming that  $Y$  has the Approximation Property),  $T$  is compact if and only if  $a_n(T) \xrightarrow{n \rightarrow \infty} 0$ .

We will also need other singular numbers (see [2], p. 49).

The *n-th Bernstein number*  $b_n(T)$  of  $T$ , defined as:

$$(1.5) \quad b_n(T) = \sup_{\substack{E \subset X \\ \dim E = n}} \inf_{x \in S_E} \|Tx\|,$$

where  $S_E = \{x \in E; \|x\| = 1\}$  is the unit sphere of  $E$ . When these numbers tend to 0,  $T$  is said to be superstrictly singular, or finitely strictly singular (see [17]).

The *n-th Gelfand number* of  $T$ , defined as:

$$(1.6) \quad c_n(T) = \inf_{\substack{L \subset Y \\ \text{codim } L < n}} \|T|_L\|,$$

One always has:

$$(1.7) \quad a_n(T) \geq c_n(T) \quad \text{and} \quad a_n(T) \geq b_n(T),$$

and, when  $X$  and  $Y$  are Hilbert spaces, one has  $a_n(T) = b_n(T) = c_n(T)$  ([16], Theorem 2.1).

## 2 Lower bounds

### 2.1 Sub-geometrical decay

We first show that, as in the Hilbertian case  $H^2$  ([10], Theorem 3.1), the approximation numbers of the composition operators on  $H^p$  cannot decrease faster than geometrically.

Though we cannot longer appeal to the Hilbertian techniques of [10], Weyl's inequality has the following generalization ([3], Proposition 2).

**Proposition 2.1 (Carl-Triebel)** *Let  $T$  be a compact operator on a complex Banach space  $E$  and  $(\lambda_n(T))_{n \geq 1}$  be the sequence of its eigenvalues, indexed such that  $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \dots$ . Then, for  $n = 1, 2, \dots$  and  $m = 0, 1, \dots, n-1$ , one has:*

$$(2.1) \quad \prod_{j=1}^n |\lambda_j(T)| \leq 16^n \|T\|^m a_{m+1}(T)^{n-m}.$$

(see [1] for an optimal result). Then, we can state:

**Theorem 2.2** *For every non-constant analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ , there exist  $0 < r \leq 1$  and  $c > 0$ , depending only on  $\varphi$ , such that the approximation numbers of the composition operator  $C_\varphi: H^p \rightarrow H^p$  satisfy:*

$$a_n(C_\varphi) \geq c r^n, \quad n = 1, 2, \dots$$

*In particular  $\liminf_{n \rightarrow \infty} [a_n(C_\varphi)]^{1/n} \geq r > 0$ .*

**Proof.** If  $C_\varphi$  is not compact, the result is trivial, with  $r = 1$ ; so we assume that  $C_\varphi$  is compact.

Before carrying on, we first recall some notation used in [10]. For every  $z \in \mathbb{D}$ , let

$$\varphi^\sharp(z) = \frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}$$

be the pseudo-hyperbolic derivative of  $\varphi$  at  $z$ , and

$$[\varphi] = \sup_{z \in \mathbb{D}} \varphi^\sharp(z).$$

By the Schwarz-Pick inequality, one has  $[\varphi] \leq 1$ . Moreover, since  $\varphi$  is not constant, one has  $[\varphi] > 0$ .

We also set, for every operator  $T: H^p \rightarrow H^p$ :

$$\beta^-(T) = \liminf_{n \rightarrow \infty} [a_n(T)]^{1/n}.$$

For every  $a \in \mathbb{D}$ , we are going to show that  $\beta^-(C_\varphi) \geq (\varphi^\sharp(a))^2$ , which will give  $\beta^-(C_\varphi) \geq [\varphi]^2$ , by taking the supremum for  $a \in \mathbb{D}$ , and the stated result, with  $0 < r < [\varphi]^2$ .

If  $\varphi^\sharp(a) = 0$ , the result is obvious, so we assume that  $\varphi^\sharp(a) > 0$ .

We consider the automorphism  $\Phi_a$ , defined by  $\Phi_a(z) = \frac{a-z}{1-\bar{a}z}$ , and set

$$\psi_a = \Phi_{\varphi(a)} \circ \varphi \circ \Phi_a.$$

One has  $\psi_a(0) = 0$  and  $|\psi'_a(0)| = \varphi^\sharp(a)$ .

Since  $C_\varphi$  is compact on  $H^p$ ,  $C_{\psi_a} = C_{\Phi_a} \circ C_\varphi \circ C_{\Phi_{\varphi(a)}}$  is also compact on  $H^p$ . But we know that this is equivalent to say that it is compact on  $H^2$ . Since  $\psi_a(0) = 0$  and  $\psi'_a(0) = \varphi^\sharp(a) \neq 0$ , we know, by the Eigenfunction Theorem ([19], p. 94), that the eigenvalues of  $C_{\psi_a}: H^2 \rightarrow H^2$  are the numbers  $(\psi'_a(0))^j$ ,  $j = 0, 1, \dots$ , and have multiplicity one. Moreover, the proof given in [19], § 6.2 shows that the eigenfunctions  $\sigma^j$  are not only in  $H^2$ , but in all  $H^q$ ,  $1 \leq q < \infty$ . Hence  $\lambda_j(C_{\psi_a}) = (\psi'_a(0))^{j-1}$ . We now use Proposition 2.1, with  $2n$  instead of  $n$  and  $m = n - 1$ ; we get:

$$\begin{aligned} |\psi'_a(0)|^{n(2n-1)} &= \prod_{j=1}^{2n} |\lambda_j(C_{\psi_a})| \leq 16^{2n} \|C_{\psi_a}\|^{n-1} a_n(C_{\psi_a})^{n+1} \\ &\leq 16^{2n} \|C_{\psi_a}\|^n a_n(C_{\psi_a})^n, \end{aligned}$$

since  $a_n(C_{\psi_a}) \leq \|C_{\psi_a}\|$ .

That implies that  $\beta^-(C_{\psi_a}) \geq |\psi'_a(0)|^2 = (\varphi^\sharp(a))^2$ .

Since  $C_{\Phi_a}$  and  $C_{\Phi_{\varphi(a)}}$  are automorphisms, we have  $\beta^-(C_\varphi) = \beta^-(C_{\psi_a})$ , hence the result.  $\square$

## 2.2 Main result

In this section, we use the fortunate fact that, though the evaluation maps at well-chosen points of  $\mathbb{D}$  can no longer be said to constitute a Riesz sequence, they will still constitute an unconditional sequence in  $H^p$  with good constants, as we are going to see, which will be sufficient for our purposes.

Recall (see [5], p. 276) that the *interpolation constant*  $\kappa_\sigma$  of a finite sequence  $\sigma = (z_1, \dots, z_n)$  of points  $z_1, \dots, z_n \in \mathbb{D}$  is defined by:

$$(2.2) \quad \kappa_\sigma = \sup_{|a_1|, \dots, |a_n| \leq 1} \inf\{\|f\|_\infty; f \in H^\infty \text{ and } f(z_j) = a_j, 1 \leq j \leq n\}.$$

Then:

**Lemma 2.3** *For every finite sequence  $\sigma = (z_1, \dots, z_n)$  of distinct points  $z_1, \dots, z_n \in \mathbb{D}$ , one has:*

$$(2.3) \quad \kappa_\sigma^{-1} \left\| \sum_{j=1}^n \lambda_j e_{z_j} \right\| \leq \left\| \sum_{j=1}^n \omega_j \lambda_j e_{z_j} \right\| \leq \kappa_\sigma \left\| \sum_{j=1}^n \lambda_j e_{z_j} \right\|$$

for all  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  and all complex numbers  $\omega_1, \dots, \omega_n$  such that  $|\omega_1| = \dots = |\omega_n| = 1$ .

**Proof.** Set  $L = \sum_{j=1}^n \lambda_j e_{z_j}$  and  $L_\omega = \sum_{j=1}^n \omega_j \lambda_j e_{z_j}$ . There exists  $h \in H^\infty$  such that  $\|h\|_\infty \leq \kappa_\sigma$  and  $h(z_j) = \omega_j$  for every  $j = 1, \dots, n$ . For every  $g \in H^p$ , one has  $L_\omega(g) = \sum_{j=1}^n \omega_j \lambda_j g(z_j) = \sum_{j=1}^n h(z_j) \lambda_j g(z_j) = L(hg)$ ; hence:

$$|L_\omega(g)| \leq \|L\| \|hg\|_p \leq \|L\| \|h\|_\infty \|g\|_p \leq \kappa_\sigma \|L\| \|g\|_p$$

and we get  $\|L_\omega\| \leq \kappa_\sigma \|L\|$ , which is the right-hand side of (2.3). The left-hand side follows, by replacing  $\lambda_1, \dots, \lambda_n$  by  $\overline{\omega_1} \lambda_1, \dots, \overline{\omega_n} \lambda_n$ .  $\square$

We now prove the following lower estimate.

**Theorem 2.4** *Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  and  $C_\varphi: H^p \rightarrow H^p$ , with  $1 \leq p < \infty$ . Let  $u_1, \dots, u_n \in \mathbb{D}$  such that  $v_1 = \varphi(u_1), \dots, v_n = \varphi(u_n)$  are distinct. Then, for some constant  $c_p$  depending only on  $p$ , we have:*

$$(2.4) \quad a_n(C_\varphi) \geq c_p \kappa_v^{-1} \left(1 + \log \frac{1}{\delta_u}\right)^{-1/\min(p,2)} \inf_{1 \leq j \leq n} \left(\frac{1 - |u_j|^2}{1 - |v_j|^2}\right)^{1/p},$$

where  $\delta_u$  is the uniform separation constant of the sequence  $u = (u_1, \dots, u_n)$  and  $\kappa_v$  the interpolation constant of  $v = (v_1, \dots, v_n)$ .

For the proof, we need to know some precisions on the constant in Carleson's embedding theorem. Recall that the *uniform separation constant*  $\delta_\sigma$  of a finite sequence  $\sigma = (z_1, \dots, z_n)$  in the unit disk  $\mathbb{D}$ , is defined by:

$$(2.5) \quad \delta_\sigma = \inf_{1 \leq j \leq n} \prod_{k \neq j} \left| \frac{z_j - z_k}{1 - \overline{z_j} z_k} \right|.$$

**Lemma 2.5** *Let  $\sigma = (z_1, \dots, z_n)$  be a finite sequence of distinct points in  $\mathbb{D}$  with uniform separation constant  $\delta_\sigma$ . Then:*

$$(2.6) \quad \sum_{j=1}^n (1 - |z_j|^2) |f(z_j)|^p \leq 12 \left[1 + \log \frac{1}{\delta_\sigma}\right] \|f\|_p^p$$

for all  $f \in H^p$ .

**Proof.** For  $a \in \mathbb{D}$ , let  $k_a(z) = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}$  be the normalized reproducing kernel. For every positive Borel measure  $\mu$  on  $\mathbb{D}$ , let:

$$\gamma_\mu = \sup_{a \in \text{supp } \mu} \int_{\mathbb{D}} |k_a(z)|^2 d\mu(z).$$

The so-called Reproducing Kernel Thesis (see [14], Lecture VII, pp. 151–158) says that there is an absolute positive constant  $A_1$  such that:

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq A_1 \gamma_\mu \|f\|_p^p$$

for every  $f \in H^p$  (that follows from the case  $p = 2$  in writing  $f = Bh^{2/p}$  where  $B$  is a Blaschke product and  $h \in H^2$ ). Actually, one can take  $A_1 = 2e$  (see [15],

Theorem 0.2). But when  $\mu$  is the discrete measure  $\sum_{j=1}^n (1 - |z_j|^2) \delta_{z_j}$ , it is not difficult to check (see [4], Lemma 1, p. 150, or [6], p. 201) that:

$$\gamma_\mu \leq 1 + 2 \log \frac{1}{\delta_\sigma}.$$

That gives the result since  $4e \leq 12$ .  $\square$

**Proof of Theorem 2.4.** We will actually work with the Bernstein numbers of  $C_\varphi^*$ . Recall that they are defined in (1.5). That will suffice since  $a_n(C_\varphi) \geq a_n(C_\varphi^*)$  (one has equality if  $C_\varphi$  is compact: see [7] or [2], pp. 89–91) and  $a_n(C_\varphi^*) \geq b_n(C_\varphi^*)$ .

Take  $u_1, \dots, u_n \in \mathbb{D}$  such that  $v_1 = \varphi(u_1), \dots, v_n = \varphi(u_n)$  are distinct. The points  $u_1, \dots, u_n$  are then also distinct and the subspace  $E = \text{span}\{e_{u_1}, \dots, e_{u_n}\}$  of  $(H^p)^*$  is  $n$ -dimensional. Let

$$L = \sum_{j=1}^n \lambda_j e_{u_j}$$

be in the unit sphere of  $E$ . We set, for  $f \in H^p$  and for  $j = 1, \dots, n$ :

$$\Lambda_j = \lambda_j \|e_{u_j}\|, \quad \text{and} \quad F_j = \|e_{u_j}\|^{-1} f(u_j),$$

and finally:

$$\Lambda = (\Lambda_1, \dots, \Lambda_n) \quad \text{and} \quad F = (F_1, \dots, F_n).$$

We will separate three cases.

**Case 1:**  $1 < p \leq 2$ .

One has  $\|C_\varphi^*(L)\| = \left\| \sum_{j=1}^n \lambda_j e_{v_j} \right\|$ . Using Lemma 2.3, we obtain for any choice of complex signs  $\omega_1, \dots, \omega_n$ :

$$(2.7) \quad \|C_\varphi^*(L)\| \geq \kappa_v^{-1} \left\| \sum_{j=1}^n \omega_j \lambda_j e_{v_j} \right\|.$$

Let now  $q$  be the conjugate exponent of  $p$ . We know that the space  $H^p$  is of type  $p$  as a subspace of  $L^p$  ([9], p. 169) and therefore its dual  $(H^p)^*$  is of cotype  $q$  ([9], p. 165), with cotype constant  $\leq \tau_p$ , the type  $p$  constant of  $L^p$  (let us note that we might use that  $(H^p)^*$  is isomorphic to the subspace  $H^q$  of  $L^q$ , but we have then to introduce the constant of this isomorphism). Hence, by averaging (2.7) over all independent choices of signs and using the cotype  $q$  property of  $(H^p)^*$ , we get:

$$\|C_\varphi^*(L)\| \geq \tau_p^{-1} \kappa_v^{-1} \left( \sum_{j=1}^n |\lambda_j|^q \|e_{v_j}\|^q \right)^{1/q} \geq \tau_p^{-1} \kappa_v^{-1} \mu_n \left( \sum_{j=1}^n |\lambda_j|^q \|e_{u_j}\|^q \right)^{1/q},$$

so that

$$(2.8) \quad \|C_\varphi^*(L)\| \geq \tau_p^{-1} \kappa_v^{-1} \mu_n \|\Lambda\|_q,$$

where:

$$\mu_n = \inf_{1 \leq j \leq n} \frac{\|e_{v_j}\|}{\|e_{u_j}\|} = \inf_{1 \leq j \leq n} \left( \frac{1 - |u_j|^2}{1 - |v_j|^2} \right)^{1/p}.$$

It remains to give a lower bound for  $\|\Lambda\|_q$ .

But, by Hölder's inequality:

$$|L(f)| = \left| \sum_{j=1}^n \lambda_j f(u_j) \right| = \left| \sum_{j=1}^n \Lambda_j F_j \right| \leq \|\Lambda\|_q \|F\|_p.$$

Since

$$\|F\|_p^p = \sum_{j=1}^n \|e_{u_j}\|^{-p} |f(u_j)|^p = \sum_{j=1}^n (1 - |u_j|^2) |f(u_j)|^p,$$

Lemma 2.5 gives:

$$|L(f)| \leq \|\Lambda\|_q \left[ 12 \left( 1 + \log \frac{1}{\delta_u} \right) \right]^{1/p} \|f\|_p.$$

Taking the supremum over all  $f$  with  $\|f\|_p \leq 1$ , we get, taking into account that  $\|L\| = 1$ :

$$(2.9) \quad \|\Lambda\|_q \geq \left[ 12 \left( 1 + \log \frac{1}{\delta_u} \right) \right]^{-1/p}.$$

By combining (2.8) and (2.9), we get:

$$\|C_\varphi^*(L)\| \geq (12)^{-1/p} \tau_p^{-1} \mu_n \kappa_v^{-1} \left( 1 + \log \frac{1}{\delta_u} \right)^{-1/p}.$$

Therefore:

$$b_n(C_\varphi^*) \geq (12)^{-1/p} \tau_p^{-1} \mu_n \kappa_v^{-1} \left( 1 + \log \frac{1}{\delta_u} \right)^{-1/p}.$$

**Case 2:**  $2 < p < \infty$ .

We follow the same route, but in this case,  $H^p$  is of type 2 and hence  $(H^p)^*$  is of cotype 2. Therefore, we get:

$$(2.10) \quad \|C_\varphi^*(L)\| \geq \tau_2^{-1} \kappa_v^{-1} \mu_n \|\Lambda\|_2$$

and, using Cauchy-Schwarz inequality:

$$(2.11) \quad \|\Lambda\|_2 \geq \left[ 12 \left( 1 + \log \frac{1}{\delta_u} \right) \right]^{-1/2};$$

so:

$$(2.12) \quad \|C_\varphi^*(L)\| \geq (12)^{-1/2} \tau_2^{-1} \mu_n \kappa_v^{-1} \left( 1 + \log \frac{1}{\delta_u} \right)^{-1/2}.$$



**Case 3:**  $p = 1$ .

In this case  $(H^1)^*$  (which is isomorphic to the space  $BMOA$ ) has no finite cotype. But, for each  $k = 1, \dots, n$ , one has, using Lemma 2.3:

$$\begin{aligned} |\lambda_k| \|e_{v_k}\| &= \frac{1}{2} \left\| \left( \sum_{j \neq k} \lambda_j e_{v_j} + \lambda_k e_{v_k} \right) - \left( \sum_{j \neq k} \lambda_j e_{v_j} - \lambda_k e_{v_k} \right) \right\| \\ &\leq \frac{1}{2} \left( \left\| \sum_{j \neq k} \lambda_j e_{v_j} + \lambda_k e_{v_k} \right\| + \left\| \sum_{j \neq k} \lambda_j e_{v_j} - \lambda_k e_{v_k} \right\| \right) \\ &\leq \kappa_v \left\| \sum_{j=1}^n \lambda_j e_{v_j} \right\|; \end{aligned}$$

hence:

$$(2.13) \quad \|C_\varphi^*(L)\| \geq \kappa_v^{-1} \mu_n \|\Lambda\|_\infty.$$

Since  $|L(F)| \leq \|\Lambda\|_\infty \|F\|_1$ , we get, as above, using Lemma 2.5:

$$(2.14) \quad \|\Lambda\|_\infty \geq \left[ 12 \left( 1 + \log \frac{1}{\delta_u} \right) \right]^{-1},$$

and therefore:

$$(2.15) \quad \|C_\varphi^*(L)\| \geq (12)^{-1} \mu_n \kappa_v^{-1} \left( 1 + \log \frac{1}{\delta_u} \right)^{-1}$$

and that finishes the proof of Theorem 2.4.  $\square$

**Example.** We will now apply this result to lens maps. We refer to [19] or [8] for their definition. For  $\theta \in (0, 1)$ , we denote:

$$(2.16) \quad \lambda_\theta(z) = \frac{(1+z)^\theta - (1-z)^\theta}{(1+z)^\theta + (1-z)^\theta}.$$

**Proposition 2.6** *Let  $\lambda_\theta$  be the lens map of parameter  $\theta$  acting on  $HP$ , with  $1 \leq p < \infty$ . Then, for positive constants  $a$  and  $b$ , depending only on  $\theta$  and  $p$ :*

$$a_n(C_{\lambda_\theta}) \geq a e^{-b\sqrt{n}}.$$

Actually, this estimate is valid for polygonal maps as well.

**Proof.** Let  $0 < \sigma < 1$  and consider  $u_j = 1 - \sigma^j$  and  $v_j = \lambda_\theta(u_j)$ ,  $1 \leq j \leq n$ . We know from [10], Lemma 6.4 and Lemma 6.5, that, for  $\alpha = \frac{\pi^2}{2}$  and  $\beta = \beta_\theta = \frac{\pi^2}{2\theta}$ :

$$\delta_u \geq e^{-\alpha/(1-\sigma)} \quad \text{and} \quad \delta_v \geq e^{-\beta/(1-\sigma)}.$$

But we know that the interpolation constant  $\kappa_\sigma$  is related to the uniform separation constant  $\delta_\sigma$  by the following inequality ([5] page 278), in which  $\Lambda$  is a positive numerical constant:

$$(2.17) \quad \frac{1}{\delta_\sigma} \leq \kappa_\sigma \leq \frac{\Lambda}{\delta_\sigma} \left( 1 + \log \frac{1}{\delta_\sigma} \right).$$

Actually, S. A. Vinogradov, E. A. Gorin and S. V. Hruščev [21] (see [13], p. 505) proved that

$$\kappa_\sigma \leq \frac{2e}{\delta_\sigma} \left(1 + 2 \log \frac{1}{\delta_\sigma}\right),$$

so we can take  $\Lambda \leq 4e \leq 12$ .

It follows that

$$(2.18) \quad \kappa_v^{-1} \geq \frac{1-\sigma}{\Lambda(\beta+1)} e^{-\beta/(1-\sigma)}.$$

Setting  $\tilde{p} = \min(p, 2)$ , we have:

$$(2.19) \quad \left(1 + \log \frac{1}{\delta_u}\right)^{-1/\tilde{p}} \geq \left(\frac{1-\sigma}{\alpha+1}\right)^{1/\tilde{p}}.$$

We now estimate  $\mu_n$ .

Since  $\lambda_\theta(0) = 0$ , Schwarz's lemma says that  $|\lambda_\theta(z)| \leq |z|$ ; hence  $\frac{1-|z|^2}{1-|\lambda_\theta(z)|^2} \geq \frac{1-|z|}{1-|\lambda_\theta(z)|}$ . But  $1 - v_j = 1 - \lambda_\theta(u_j) = \frac{2\sigma^{j\theta}}{(2-\sigma^j)^\theta + \sigma^{j\theta}}$ ; hence (since  $u_j$  and  $v_j$  are real):

$$\frac{1-|u_j|^2}{1-|v_j|^2} \geq \frac{1-u_j}{1-v_j} = \frac{\sigma^j}{2\sigma^{j\theta}} [(2-\sigma^j)^\theta + \sigma^{j\theta}].$$

Since the function  $f(x) = (2-x)^\theta + x^\theta$  increases on  $[0, 1]$ , one gets:

$$\frac{1-|u_j|^2}{1-|v_j|^2} \geq \left(\frac{1}{2}\sigma^j\right)^{1-\theta},$$

and therefore:

$$(2.20) \quad \mu_n \geq \left(\frac{1}{2}\sigma^n\right)^{(1-\theta)/p}.$$

Applying now Theorem 2.4 and using (2.18), (2.19) and (2.20), we get:

$$a_n(C_{\lambda_\theta}) \geq \alpha_{p,\theta} e^{-\beta/(1-\sigma)} (1-\sigma)^{1/\tilde{p}} \sigma^{n(1-\theta)/p}$$

with  $\alpha_{p,\theta} = \frac{c_p}{\Lambda(\beta+1)(\alpha+1)^{1/\tilde{p}2(1-\theta)/p}}$ .

Taking  $\sigma = e^{-\varepsilon}$  where  $0 < \varepsilon < 1$ , we get, since  $1 - e^{-\varepsilon} \geq \varepsilon/2$ :

$$a_n(C_{\lambda_\theta}) \geq \alpha_{p,\theta} e^{-2\beta/\varepsilon} \left(\frac{\varepsilon}{2}\right)^{1/\tilde{p}} e^{-\varepsilon n(1-\theta)/p}.$$

Optimizing by taking  $\varepsilon = \sqrt{\frac{3\beta p}{1-\theta}} \frac{1}{\sqrt{n}}$  gives, for  $n$  large enough (in order to have  $\varepsilon < 1$ ):

$$(2.21) \quad a_n(C_{\lambda_\theta}) \geq \alpha'_{p,\theta} n^{-1/(2\tilde{p})} e^{-\beta_{p,\theta}\sqrt{n}}$$

with  $\alpha'_{p,\theta} = \alpha_{p,\theta} \left(\frac{\beta p}{2(1-\theta)}\right)^{1/(2\tilde{p})}$  and  $\beta_{p,\theta} = \sqrt{\frac{2\beta(1-\theta)}{p}}$ .

We get Theorem 2.6, with  $b > \beta_{p,\theta}$ .  $\square$

Let us note that  $\beta_{p,\theta} = \frac{2^{1-\theta}}{\sqrt{p}} \pi \sqrt{\frac{1-\theta}{\theta}}$  tends to 0 when  $\theta$  goes to 1 and tends to infinity when  $\theta$  goes to 0.

### 2.3 A minoration depending on the radial behaviour of $\varphi$

We are using Theorem 2.4 to give, as in [11], Theorem 3.2, a lower bound for  $a_n(C_\varphi)$  which depends on the behaviour of  $\varphi$  near  $\partial\mathbb{D}$ .

We recall first (see [11], Section 3) that an analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  is said to be *real* if it takes real values on  $] -1, 1[$ . If  $\omega: [0, 1] \rightarrow [0, 2]$  is a modulus of continuity (meaning that  $\omega$  is continuous, increasing, sub-additive, vanishing at 0, and concave),  $\varphi$  is said to be an  $\omega$ -*radial symbol* if it is real and:

$$(2.22) \quad 1 - \varphi(r) \leq \omega(1 - r), \quad 0 \leq r < 1.$$

We have the following result.

**Theorem 2.7** *Let  $\varphi$  be an  $\omega$ -radial symbol. Then, for  $1 \leq p < \infty$ , the approximation numbers of the composition operator  $C_\varphi: H^p \rightarrow H^p$  satisfy:*

$$(2.23) \quad a_n(C_\varphi) \geq c'_p \sup_{0 < \sigma < 1} \left[ \left( \frac{\omega^{-1}(a\sigma^n)}{a\sigma^n} \right)^{1/p} (1 - \sigma)^{1/\max(p^*, 2)} \exp\left(-\frac{5}{1 - \sigma}\right) \right],$$

where  $c'_p$  is a constant depending only on  $p$ ,  $p^*$  is the conjugate exponent of  $p$ , and  $a = 1 - \varphi(0) > 0$ .

**Proof.** As in [11], p. 556, we fix  $0 < \sigma < 1$  and define inductively  $u_j \in [0, 1)$  by  $u_0 = 0$  and, using the intermediate value theorem:

$$1 - \varphi(u_{j+1}) = \sigma [1 - \varphi(u_j)], \quad \text{with } 1 > u_{j+1} > u_j.$$

We set  $v_j = \varphi(u_j)$ . We have  $-1 < v_j < 1$  and  $1 - v_n = a\sigma^n$ . We proved in [11], p. 556, that:

$$(2.24) \quad \frac{1 - |u_j|^2}{1 - |v_j|^2} \geq \frac{1}{2} \frac{\omega^{-1}(a\sigma^n)}{a\sigma^n}.$$

Moreover, we proved in [11], p. 557, that the uniform separation constant of  $v = (v_1, \dots, v_n)$  is such that:

$$(2.25) \quad \delta_v \geq \exp\left(-\frac{5}{1 - \sigma}\right).$$

Since  $\delta_u \geq \delta_v$ , we get, from (2.17), that:

$$(2.26) \quad \kappa_u \leq 12 \left( \frac{6 - \sigma}{1 - \sigma} \right) \exp\left(\frac{5}{1 - \sigma}\right) \leq 60 \left( \frac{1}{1 - \sigma} \right) \exp\left(\frac{5}{1 - \sigma}\right).$$

Using now (2.4) of Theorem 2.4 and combining (2.24), (2.25) and (2.26), we get Theorem 2.7.  $\square$

**Example 1: lens maps.** Let us come back to the lens maps  $\lambda_\theta$  for testing Theorem 2.7. We have  $\omega^{-1}(h) \approx h^{1/\theta}$  (see [8], Lemma 2.5) and  $a = 1 - \lambda_\theta(0) = 1$ . Setting  $K = \frac{1}{10\sqrt{p}} \sqrt{\frac{1-\theta}{\theta}}$  and taking, for  $n$  large enough,  $\sigma = 1 - \frac{1}{K\sqrt{n}}$ , we have, using that  $e^{-s} \leq 1 - \frac{4}{5}s$  for  $s > 0$  small enough,  $\sigma^n \geq \exp(-\frac{5}{4K} \sqrt{n})$  and hence:

$$a_n(C_{\lambda_\theta}) \geq c_{\theta,p} n^{-\frac{1}{2 \max(p^*, 2)}} \exp \left[ -\frac{5}{\sqrt{p}} \sqrt{\frac{1-\theta}{\theta}} \sqrt{n} \right].$$

Note that the coefficient of  $\sqrt{n}$  in the exponential is slightly different of that in (2.21), but of the same order.

**Example 2: cusp map.** We refer to [11], Section 4, for its definition and properties. It is the conformal mapping  $\chi$  from  $\mathbb{D}$  onto the domain represented on Fig. 1 such that  $\chi(1) = 1$ ,  $\chi(-1) = 0$ ,  $\chi(i) = (1+i)/2$  and  $\chi(-i) = (1-i)/2$ . We proved in [11], Lemma 4.2, that, for  $0 \leq r < 1$ , one has:

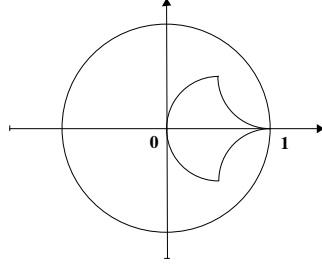


Figure 1: *Cusp map domain*

$$1 - \chi(r) = \frac{1}{1 + \frac{2}{\pi} \log \left[ \frac{1}{2} \arctan \left( \frac{1-r}{1+r} \right) \right]}.$$

Since  $1 - \frac{2}{\pi} \log 2 > 0$  and  $\arctan x \leq x$  for  $x \geq 0$ , we get that:

$$1 - \chi(r) \leq \frac{\pi}{2} \frac{1}{\log \left( \frac{1+r}{1-r} \right)} \leq \frac{\pi}{2} \frac{1}{\log \left( \frac{1}{1-r} \right)} \leq 2 \frac{1}{\log \left( \frac{1}{1-r} \right)}.$$

Hence  $\chi$  is an  $\omega$ -radial symbol with  $\omega(x) = 2/\log(1/x)$ . Then  $\omega^{-1}(h) = e^{-2/h}$ . By choosing  $\sigma = 1 - \frac{\log n}{4n}$  in (2.23), we get, using that  $\log(1-x) \geq -2x$  for  $x > 0$  small enough, that, for  $n$  large enough,  $\sigma^n \geq 1/\sqrt{n}$ ; hence:

$$a_n(C_\chi) \geq c_p'' \left( \sqrt{n} \exp \left[ - (2a) \sqrt{n} \right] \right)^{1/p} \left( \frac{\log n}{n} \right)^{1/\max(p^*, 2)} \exp \left( -\frac{20n}{\log n} \right).$$

It follows that, for some constant  $C_p > 0$  depending only on  $p$ , we have:

$$(2.27) \quad a_n(C_\chi) \geq C_p \exp \left( -\frac{25n}{\log n} \right).$$

It has to be stressed that the term in the exponential does not depend on  $p$ .

**Example 3: Shapiro-Taylor's maps.** These maps  $\varsigma_\theta$ , for  $\theta > 0$ , were defined in [20]. Let us recall their definition. For  $\varepsilon > 0$ , we set  $V_\varepsilon = \{z \in \mathbb{C}; \Re z > 0 \text{ and } |z| < \varepsilon\}$ . For  $\varepsilon = \varepsilon_\theta > 0$  small enough, one can define

$$(2.28) \quad f_\theta(z) = z(-\log z)^\theta,$$

for  $z \in V_\varepsilon$ , where  $\log z$  will be the principal determination of the logarithm. Let now  $g_\theta$  be the conformal mapping from  $\mathbb{D}$  onto  $V_\varepsilon$ , which maps  $\mathbb{T} = \partial\mathbb{D}$  onto  $\partial V_\varepsilon$ , defined by  $g_\theta(z) = \varepsilon \varphi_0(z)$ , where  $\varphi_0$  is the conformal map from  $\mathbb{D}$  onto  $V_1$ , given by:

$$(2.29) \quad \varphi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1}.$$

Then, we define:

$$(2.30) \quad \varsigma_\theta = \exp(-f_\theta \circ g_\theta).$$

We saw in [11], p. 560, that  $\omega^{-1}(h) = K_\theta h(\log(1/h))^{-\theta}$ . Hence, choosing  $\sigma = 1/(e\alpha_\theta^{1/n})$ , where  $\alpha_\theta = 1 - \varsigma_\theta(0)$ , we get that:

$$(2.31) \quad a_n(C_{\varsigma_\theta}) \geq c_{p,\theta} \frac{1}{n^{\theta/2p}}.$$

However, we already remarked in [11], Section 4.2, that, even for  $p = 2$ , this result is not optimal.

### 3 Upper bound

For upper bounds, there is essentially no change with regard to the case  $p = 2$ . Hence we essentially only state some results.

We have the following upper bound, which can be obtained with the same proof as in [8].

**Theorem 3.1** *Let  $C_\varphi: H^p \rightarrow H^p$ ,  $1 \leq p < \infty$ , a composition operator, and  $n \geq 1$ . Then, for every Blaschke product  $B$  with (strictly) less than  $n$  zeros, each counted with its multiplicity, one has:*

$$a_n(C_\varphi) \leq C\sqrt{n} \left( \sup_{\substack{0 < h < 1 \\ \xi \in \mathbb{T}}} \frac{1}{h} \int_{S(\xi,h)} |B|^p dm_\varphi \right)^{1/p},$$

where  $m_\varphi$  is the pullback measure of  $m$ , the normalized Lebesgue measure on  $\mathbb{T}$ , under  $\varphi$  and  $S(\xi, h) = \mathbb{D} \cap D(\xi, h)$  is the Carleson window of size  $h$  centered at  $\xi \in \mathbb{T}$ .

**Proof.** We first estimate the Gelfand number  $c_n(C_\varphi)$  by restricting to the subspace  $BH^p$  which is of codimension  $< n$ . As in [8], Lemma 2.4:

$$c_n(C_\varphi) \lesssim \left( \sup_{\substack{0 < h < 1 \\ \xi \in \mathbb{T}}} \frac{1}{h} \int_{S(\xi, h)} |B|^p dm_\varphi \right)^{1/p}.$$

Now (see [2], Proposition 2.4.3), one has  $a_n(C_\varphi) \leq \sqrt{2n} c_n(C_\varphi)$ , hence the result.  $\square$

We can then deduce, with the same proof, the following version of [11], Theorem 2.3.

Recall ([11], Definition 2.2) that a symbol  $\varphi \in A(\mathbb{D})$  (i.e.  $\varphi: \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  is continuous and analytic in  $\mathbb{D}$ ) is said to be *globally regular* if  $\varphi(\overline{\mathbb{D}}) \cap \partial\mathbb{D} = \{\xi_1, \dots, \xi_l\}$  and there exists a modulus of continuity  $\omega$  (i.e. a continuous, increasing and sub-additive function  $\omega: [0, A] \rightarrow \mathbb{R}^+$ , which vanishes at zero, and that we may assume to be concave), such that, writing  $E_{\xi_j} = \{t; \gamma(t) = \xi_j\}$ , one has  $\mathbb{T} = \bigcup_{j=1}^l (E_{\xi_j} + [-r_j, r_j])$  for some  $r_1, \dots, r_l > 0$ , and for some positive constants  $C, c > 0$ :

$$(3.1) \quad |\gamma(t) - \gamma(t_j)| \leq C(1 - |\gamma(t)|)$$

$$(3.2) \quad c \omega(|t - t_j|) \leq |\gamma(t) - \gamma(t_j)|$$

for  $j = 1, \dots, l$ , all  $t_j \in E_{\xi_j}$  with  $|t - t_j| \leq r_j$ .

**Theorem 3.2** *Let  $\varphi$  be a symbol in  $A(\mathbb{D})$  whose image touches  $\partial\mathbb{D}$  exactly at the points  $\xi_1, \dots, \xi_l$  and which is globally-regular. Then there are constants  $\kappa, K, L > 0$ , depending only on  $\varphi$ , such that, for every  $k \geq 1$ :*

$$(3.3) \quad a_k(C_\varphi) \leq K \left[ \frac{\omega^{-1}(\kappa 2^{-N_k})}{\kappa 2^{-N_k}} \right]^{1/p},$$

where  $N_k$  is the largest integer such that  $lNd_N < k$  and  $d_N$  is the integer part of  $\lceil \log \frac{\kappa 2^{-N}}{\omega^{-1}(\kappa 2^{-N})} / \log(\chi^{-p}) \rceil + 1$ , with  $0 < \chi < 1$  an absolute constant.

As a corollary, we get for lens maps  $\lambda_\theta$  (as well as for polygonal maps), in the same way as Theorem 2.4 in [11], p. 550 (recall that then  $\omega(h) \approx h^\theta$ ), the following upper bound.

**Theorem 3.3** *Let  $\varphi = \lambda_\theta$  be the lens map of parameter  $\theta$  acting on  $H^p$ ,  $1 < p < \infty$ . Then, for positive constants  $b$  and  $c$  depending only on  $\theta$  and  $p$ :*

$$a_n(C_{\lambda_\theta}) \leq c e^{-b\sqrt{n}}.$$

For the cusp map, we also have as in [11], Theorem 4.3 (here,  $\omega(h) \approx 1/\log(1/h)$ ).

**Theorem 3.4** *Let  $\varphi = \chi$  be the cusp map. For some positive constants  $b$  and  $c$  depending only on  $p$ , one has:*

$$a_n(C_\chi) \leq c e^{-bn/\log n}.$$

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