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► **To cite this version:**

Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza. Two remarks on composition operators on the Dirichlet space. 2014. hal-00982257v2

**HAL Id: hal-00982257**

**<https://univ-artois.hal.science/hal-00982257v2>**

Preprint submitted on 10 Jul 2014

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# Two remarks on composition operators on the Dirichlet space

*Daniel Li, Hervé Queffélec,  
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July 10, 2014

**Abstract.** *We show that the decay of approximation numbers of compact composition operators on the Dirichlet space  $\mathcal{D}$  can be as slow as we wish. We also prove the optimality of a result of O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi on boundedness on  $\mathcal{D}$  of self-maps of the disk all of whose powers are norm-bounded in  $\mathcal{D}$ .*

**Mathematics Subject Classification.** Primary: 47B33 – Secondary: 46E22; 47B06 ; 47B32

**Key-words.** approximation numbers – Carleson embedding – composition operator – cusp map – Dirichlet space

## 1 Introduction

Recall that if  $\varphi$  is an analytic self-map of  $\mathbb{D}$ , a so-called *Schur function*, the composition operator  $C_\varphi$  associated to  $\varphi$  is formally defined by

$$C_\varphi(f) = f \circ \varphi.$$

The Littlewood subordination principle ([4], p. 30) tells us that  $C_\varphi$  maps the Hardy space  $H^2$  to itself for every Schur function  $\varphi$ . Also recall that if  $H$  is a Hilbert space and  $T: H \rightarrow H$  a bounded linear operator, the  $n$ -th approximation number  $a_n(T)$  of  $T$  is defined as

$$(1.1) \quad a_n(T) = \inf\{\|T - R\|; \text{rank } R < n\}, \quad n = 1, 2, \dots$$

In [12], working on that Hardy space  $H^2$  (and also on some weighted Bergman spaces), we have undertaken the study of approximation numbers  $a_n(C_\varphi)$  of composition operators  $C_\varphi$ , and proved among other facts the following:

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\*Supported by a Spanish research project MTM 2012-05622.

**Theorem 1.1** *Let  $(\varepsilon_n)_{n \geq 1}$  be a non-increasing sequence of positive numbers tending to 0. Then, there exists a compact composition operator  $C_\varphi$  on  $H^2$  such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

*As a consequence, there are composition operators on  $H^2$  which are compact but in no Schatten class.*

The last item had been previously proved by Carroll and Cowen ([3]), the above statement with approximation numbers being more precise.

For the Dirichlet space, the situation is more delicate because not every analytic self-map of  $\mathbb{D}$  generates a bounded composition operator on  $\mathcal{D}$ . When this is the case, we will say that  $\varphi$  is a *symbol* (understanding “of  $\mathcal{D}$ ”). Note that every symbol is necessarily in  $\mathcal{D}$ .

In [11], we have performed a similar study on that Dirichlet space  $\mathcal{D}$ , and established several results on approximation numbers in that new setting, in particular the existence of symbols  $\varphi$  for which  $C_\varphi$  is compact without being in any Schatten class  $S_p$ . But we have not been able in [11] to prove a full analogue of Theorem 1.1. Using a new approach, essentially based on Carleson embeddings and the Schur test, we are now able to prove that analogue.

**Theorem 1.2** *For every sequence  $(\varepsilon_n)_{n \geq 1}$  of positive numbers tending to 0, there exists a compact composition operator  $C_\varphi$  on the Dirichlet space  $\mathcal{D}$  such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

Turning now to the question of necessary or sufficient conditions for a Schur function  $\varphi$  to be a symbol, we can observe that, since  $(z^n/\sqrt{n})_{n \geq 1}$  is an orthonormal sequence in  $\mathcal{D}$  and since formally  $C_\varphi(z^n) = \varphi^n$ , a necessary condition is as follows:

$$(1.2) \quad \varphi \text{ is a symbol} \implies \|\varphi^n\|_{\mathcal{D}} = O(\sqrt{n}).$$

It is worth noting that, for any Schur function, one has:

$$\varphi \in \mathcal{D} \implies \|\varphi^n\|_{\mathcal{D}} = O(n)$$

(of course, this is an equivalence). Indeed, anticipating on the next section, we have for any integer  $n \geq 1$ :

$$\begin{aligned} \|\varphi^n\|_{\mathcal{D}}^2 &= |\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2 |\varphi(z)|^{2(n-1)} |\varphi'(z)|^2 dA(z) \\ &\leq |\varphi(0)|^2 + \int_{\mathbb{D}} n^2 |\varphi'(z)|^2 dA(z) \leq n^2 \|\varphi\|_{\mathcal{D}}^2, \end{aligned}$$

giving the result.

Now, the following sufficient condition was given in [5]:

$$(1.3) \quad \|\varphi^n\|_{\mathcal{D}} = O(1) \implies \varphi \text{ is a symbol.}$$

In view of (1.2), one might think of improving this condition, but it turns out to be optimal, as says the second main result of that paper.

**Theorem 1.3** *Let  $(M_n)_{n \geq 1}$  be an arbitrary sequence of positive numbers tending to  $\infty$ . Then, there exists a Schur function  $\varphi \in \mathcal{D}$  such that:*

- 1)  $\|\varphi^n\|_{\mathcal{D}} = O(M_n)$  as  $n \rightarrow \infty$ ;
- 2)  $\varphi$  is not a symbol on  $\mathcal{D}$ .

The organization of that paper will be as follows: in Section 2, we give the notation and background. In Section 3, we prove Theorem 1.2; in Section 3.1, we prove Theorem 1.3; and we end with a section of remarks and questions.

## 2 Notation and background.

We denote by  $\mathbb{D}$  the open unit disk of the complex plane and by  $A$  the normalized area measure  $dx dy/\pi$  of  $\mathbb{D}$ . The unit circle is denoted by  $\mathbb{T} = \partial\mathbb{D}$ . The notation  $A \lesssim B$  indicates that  $A \leq cB$  for some positive constant  $c$ .

A Schur function is an analytic self-map of  $\mathbb{D}$  and the associated composition operator is defined, formally, by  $C_\varphi(f) = f \circ \varphi$ . The operator  $C_\varphi$  maps the space  $\mathcal{H}ol(\mathbb{D})$  of holomorphic functions on  $\mathbb{D}$  into itself.

The Dirichlet space  $\mathcal{D}$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that

$$(2.1) \quad \|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty.$$

If  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ , one has:

$$(2.2) \quad \|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n |c_n|^2.$$

Then  $\|\cdot\|_{\mathcal{D}}$  is a norm on  $\mathcal{D}$ , making  $\mathcal{D}$  a Hilbert space, and  $\|\cdot\|_{H^2} \leq \|\cdot\|_{\mathcal{D}}$ . For further information on the Dirichlet space, the reader may see [1] or [16].

The Bergman space  $\mathfrak{B}$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that:

$$\|f\|_{\mathfrak{B}}^2 := \int_{\mathbb{D}} |f(z)|^2 dA(z) < +\infty.$$

If  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ , one has  $\|f\|_{\mathfrak{B}}^2 = \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}$ . If  $f \in \mathcal{D}$ , one has by definition:

$$\|f\|_{\mathcal{D}}^2 = \|f'\|_{\mathfrak{B}}^2 + |f(0)|^2.$$

Recall that, whereas every Schur function  $\varphi$  generates a bounded composition operator  $C_\varphi$  on Hardy and Bergman spaces, it is no longer the case for the Dirichlet space (see [14], Proposition 3.12, for instance).

We denote by  $b_n(T)$  the  $n$ -th *Bernstein number* of the operator  $T: H \rightarrow H$ , namely:

$$(2.3) \quad b_n(T) = \sup_{\dim E=n} \left( \inf_{f \in S_E} \|Tx\| \right)$$

where  $S_E$  denotes the unit sphere of  $E$ . It is easy to see ([11]) that

$$b_n(T) = a_n(T) \quad \text{for all } n \geq 1.$$

(recall that the approximation numbers are defined in (1.1)).

If  $\varphi$  is a Schur function, let

$$(2.4) \quad n_\varphi(w) = \#\{z \in \mathbb{D}; \varphi(z) = w\} \geq 0$$

be the associated *counting function*. If  $f \in \mathcal{D}$  and  $g = f \circ \varphi$ , the change of variable formula provides us with the useful following equation ([17], [11]):

$$(2.5) \quad \int_{\mathbb{D}} |g'(z)|^2 dA(z) = \int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) dA(w)$$

(the integrals might be infinite). In those terms, a necessary and sufficient condition for  $\varphi$  to be a symbol is as follows ([17], Theorem 1). Let:

$$(2.6) \quad \rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} \int_{S(\xi, h)} n_\varphi dA$$

where  $S(\xi, h) = \mathbb{D} \cap D(\xi, h)$  is the Carleson window centered at  $\xi$  and of size  $h$ . Then  $\varphi$  is a symbol if and only if:

$$(2.7) \quad \sup_{0 < h < 1} \frac{1}{h^2} \rho_\varphi(h) < \infty.$$

This is not difficult to prove. In view of (2.5), the boundedness of  $C_\varphi$  amounts to the existence of a constant  $C$  such that:

$$\int_{\mathbb{D}} |f'(w)|^2 n_\varphi(w) dA(w) \leq C \int_{\mathbb{D}} |f'(z)|^2 dA(z), \quad \forall f \in \mathcal{D}.$$

Since  $f' = h$  runs over  $\mathfrak{B}$  as  $f$  runs over  $\mathcal{D}$ , and with equal norms, the above condition reads:

$$\int_{\mathbb{D}} |h(w)|^2 n_\varphi(w) dA(w) \leq C \int_{\mathbb{D}} |h(z)|^2 dA(z), \quad \forall h \in \mathfrak{B}.$$

This exactly means that the measure  $n_\varphi dA$  is a Carleson measure for  $\mathfrak{B}$ . Such measures have been characterized in [7] and that characterization gives (2.7).

But this condition is very abstract and difficult to test, and sometimes more “concrete” sufficient conditions are desirable. In [11], we proved that, even if the Schur function extends continuously to  $\overline{\mathbb{D}}$ , no Lipschitz condition of order  $\alpha$ ,  $0 < \alpha < 1$ , on  $\varphi$  is sufficient for ensuring that  $\varphi$  is a symbol. It is worth noting that the limiting case  $\alpha = 1$ , so restrictive it is, guarantees the result.

**Proposition 2.1** *Suppose that the Schur function  $\varphi$  is in the analytic Lipschitz class on the unit disk, i.e. satisfies:*

$$|\varphi(z) - \varphi(w)| \leq C |z - w|, \quad \forall z, w \in \mathbb{D}.$$

Then  $C_\varphi$  is bounded on  $\mathcal{D}$ .

**Proof.** Let  $f \in \mathcal{D}$ ; one has:

$$\begin{aligned} \|C_\varphi(f)\|_{\mathcal{D}}^2 &= |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z) \\ &\leq |f(\varphi(0))|^2 + \|\varphi'\|_{\infty}^2 \int_{\mathbb{D}} |f'(\varphi(z))|^2 dA(z). \end{aligned}$$

This integral is nothing but  $\|C_\varphi(f')\|_{\mathfrak{B}}^2$  and hence, since  $C_\varphi$  is bounded on the Bergman space  $\mathfrak{B}$ , we have, for some constant  $K_1$ :

$$\int_{\mathbb{D}} |f'(\varphi(z))|^2 dA(z) \leq K_1^2 \|f'\|_{\mathfrak{B}}^2 \leq K_1^2 \|f\|_{\mathcal{D}}^2.$$

On the other hand,

$$|f(\varphi(0))| \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{H^2} \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{\mathcal{D}},$$

and we get

$$\|C_\varphi(f)\|_{\mathcal{D}}^2 \leq K^2 \|f\|_{\mathcal{D}}^2,$$

with  $K^2 = K_1^2 + (1 - |\varphi(0)|^2)^{-1}$ . □

### 3 Proof of Theorem 1.2

We are going to prove Theorem 1.2 mentioned in the Introduction, which we recall here.

**Theorem 3.1** *For every sequence  $(\varepsilon_n)$  of positive numbers with limit 0, there exists a compact composition operator  $C_\varphi$  on  $\mathcal{D}$  such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0.$$

Before entering really in the proof, we may remark that, without loss of generality, by replacing  $\varepsilon_n$  with  $\inf(2^{-8}, \sup_{k \geq n} \varepsilon_k)$ , we can, and do, assume that  $(\varepsilon_n)_n$  decreases and  $\varepsilon_1 \leq 2^{-8}$ .

Moreover, we can assume that  $(\varepsilon_n)_n$  decreases “slowly”, as said in the following lemma.

**Lemma 3.2** *Let  $(\varepsilon_i)$  be a decreasing sequence with limit zero and let  $0 < \rho < 1$ . Then, there exists another sequence  $(\widehat{\varepsilon}_i)$ , decreasing with limit zero, such that  $\widehat{\varepsilon}_i \geq \varepsilon_i$  and  $\widehat{\varepsilon}_{i+1} \geq \rho \widehat{\varepsilon}_i$ , for every  $i \geq 1$ .*

**Proof.** We define inductively  $\widehat{\varepsilon}_i$  by  $\widehat{\varepsilon}_1 = \varepsilon_1$  and

$$\widehat{\varepsilon}_{i+1} = \max(\rho \widehat{\varepsilon}_i, \varepsilon_{i+1}).$$

It is seen by induction that  $\widehat{\varepsilon}_i \geq \varepsilon_i$  and that  $\widehat{\varepsilon}_i$  decreases to a limit  $a \geq 0$ . If  $\widehat{\varepsilon}_i = \varepsilon_i$  for infinitely many indices  $i$ , we have  $a = 0$ . In the opposite case,  $\widehat{\varepsilon}_{i+1} = \rho \widehat{\varepsilon}_i$  from some index  $i_0$  onwards, and again  $a = 0$  since  $\rho < 1$ .  $\square$

We will take  $\rho = 1/2$  and assume for the sequel that  $\varepsilon_{i+1} \geq \varepsilon_i/2$ .

**Proof of Theorem 3.1.** We first construct a subdomain  $\Omega = \Omega_\theta$  of  $\mathbb{D}$  defined by a cuspidal inequality:

$$(3.1) \quad \Omega = \{z = x + iy \in \mathbb{D}; |y| < \theta(1-x), 0 < x < 1\},$$

where  $\theta: [0, 1] \rightarrow [0, 1[$  is a continuous increasing function such that

$$(3.2) \quad \theta(0) = 0 \quad \text{and} \quad \theta(1-x) \leq 1-x.$$

Note that since  $1-x \leq \sqrt{1-x^2}$ , the condition  $|y| < \theta(1-x)$  implies that  $z = x + iy \in \mathbb{D}$ . Note also that  $1 \in \overline{\Omega}$  and that  $\Omega$  is a Jordan domain.

We introduce a parameter  $\delta$  with  $\varepsilon_1 \leq \delta \leq 1 - \varepsilon_1$ . We put:

$$(3.3) \quad \theta(\delta^j) = \varepsilon_j \delta^j$$

and we extend  $\theta$  to an increasing continuous function from  $(0, 1)$  into itself (piecewise linearly, or more smoothly, as one wishes). We claim that:

$$(3.4) \quad \theta(h) \leq h \quad \text{and} \quad \theta(h) = o(h) \quad \text{as} \quad h \rightarrow 0.$$

Indeed, if  $\delta^{j+1} \leq h < \delta^j$ , we have  $\theta(h)/h \leq \theta(\delta^j)/\delta^{j+1} = \varepsilon_j/\delta$ , which is  $\leq \varepsilon_1/\delta \leq 1$  and which tends to 0 with  $h$ .

We define now  $\varphi = \varphi_\theta: \overline{\mathbb{D}} \rightarrow \overline{\Omega}$  as a continuous map which is a Riemann map from  $\mathbb{D}$  onto  $\Omega$ , and with  $\varphi(1) = 1$  (a cusp-type map). Since  $\varphi$  is univalent, one has  $n_\varphi = \mathbb{1}_\Omega$ , and since  $\Omega$  is bounded,  $\varphi$  defines a symbol on  $\mathcal{D}$ , by (2.7). Moreover, (3.4) implies that  $A[S(\xi, h) \cap \Omega] \leq h\theta(h)$  for every  $\xi \in \mathbb{T}$ ; hence,  $\rho_\varphi$  being defined in (2.6), one has  $\rho_\varphi(h) = o(h^2)$  as  $h \rightarrow 0^+$ . In view of [17], this little-oh condition guarantees the compactness of  $C_\varphi: \mathcal{D} \rightarrow \mathcal{D}$ .

It remains to minorate its approximation numbers.

The measure  $\mu = n_\varphi dA$  is a Carleson measure for the Bergman space  $\mathfrak{B}$ , and it was proved in [10] that  $C_\varphi^* C_\varphi$  is unitarily equivalent to the Toeplitz operator  $T_\mu = I_\mu^* I_\mu: \mathfrak{B} \rightarrow \mathfrak{B}$  defined by:

$$(3.5) \quad T_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2} dA(w) = \int_{\mathbb{D}} f(w) K_w(z) dA(w),$$

where  $I_\mu: \mathfrak{B} \rightarrow L^2(\mu)$  is the canonical inclusion and  $K_w$  the reproducing kernel of  $\mathfrak{B}$  at  $w$ , i.e.  $K_w(z) = \frac{1}{(1-\bar{w}z)^2}$ .

Actually, we can get rid of the analyticity constraint in considering, instead of  $T_\mu$ , the operator  $S_\mu = I_\mu I_\mu^*: L^2(\mu) \rightarrow L^2(\mu)$ , which corresponds to the arrows:

$$L^2(\mu) \xrightarrow{I_\mu^*} \mathfrak{B} \xrightarrow{I_\mu} L^2(\mu).$$

We use the relation (3.5) which implies:

$$(3.6) \quad a_n(C_\varphi) = a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)}.$$

We set:

$$(3.7) \quad c_j = 1 - 2\delta^j \quad \text{and} \quad r_j = \varepsilon_j \delta^j$$

One has  $r_j = \varepsilon_j(1 - c_j)/2$ .

**Lemma 3.3** *The disks  $\Delta_j = D(c_j, r_j)$ ,  $j \geq 1$ , are disjoint and contained in  $\Omega$ .*

**Proof.** If  $z = x + iy \in \Delta_j$ , then  $1 - x > 1 - c_j - r_j = (1 - c_j)(1 - \varepsilon_j/2) = 2\delta^j(1 - \varepsilon_j/2) \geq \delta^j$  and  $|y| < r_j = \theta(\delta^j)$ ; hence  $|y| < \theta(\delta^j) \leq \theta(1 - x)$  and  $z \in \Omega$ . On the other hand,  $c_{j+1} - c_j = 2(\delta^j - \delta^{j+1}) = 2(1 - \delta)\delta^j \geq 2\varepsilon_1\delta^j \geq 2\varepsilon_j\delta^j = 2r_j > r_j + r_{j+1}$ ; hence  $\Delta_j \cap \Delta_{j+1} = \emptyset$ .  $\square$

We will next need a description of  $S_\mu$ .

**Lemma 3.4** *For every  $g \in L^2(\mu)$  and every  $z \in \mathbb{D}$ :*

$$(3.8) \quad I_\mu^*g(z) = \int_{\Omega} \frac{g(w)}{(1 - \bar{w}z)^2} dA(w)$$

$$(3.9) \quad S_\mu g(z) = \left( \int_{\Omega} \frac{g(w)}{(1 - \bar{w}z)^2} dA(w) \right) \mathbb{1}_{\Omega}(z).$$

**Proof.**  $K_w$  being the reproducing kernel of  $\mathfrak{B}$ , we have for any pair of functions  $f \in \mathfrak{B}$  and  $g \in L^2(\mu)$ :

$$\begin{aligned} \langle I_\mu^*g, f \rangle_{\mathfrak{B}} &= \langle g, I_\mu f \rangle_{L^2(\mu)} = \int_{\Omega} g(w) \overline{f(w)} dA(w) = \int_{\Omega} g(w) \langle K_w, f \rangle_{\mathfrak{B}} dA(w) \\ &= \left\langle \int_{\Omega} g(w) K_w dA(w), f \right\rangle_{\mathfrak{B}}, \end{aligned}$$

so that  $I_\mu^*g = \int_{\Omega} g(w) K_w dA(w)$ , giving the result.  $\square$

In the rest of the proof, we fix a positive integer  $n$  and put:

$$(3.10) \quad f_j = \frac{1}{r_j} \mathbb{1}_{\Delta_j}, \quad j = 1, \dots, n.$$

Let:

$$E = \text{span}(f_1, \dots, f_n).$$



This is an  $n$ -dimensional subspace of  $L^2(\mu)$ .

The  $\Delta_j$ 's being disjoint, the sequence  $(f_1, \dots, f_n)$  is orthonormal in  $L^2(\mu)$ . Indeed, those functions have disjoint supports, so are orthogonal, and:

$$\int f_j^2 d\mu = \int f_j^2 n_\varphi dA = \int_{\Delta_j} \frac{1}{r_j^2} dA = 1.$$

We now estimate from below the Bernstein numbers of  $I_\mu^*$ . To that effect, we compute the scalar products  $m_{i,j} = \langle I_\mu^*(f_i), I_\mu^*(f_j) \rangle$ . One has:

$$\begin{aligned} m_{i,j} &= \langle f_i, S_\mu(f_j) \rangle = \int_{\Omega} f_i(z) \overline{S_\mu f_j(z)} dA(z) \\ &= \iint_{\Omega \times \Omega} \frac{f_i(z) \overline{f_j(w)}}{(1 - w\bar{z})^2} dA(z) dA(w) \\ &= \frac{1}{r_i r_j} \iint_{\Delta_i \times \Delta_j} \frac{1}{(1 - w\bar{z})^2} dA(z) dA(w). \end{aligned}$$

**Lemma 3.5** *We have*

$$(3.11) \quad m_{i,i} \geq \frac{\varepsilon_i^2}{32}, \quad \text{and} \quad |m_{i,j}| \leq \varepsilon_i \varepsilon_j \delta^{j-i} \quad \text{for } i < j.$$

**Proof.** Set  $\varepsilon'_i = \frac{r_i}{1-c_i^2} = \frac{\varepsilon_i}{2(1+c_i)}$ . One has  $\frac{\varepsilon_i}{4} \leq \varepsilon'_i \leq \frac{\varepsilon_i}{2}$ . We observe that (recall that  $A(\Delta_i) = r_i^2$ ):

$$m_{i,i} - \varepsilon_i'^2 = \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \left[ \frac{1}{(1 - w\bar{z})^2} - \frac{1}{(1 - c_i^2)^2} \right] dA(z) dA(w).$$

Therefore, using the fact that, for  $z \in \Delta_i$  and  $w \in \mathbb{D}$ :

$$|1 - w\bar{z}| \geq 1 - |z| \geq 1 - c_i - r_i = 1 - c_i - \varepsilon_i \left( \frac{1 - c_i}{2} \right) \geq (1 - c_i) \left( 1 - \frac{\varepsilon_i}{2} \right) \geq \frac{1 - c_i}{2}$$

and then the mean-value theorem, we get:

$$\begin{aligned} |m_{i,i} - \varepsilon_i'^2| &\leq \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \left| \frac{1}{(1 - w\bar{z})^2} - \frac{1}{(1 - c_i^2)^2} \right| dA(z) dA(w) \\ &\leq \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \frac{32 r_i}{(1 - c_i)^3} dA(z) dA(w) \\ &= \frac{32 r_i^3}{(1 - c_i)^3} \leq 32 \times 8 \varepsilon_i'^3 \leq \frac{\varepsilon_i'^2}{2}, \end{aligned}$$

since  $\varepsilon_i \leq \varepsilon_1 \leq 2^{-8}$  implies that  $\varepsilon'_i \leq 1/(32 \times 16)$ . This gives us the lower bound  $m_{i,i} \geq \varepsilon_i'^2/2 \geq \varepsilon_i^2/32$ .

Next, for  $i < j$ :

$$\begin{aligned} |m_{i,j}| &\leq \frac{1}{r_i r_j} \iint_{\Delta_i \times \Delta_j} \left| \frac{1}{(1-w\bar{z})^2} \right| dA(z) dA(w) \leq \frac{1}{r_i r_j} \frac{4}{(1-c_i)^2} r_i^2 r_j^2 \\ &= \frac{4 \varepsilon_i \varepsilon_j \delta^{i+j}}{4 \delta^{2i}} = \varepsilon_i \varepsilon_j \delta^{j-i}, \end{aligned}$$

and that ends the proof of Lemma 3.5.  $\square$

We further write the  $n \times n$  matrix  $M = (m_{i,j})_{1 \leq i,j \leq n}$  as  $M = D + R$  where  $D$  is the diagonal matrix  $m_i = m_{i,i}$  with  $m_i \geq \frac{\varepsilon_i^2}{32}$ ,  $1 \leq i \leq n$ . Observe that  $M$  is nothing but the matrix of  $S_\mu$  on the orthonormal basis  $(f_1, \dots, f_n)$  of  $E$ , so that we can identify  $M$  and  $S_\mu$  on  $E$ .

Now the following lemma will end the proof of Theorem 3.1.

**Lemma 3.6** *If  $\delta \leq 1/200$ , we have:*

$$(3.12) \quad \|D^{-1}R\| \leq 1/2.$$

Indeed, by the ideal property of Bernstein numbers, Neumann's lemma and the relations:

$$M = D(I + D^{-1}R), \quad \text{and} \quad D = MQ \quad \text{with} \quad \|Q\| \leq 2,$$

we have  $b_n(D) \leq b_n(M) \|Q\| \leq 2 b_n(M)$ , that is:

$$a_n(S_\mu) = b_n(S_\mu) \geq b_n(M) \geq \frac{b_n(D)}{2} = \frac{m_{n,n}}{2} \geq \frac{\varepsilon_n^2}{64},$$

since the  $n$  first approximation numbers of the diagonal matrix  $D$  (the matrices being viewed as well as operators on the Hilbertian space  $\mathbb{C}^n$  with its canonical basis) are  $m_{1,1}, \dots, m_{n,n}$ . It follows that, using (3.6):

$$(3.13) \quad a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)} \geq \frac{\varepsilon_n}{8}.$$

In view of (3.6), we have as well  $a_n(C_\varphi) \geq \varepsilon_n/8$ , and we are done.  $\square$

**Proof of Lemma 3.6.** Write  $M = (m_{i,j}) = D(I + N)$  with  $N = D^{-1}R$ . One has:

$$(3.14) \quad N = (\nu_{i,j}), \quad \text{with} \quad \nu_{i,i} = 0 \quad \text{and} \quad \nu_{i,j} = \frac{m_{i,j}}{m_{i,i}} \text{ for } j \neq i.$$

We shall show that  $\|N\| \leq 1/2$  by using the (unweighted) Schur test, which we recall ([6], Problem 45):

**Proposition 3.7** *Let  $(a_{i,j})_{1 \leq i,j \leq n}$  be a matrix of complex numbers. Suppose that there exist two positive numbers  $\alpha, \beta > 0$  such that:*

1.  $\sum_{j=1}^n |a_{i,j}| \leq \alpha$  for all  $i$ ;
2.  $\sum_{i=1}^n |a_{i,j}| \leq \beta$  for all  $j$ .

*Then, the (Hilbertian) norm of this matrix satisfies  $\|A\| \leq \sqrt{\alpha\beta}$ .*

It is essential for our purpose to note that:

$$(3.15) \quad i < j \implies |\nu_{i,j}| \leq 32 \delta^{j-i},$$

$$(3.16) \quad i > j \implies |\nu_{i,j}| \leq 32 (2\delta)^{i-j}.$$

Indeed, we see from (3.11) and (3.14) that, for  $i < j$ :

$$|\nu_{i,j}| = \frac{|m_{i,j}|}{m_{i,i}} \leq 32 \varepsilon_i \varepsilon_j \varepsilon_i^{-2} \delta^{j-i} \leq 32 \delta^{j-i}$$

since  $\varepsilon_j \leq \varepsilon_i$ . Secondly, using  $\varepsilon_j/\varepsilon_i \leq 2^{i-j}$  for  $i > j$  (recall that we assumed that  $\varepsilon_{k+1} \geq \varepsilon_k/2$ ), as well as  $|m_{i,j}| = |m_{j,i}|$ , we have, for  $i > j$ :

$$|\nu_{i,j}| = \frac{|m_{j,i}|}{m_{i,i}} \leq 32 \frac{\varepsilon_j}{\varepsilon_i} \delta^{i-j} \leq 32 (2\delta)^{i-j}.$$

Now, for fixed  $i$ , (3.15) gives:

$$\begin{aligned} \sum_{j=1}^n |\nu_{i,j}| &= \sum_{j>i} |\nu_{i,j}| + \sum_{j<i} |\nu_{i,j}| \leq 32 \left( \sum_{j>i} \delta^{j-i} + \sum_{j<i} (2\delta)^{i-j} \right) \\ &\leq 32 \left( \frac{\delta}{1-\delta} + \frac{2\delta}{1-2\delta} \right) \leq 32 \frac{3\delta}{1-2\delta} \leq \frac{96}{198} \leq \frac{1}{2}, \end{aligned}$$

since  $\delta \leq 1/200$ . Hence:

$$(3.17) \quad \sup_i \left( \sum_j |\nu_{i,j}| \right) \leq 1/2.$$

In the same manner, but using (3.16) instead of (3.15), one has:

$$(3.18) \quad \sup_j \left( \sum_i |\nu_{i,j}| \right) \leq 1/2.$$

Now, (3.17), (3.18) and the Schur criterion recalled above give:

$$\|N\| \leq \sqrt{1/2 \times 1/2} = 1/2,$$

as claimed.  $\square$

**Remark.** We could reverse the point of view in the preceding proof: start from  $\theta$  and see what lower bound for  $a_n(C_\varphi)$  emerges. For example, if  $\theta(h) \approx h$  as is the case for lens maps (see [11]), we find again that  $a_n(C_\varphi) \geq \delta_0 > 0$  and that  $C_\varphi$  is not compact. But if  $\theta(h) \approx h^{1+\alpha}$  with  $\alpha > 0$ , the method only gives  $a_n(C_\varphi) \gtrsim e^{-\alpha n}$  (which is always true: see [11], Theorem 2.1), whereas the methods of [11] easily give  $a_n(C_\varphi) \gtrsim e^{-\alpha\sqrt{n}}$ . Therefore, this  $\mu$ -method seems to be sharp when we are close to non-compactness, and to be beaten by those of [11] for “strongly compact” composition operators.

### 3.1 Optimality of the EKSJ result

El Fallah, Kellay, Shabankhah and Youssfi proved in [5] the following: if  $\varphi$  is a Schur function such that  $\varphi \in \mathcal{D}$  and  $\|\varphi^p\|_{\mathcal{D}} = O(1)$  as  $p \rightarrow \infty$ , then  $\varphi$  is a symbol on  $\mathcal{D}$ . We have the following theorem, already stated in the Introduction, which shows the optimality of their result.

**Theorem 3.8** *Let  $(M_p)_{p \geq 1}$  be an arbitrary sequence of positive numbers such that  $\lim_{p \rightarrow \infty} M_p = \infty$ . Then, there exists a Schur function  $\varphi \in \mathcal{D}$  such that:*

- 1)  $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$  as  $p \rightarrow \infty$ ;
- 2)  $\varphi$  is not a symbol on  $\mathcal{D}$ .

**Remark.** We first observe that we cannot replace  $\lim$  by  $\limsup$  in Theorem 3.8. Indeed, since  $\varphi \in \mathcal{D}$ , the measure  $\mu = n_\varphi dA$  is finite, and

$$\|\varphi^p\|_{\mathcal{D}}^2 = p^2 \int_{\mathbb{D}} |w|^{2p-2} d\mu(w) \geq c p^2 \left( \int_{\mathbb{D}} |w|^2 d\mu(w) \right)^{p-1} \geq c \delta^p,$$

where  $c$  and  $\delta$  are positive constants.

**Proof of Theorem 3.8.** We may, and do, assume that  $(M_p)$  is non-decreasing and integer-valued. Let  $(l_n)_{n \geq 1}$  be an non-decreasing sequence of positive integers tending to infinity, to be adjusted. Let  $\Omega$  be the subdomain of the right half-plane  $\mathbb{C}_0$  defined as follows. We set:

$$\varepsilon_n = -\log(1 - 2^{-n}) \sim 2^{-n},$$

and we consider the (essentially) disjoint boxes ( $k = 0, 1, \dots$ ):

$$B_{k,n} = B_{0,n} + 2k\pi i,$$

with:

$$B_{0,n} = \{u \in \mathbb{C}; \varepsilon_{n+1} \leq \Re u \leq \varepsilon_n \text{ and } |\Im u| \leq 2^{-n}\pi\},$$

as well as the union

$$T_n = \bigcup_{0 < k < l_n} B_{k,2n},$$

which is a kind of broken tower above the "basis"  $B_{0,2n}$  of even index.

We also consider, for  $1 \leq k \leq l_n - 1$ , very thin vertical pipes  $P_{k,n}$  connecting  $B_{k,2n}$  and  $B_{k-1,2n}$ , of side lengths  $4^{-2n}$  and  $2\pi(1 - 2^{-2n})$  respectively:

$$P_{k,n} = P_{0,n} + 2k\pi i,$$

and we set:

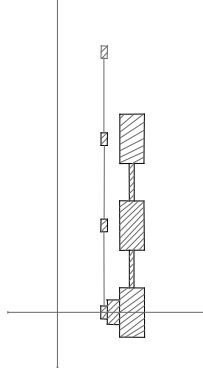
$$P_n = \bigcup_{1 \leq k < l_n} P_{k,n}$$

Finally, we set:

$$F = \left( \bigcup_{n=2}^{\infty} B_{0,n} \right) \cup \left( \bigcup_{n=1}^{\infty} T_n \right) \cup \left( \bigcup_{n=1}^{\infty} P_n \right)$$

and:

$$\Omega = \overset{\circ}{F}$$



Then  $\Omega$  is a simply connected domain. Indeed, it is connected thanks to the  $B_{0,n}$  and the  $P_n$ , since the  $P_{k,n}$  were added to ensure that. Secondly, its unbounded complement is connected as well, since we take one value of  $n$  out of two in the union of sets  $B_{k,n}$  defining  $F$ .

Let now  $f: \mathbb{D} \rightarrow \Omega$  be a Riemann map, and  $\varphi = e^{-f}: \mathbb{D} \rightarrow \mathbb{D}$ .

We introduce the Carleson window  $W = W(1, h)$  defined as:

$$W(1, h) = \{z \in \mathbb{D}; 1 - h \leq |z| < 1 \text{ and } |\arg z| < \pi h\}.$$

This is a variant of the sets  $S(1, h)$  of Section 2. We also introduce the Hastings-Luecking half-windows  $W'_n$  defined by:

$$W'_n = \{z \in \mathbb{D}; 1 - 2^{-n} < |z| < 1 - 2^{-n-1} \text{ and } |\arg z| < \pi 2^{-n}\}.$$

We will also need the sets:

$$E_n = e^{-(T_n \cup B_{0,2n+1} \cup P_n)} = e^{-(B_{0,2n} \cup B_{0,2n+1} \cup P_{0,n})},$$

for which one has:

$$\varphi(\mathbb{D}) \subseteq \bigcup_{n=1}^{\infty} E_n.$$

Next, we consider the measure  $\mu = n_\varphi dA$ , and a Carleson window  $W = W(1, h)$  with  $h = 2^{-2N}$ . We observe that  $W'_{2N} \subseteq W$  and claim that:

**Lemma 3.9** *One has:*

- 1)  $w \in W'_{2N} \implies n_\varphi(w) \geq l_N;$
- 2)  $\|\varphi^p\|_{\mathcal{D}}^2 \lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} e^{-p^4 n^{-n}}.$

**Proof of Lemma 3.9.** 1) Let  $w = r e^{i\theta} \in W'_{2N}$  with  $1 - 2^{-2N} < r < 1 - 2^{-2N-1}$  and  $|\theta| < \pi 2^{-2N}$ . As  $-(\log r + i\theta) \in B_{0,2N}$ , one has  $-(\log r + i\theta) = f(z_0)$  for some  $z_0 \in \mathbb{D}$ . Similarly,  $-(\log r + i\theta) + 2k\pi i$ , for  $1 \leq k < l_N$ , belongs to  $B_{k,2N}$  and can be written as  $f(z_k)$ , with  $z_k \in \mathbb{D}$ . The  $z_k$ 's,  $0 \leq k < l_N$ , are distinct and satisfy  $\varphi(z_k) = e^{-f(z_k)} = e^{-f(z_0)} = w$  for  $0 \leq k < l_N$ , thanks to the  $2\pi i$ -periodicity of the exponential function.

2) We have  $A(E_n) \lesssim e^{-2\varepsilon_{2n+2}} 4^{-2n} \leq 4^{-2n}$  (the term  $e^{-2\varepsilon_{2n+2}}$  coming from the Jacobian of  $e^{-z}$ ) and we observe that

$$w \in E_n \implies |w|^{2p-2} \leq (1 - 2^{-2n-1})^{2p-2} \lesssim e^{-p4^{-n}}.$$

It is easy to see that  $n_\varphi(w) \leq l_n$  for  $w \in E_n$ ; thus we obtain, forgetting the constant term  $|\varphi(0)|^{2p} \leq 1$ , using (2.5) and keeping in mind the fact that  $n_\varphi(w) = 0$  for  $w \notin \varphi(\mathbb{D})$ :

$$\begin{aligned} \|\varphi^p\|_{\mathcal{D}}^2 &= p^2 \int_{\varphi(\mathbb{D})} |w|^{2p-2} n_\varphi(w) dA(w) \\ &\leq p^2 \left( \sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} n_\varphi(w) dA(w) \right) \\ &\leq p^2 \left( \sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2} l_n dA(w) \right) \\ &\lesssim p^2 \sum_{n=1}^{\infty} l_n 16^{-n} e^{-p4^{-n}}, \end{aligned}$$

ending the proof of Lemma 3.9. □

*End of the proof of Theorem 3.8.* Note that, as a consequence of the first part of the proof of Lemma 3.9, one has

$$\mu(W) \geq \mu(W'_{2N}) = \int_{W'_{2N}} n_\varphi dA \geq l_N A(W'_{2N}) \gtrsim l_N h^2,$$

which implies that  $\sup_{0 < h < 1} h^{-2} \mu[W(1, h)] = +\infty$  and shows that  $C_\varphi$  is not bounded on  $\mathcal{D}$  by Zorboska's criterion ([17], Theorem 1), recalled in (2.7).

It remains now to show that we can adjust the non-decreasing sequence of integers  $(l_n)$  so as to have  $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$ . To this effect, we first observe that, if one sets  $F(x) = x^2 e^{-x}$ , we have:

$$p^2 \sum_{n=1}^{\infty} 16^{-n} e^{-p4^{-n}} = \sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim 1.$$

Indeed, let  $s$  be the integer such that  $4^s \leq p < 4^{s+1}$ . We have:

$$\sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim \sum_{n=1}^s \frac{4^n}{p} + \sum_{n>s} F(4^{-(n-s-1)}) \lesssim 1 + \sum_{n=0}^{\infty} F(4^{-n}) < \infty,$$

where we used that  $F$  is increasing on  $(0, 1)$  and satisfies  $F(x) \lesssim \min(x^2, 1/x)$  for  $x > 0$ . We finally choose the non-decreasing sequence  $(l_n)$  of integers as:

$$l_n = \min(n, M_n^2).$$

In view of Lemma 3.9 and of the previous observation, we obtain:

$$\begin{aligned} \|\varphi^p\|_{\mathcal{D}}^2 &\lesssim p^2 \sum_{n=1}^{\infty} 16^{-n} e^{-p4^{-n}} l_n \\ &\leq p^2 \sum_{n=1}^p 16^{-n} e^{-p4^{-n}} l_p + p^2 \sum_{n>p} 16^{-n} l_n \\ &\lesssim l_p + p^2 \sum_{n>p} 4^{-n} \lesssim l_p + p^2 4^{-p} \lesssim M_p^2, \end{aligned}$$

as desired. This choice of  $(l_n)$  gives us an unbounded composition operator on  $\mathcal{D}$  such that  $\|\varphi^p\|_{\mathcal{D}} = O(M_p)$ , which ends the proof of Theorem 3.8.  $\square$

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