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► To cite this version:

Chataur David, Martintxo Saralegi-Aranguren, Tanré Daniel. Steenrod squares on Intersection cohomology and a conjecture of M. Goresky and W. Pardon. Algebraic and Geometric Topology, 2016, 16 (4), pp.1851-1904. 10.2140/agt.2016.16.1851 . hal-00787988

HAL Id: hal-00787988

<https://univ-artois.hal.science/hal-00787988>

Submitted on 9 Jan 2024

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STEENROD SQUARES ON INTERSECTION COHOMOLOGY AND A CONJECTURE OF M. GORESKY AND W. PARDON

DAVID CHATAUR, MARTINTXO SARALEGI-ARANGUREN, AND DANIEL TANRÉ

ABSTRACT. We prove a conjecture raised by M. Goresky and W. Pardon, concerning the range of validity of the perverse degree of Steenrod squares in intersection cohomology. This answer turns out to be of importance for the definition of characteristic classes in the framework of intersection cohomology.

For this purpose, we present a construction of cup_i -products on the cochain complex, built on the blow-up of some singular simplices and introduced in a previous work. We extend to this setting the classical properties of the associated Steenrod squares, including Adem and Cartan relations, for any loose perversities. In the case of a PL-pseudomanifold and range $2\bar{p}$, we prove that our definition coincides with M. Goresky's definition. We show also that our Steenrod squares are topological invariants which do not depend on the choice of a stratification of X .

Several examples of concrete computation of perverse Steenrod squares are given, including the case of isolated singularities and, more especially, we describe the Steenrod squares on the Thom space of a vector bundle, as a function of the Steenrod squares of the base space and the Stiefel-Whitney classes of the bundle. We detail also an example of a non-trivial square, $\text{Sq}^2: H_{\bar{p}} \rightarrow H_{\bar{p}+2}$, whose information is lost if we consider it as taking values in $H_{2\bar{p}}$, showing the interest of the Goresky-Pardon conjecture.

Intersection cohomology was introduced by M. Goresky and R. MacPherson in [11] and [12], in order to adapt Poincaré duality to singular manifolds and extend characteristic classes to this paradigm. Steenrod squares on the intersection cohomology of a pseudomanifold, X , were already defined and studied by M. Goresky in [10]. For that, he uses a sheaf introduced by Deligne and proves that the Steenrod construction of cup_i -products induces a morphism, $\text{Sq}_G^i: H_{\bar{p}}^r(X; \mathbb{F}_2) \rightarrow H_{2\bar{p}}^{r+i}(X; \mathbb{F}_2)$, for any Goresky-MacPherson perversity $\bar{p}$ such that $2\bar{p}(\ell) \leq \ell - 2$ for any ℓ and with $\mathbb{F}_2$ the field with two elements.

Here, we consider the blow-up, $\tilde{N}^*(X)$, of the normalized cochain complex on a filtered version of the singular simplicial set associated to X . This notion of blow-up, defined in [4] and recalled in Section 1, comes from a version adapted to differential forms already existent in [3]. The elements of $\tilde{N}^*(X)$ have a perverse degree (see Definition 1.2) which allows the definition of a complex, $\tilde{N}_{\bar{p}}^*(X)$, for any loose perversity $\bar{p}$. In [4], we have proved that the blow-up, $\tilde{C}^*(X)$, gives the Goresky-MacPherson intersection cohomology of the pseudomanifold X , for the complementary perversity, when we are working over

Date: September 19, 2018.

2000 *Mathematics Subject Classification.* 55N33, 55S10, 57N80.

The third author is partially supported by the MICINN grant MTM2010-18089, ANR-11-BS01-002-01 “HOGT” and ANR-11-LABX-0007-01 “CEMPI”.

a field. With Proposition 1.5, the blow-up $\tilde{N}^*(X)$ inherits this property; we denote its cohomology by $H_{\text{TW},\bullet}^*(X; \mathbb{F}_2)$.

When the coefficients of $\tilde{N}^*(X)$ are in $\mathbb{F}_2$, we define a structure of cup $_i$ -products, $\cup_i: \tilde{N}_{\bar{p}}^*(X) \otimes \tilde{N}_{\bar{q}}^*(X) \rightarrow \tilde{N}_{\bar{p}+\bar{q}}^*(X)$, for any loose perversities $\bar{p}, \bar{q}$. This is done following the work of C. Berger and B. Fresse in [1] (see also [18]): we consider a normalized, homogeneous Bar resolution, $\mathcal{E}(2)$, of the symmetric group Σ_2 and prove that there exists a Σ_2 -equivariant cochain map, $\psi_2: \mathcal{E}(2) \otimes \tilde{N}_{\bar{p}}^*(X) \otimes \tilde{N}_{\bar{q}}^*(X) \rightarrow \tilde{N}_{\bar{p}+\bar{q}}^*(X)$. Such a map is called a *structure of perverse $\mathcal{E}(2)$ -algebra on $\tilde{N}_{\bullet}^*(X)$* ; its construction comes from the existence of a diagonal on $\mathcal{E}(2)$, established in [1]. Moreover, we prove in Theorem A that the cup $_i$ -products arising from the existence of ψ_2 verify the two following properties, $a \cup_{|a|} a = a$ and $a \cup_i a' = 0$, if $i \geq \min(|a|, |a'|)$ where $|a|, |a'|$ are the respective degrees of a and a' .

The definition of perverse $\mathcal{E}(2)$ -algebras can be extended to perverse $\mathcal{E}(n)$ -algebras, for any n . As this work is concerned with Steenrod squares, we consider only perverse $\mathcal{E}(2)$ -algebras over $\mathbb{F}_2$. Nevertheless, it is clear that our methods of proof can be enhanced to give a structure of perverse E_∞ -algebras over $\mathbb{Z}$ on $\tilde{N}_{\bullet}^*(X)$. We will come back on these points in a forthcoming paper.

As usual, Steenrod squares are defined on $H_{\text{TW},\bar{p}}^k(X; \mathbb{F}_2)$ by $\text{Sq}^i(a) = a \cup_{k-i} a$. Using May's presentation of Steenrod squares in [18], we see that the classical properties of Steenrod squares are direct consequences of the structure of perverse $\mathcal{E}(2)$ -algebra. We collect them, together with Adem and Cartan relations, in Theorem B. (One may observe that the proof of the Adem relation on a tensor product needs a brief incursion in the world of perverse $\mathcal{E}(4)$ -algebras over $\mathbb{F}_2$.)

In Theorem B, we also answer positively to the problem asked by M. Goresky in [10, Page 493] and to the conjecture made by M. Goresky and W. Pardon in [13, Conjecture 7.5]. This problem concerns the range of the perversities: with the definition of Steenrod squares via the cup $_i$ -products, it is clear that Sq^i sends $H_{\text{TW},\bar{p}}^k(X; \mathbb{F}_2)$ into $H_{\text{TW},2\bar{p}}^{k+i}(X; \mathbb{F}_2)$. We prove that, in fact, there is a lifting as a map, $\text{Sq}^i: H_{\text{TW},\bar{p}}^k(X; \mathbb{F}_2) \rightarrow H_{\text{TW},\mathcal{L}(\bar{p},i)}^{k+i}(X; \mathbb{F}_2)$, where $\mathcal{L}(\bar{p},i)$ is the loose perversity defined by $\mathcal{L}(\bar{p},i)(\ell) = \min(2\bar{p}(\ell), \bar{p}(\ell) + i)$, which is exactly [13, Conjecture 7.5]. This reveals an important fact because it allows the lifting of Wu classes in intersection cohomology, in a lower part of the poset of perversities.

In Theorem C, we prove that our definition of Steenrod squares coincides with Goresky's definition introduced in [10]. For doing that, we transform the blow-up, $\tilde{N}_{\bullet}^*$, into a sheaf $\mathbf{IN}_{\bullet}^*$ on X and prove that $\mathbf{IN}_{\bullet}^*$ is isomorphic to the Deligne sheaf, in the derived category of sheaves on X . The rest of the proof comes from a unicity theorem for Steenrod squares defined on an injective sheaf, established by M. Goresky, [10].

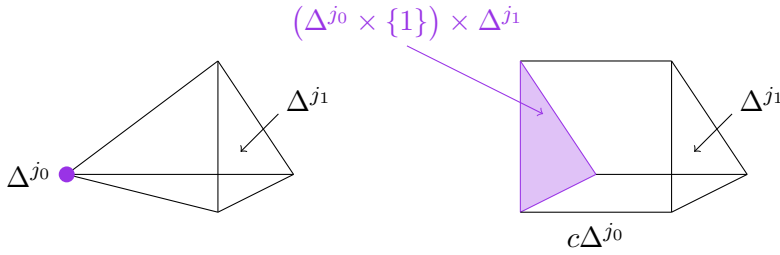
We end this part of the work with examples of concrete computation of perverse Steenrod squares, beginning with the case of isolated singularities. From it, we are able to write the Steenrod squares on the intersection cohomology of the Thom space associated to a vector bundle, as a function of the Steenrod squares of the base space and the Stiefel-Whitney classes of the bundle. We detail also an example of a non-trivial square, $\text{Sq}^2: H_{\text{TW},\bar{p}}^k(X; \mathbb{F}_2) \rightarrow H_{\text{TW},\mathcal{L}(\bar{p},2)}^{k+2}(X; \mathbb{F}_2)$, whose information is lost if we consider

it as values in $H_{\text{TW}, 2\overline{p}}$, showing the interest of the Goresky-Pardon conjecture. This last example can also be seen as a tubular neighborhood of a stratum, which is the first step in the study of intersection cohomology of pseudomanifolds.

In Theorem D, we prove that Steenrod squares, $\text{Sq}^i: H_{\text{TW}, \overline{p}}^r(X; \mathbb{F}_2) \rightarrow H_{\text{TW}, \mathcal{L}(\overline{p}, i)}^{r+i}(X; \mathbb{F}_2)$, are topological invariants when X is a PL-pseudomanifold. This completes the result of [10] that the Steenrod squares are topological invariants, as homomorphisms $H_{\overline{p}}^r(X; \mathbb{F}_2) \rightarrow H_{2\overline{p}}^{r+i}(X; \mathbb{F}_2)$. The proof is combinatorial, using the description of Steenrod squares made by Steenrod in [22].

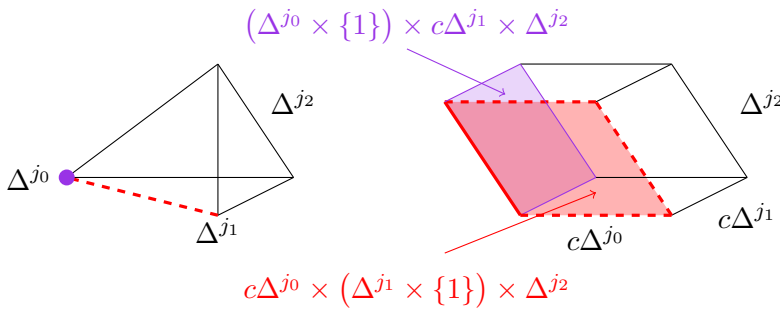
We emphasize now some particularities which are important in the process of the proof of the Goresky-Pardon conjecture. The main point is that our technique allows an explicit construction of the cup _{i} -products at the level of cochain complexes, without requiring the derived category for their definition. In the context of filtered objects, observe first that the notion of filtered singular simplices is a natural one, see Remark 1.7.

The second modus operandi is the blow-up of these simplices. In differential geometry, a blow-up is the replacement of a sub manifold N of a manifold M by the boundary of a tubular neighborhood of N in M . Its simplicial version can be illustrated as follows in the case of $\Delta = \Delta^{j_0} * \Delta^{j_1}$: we cut off a small open neighborhood of Δ^{j_0} in Δ to get $\tilde{\Delta} = c\Delta^{j_0} \times \Delta^{j_1}$. For instance,



$$\Delta = \Delta^{j_0} * \Delta^{j_1} \quad \text{has for blow-up} \quad \tilde{\Delta} = c\Delta^{j_0} \times \Delta^{j_1}.$$

In the general case of $\Delta = \Delta^{j_0} * \dots * \Delta^{j_n}$, we use an inductive process which consists in cutting off a small open neighborhood of the smallest stratum. As an illustration,



$$\Delta = \Delta^{j_0} * \Delta^{j_1} * \Delta^{j_2} \quad \text{has for blow-up} \quad \tilde{\Delta} = c\Delta^{j_0} \times c\Delta^{j_1} \times \Delta^{j_2}.$$

The faces containing $\Delta^{j_i} \times \{1\}$ as a factor, which play a fundamental role in the definition of the perverse degree (see Definition 1.2), have been shadowed in the previous drawings.

The motivation for such process occurs when one determines the intersection cohomology of a pseudomanifold with differential forms : as these forms cannot be defined on the singular strata, the only possibility is to define them on the regular part and ask for some control in the neighborhood of strata. That is what we do here for cochains. As observed in [8], by G. Friedman and J.E. McClure, the classical way for the definition of a cup-product (with back and front faces) does not fit with perverse degrees. But, one advantage of the blow-up is that we can define the cup-product (and more generally the cup_i-products) stratum after stratum, on each factor of the product $c\Delta^{j_0} \times \dots \times c\Delta^{j_{n-1}} \times \Delta^{j_n}$, from the classical definition and in a compatible way with the perverse structure. Finally, this procedure reveals itself of an easy use and does not lose any information in cohomology; it gives the same structure on cohomology as Goresky's definition, as it is established in Section 4.

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In Section 1, we recall basic notions concerning filtered face sets and their intersection cohomology. Section 2 is devoted to the construction of a structure of perverse $\mathcal{E}(2)$ -algebra on the blow-up, $\tilde{N}^*(X)$, which corresponds to the building of cup_i-products. In Section 3, we establish the main properties of perverse Steenrod squares, including the proof of the perverse range conjecture of M. Goresky and W. Pardon. The comparison between our definition and Goresky's definition of Steenrod squares, in the case of a pseudomanifold, is done in Section 4. The particular case of isolated singularities and the treatment of Steenrod squares in the intersection cohomology of a Thom space are presented in Section 5. An example of a square, Sq^2 , in the intersection cohomology of the total space of a fibration whose fiber is a cone is given in Section 6. This example shows the interest of having a range of perversity in $\mathcal{L}(\bar{p}, i)$ instead of $2\bar{p}$. Finally, Section 7 is devoted to the topological invariance of our Steenrod squares.

All the cohomology groups appearing in this text are over the field with two elements, $\mathbb{F}_2$. If there is no ambiguity, we simplify the notation $H^*(X; \mathbb{F}_2)$ in $H^*(X)$.

We thank the anonymous referee for her/his comments and suggestions which have contributed to improve the organization and the writing.

1. BLOW-UP AND PERVERSITY

In this section, we recall the basics of a simplicial version of intersection cohomology, already introduced in [4].

Let Δ^k be the standard simplex of $\mathbb{R}^{k+1}$, whose vertices, $v_0, \dots, v_k$, verify $v_i = (t_0, \dots, t_k)$, with $t_j = 0$ if $j \neq i$ and $t_i = 1$. Let $\delta_i: \{0, 1, \dots, k-1\} \rightarrow \{0, 1, \dots, k\}$ defined by

$$\delta_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i. \end{cases}$$

Such maps generate linear applications, still denoted $\delta_i: \Delta^{k-1} \rightarrow \Delta^k$ and defined by $\delta_i(v_j) = v_{\delta_i(j)}$. More generally, any map $\sigma: \{0, 1, \dots, \ell\} \rightarrow \{0, 1, \dots, k\}$ generates a linear application $\sigma: \Delta^\ell \rightarrow \Delta^k$.

We fix an integer n and consider the category $\Delta_{\mathcal{F}}^{[n]}$ whose

- objects are the joins, $\Delta = \Delta^{j_0} * \Delta^{j_1} * \dots * \Delta^{j_n}$, where Δ^{j_i} is the simplex of dimension j_i , possibly empty, with the conventions $\Delta^{-1} = \emptyset$ and $\emptyset * X = X$,
- maps are the $\sigma: \Delta = \Delta^{j_0} * \Delta^{j_1} * \dots * \Delta^{j_n} \rightarrow \Delta' = \Delta^{k_0} * \Delta^{k_1} * \dots * \Delta^{k_n}$, of the shape $\sigma = *_{i=0}^n \sigma_i$, with $\sigma_i: \{0, 1, \dots, j_i\} \rightarrow \{0, 1, \dots, k_i\}$ an injective order-preserving map for each i .

The category $\Delta_{\mathcal{F}}^{[n],+}$ is the full subcategory of $\Delta_{\mathcal{F}}^{[n]}$ whose objects are the joins $\Delta^{j_0} * \Delta^{j_1} * \dots * \Delta^{j_n}$ with $\Delta^{j_n} \neq \emptyset$, i.e., $j_n \geq 0$. To any such element, we associate its *blow-up* which is the map

$$\mu: \tilde{\Delta} = c\Delta^{j_0} \times \dots \times c\Delta^{j_{n-1}} \times \Delta^{j_n} \rightarrow \Delta = \Delta^{j_0} * \dots * \Delta^{j_n},$$

defined by

$$\begin{aligned} \mu([y_0, s_0], \dots, [y_{n-1}, s_{n-1}], y_n) &= s_0 y_0 + (1 - s_0) s_1 y_1 + \dots \\ &\quad + (1 - s_0) \dots (1 - s_{n-2}) s_{n-1} y_{n-1} \\ &\quad + (1 - s_0) \dots (1 - s_{n-2}) (1 - s_{n-1}) y_n, \end{aligned}$$

where $y_i \in \Delta^{j_i}$ and $[y_i, s_i] \in c\Delta^{j_i} = (\Delta^{j_i} \times [0, 1]) / (\Delta^{j_i} \times \{0\})$. The prism $\tilde{\Delta}$ is sometimes also called the blow-up of Δ .

Observe that this blow-up is well defined thanks to the restriction to the subcategory $\Delta_{\mathcal{F}}^{[n],+}$. In the topological setting (see Remark 1.7) this restriction means that we do not consider simplices entirely included in the singular part.

Definition 1.1. A *filtered face set*, of formal dimension n , is a contravariant functor, $\underline{K}$, from the category $\Delta_{\mathcal{F}}^{[n]}$ to the category of sets, i.e., $(j_0, \dots, j_n) \mapsto \underline{K}_{(j_0, \dots, j_n)}$. The restriction of the filtered face set, $\underline{K}$, to $\Delta_{\mathcal{F}}^{[n],+}$ is denoted $\underline{K}_+$.

If $\underline{K}$ and $\underline{K}'$ are filtered face sets, a *filtered face map*, $\underline{f}: \underline{K} \rightarrow \underline{K}'$, is a natural transformation between the two functors $\underline{K}$ and $\underline{K}'$. We denote by $\Delta_{\mathcal{F}}^{[n]}-\text{Sets}$ the category of filtered face sets.

To any simplicial set, Y , we can associate the $\mathbb{F}_2$ -vector space $C_d(Y)$ generated by the d -dimensional simplices of Y . The normalized chain complex, $N_d(Y)$, is the quotient of

$C_d(Y)$ by the degeneracies $\mathfrak{s}_i$,

$$N_d(Y) = C_d(Y)/\mathfrak{s}_0 C_{d-1}(Y) + \cdots + \mathfrak{s}_{d-1} C_{d-1}(Y).$$

We consider also the duals $N^*(Y) = \text{hom}_{\mathbb{F}_2}(N_*(Y), \mathbb{F}_2)$ and $C^*(Y) = \text{hom}_{\mathbb{F}_2}(C_*(Y), \mathbb{F}_2)$.

Any face operator, $\delta_i: \Delta^{j_\ell} \rightarrow \Delta^{j_\ell+1}$, for some $\ell \in \{0, \dots, n-1\}$, induces a chain map, $\delta_i^*: N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_{\ell+1}}) \otimes \cdots \otimes N^*(\Delta^{j_n}) \rightarrow N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_\ell}) \otimes \cdots \otimes N^*(\Delta^{j_n})$, defined by the identity on the factors in Δ^{j_i} for $i \neq \ell$.

We denote also by $\delta_i: \Delta^{j_0} * \cdots * \Delta^{j_\ell} * \cdots * \Delta^{j_n} \rightarrow \Delta^{j_0} * \cdots * \Delta^{j_{\ell+1}} * \cdots * \Delta^{j_n}$ the operator defined by $\delta_i: \Delta^{j_\ell} \rightarrow \Delta^{j_\ell+1}$ and the identity maps. For any simplex, $\sigma: \Delta^{j_0} * \cdots * \Delta^{j_{\ell+1}} * \cdots * \Delta^{j_n} \rightarrow \underline{K}_+$, we define a simplex, $\partial_i \sigma: \Delta^{j_0} * \cdots * \Delta^{j_\ell} * \cdots * \Delta^{j_n} \rightarrow \underline{K}$, by $\partial_i \sigma = \sigma \circ \delta_i$, and a complex,

$$\tilde{N}_\sigma^* = N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n}).$$

These previous considerations on face operators can easily be adapted to the case $\ell = n$.

A *global section (or cochain)* on $\underline{K}$ is a function which assigns to each simplex $\sigma \in \underline{K}_+$ an element $c_\sigma \in \tilde{N}_\sigma^*$ such that $c_{\partial_i \sigma} = \delta_i^*(c_\sigma)$ for all $\sigma \in \underline{K}_+$ and all $\delta_i \in \Delta_{\mathcal{F}}^{[n],+}$. (The restriction to $\Delta_{\mathcal{F}}^{[n],+}$ implies $\Delta^{j_n} \neq \emptyset$.)

The space of global sections is denoted by $\tilde{N}^*(\underline{K})$ and called *the blow-up* of N^* over the filtered face set $\underline{K}$. Global sections have an extra degree, called the *perverse degree*, that we describe now.

Let $\sigma: \Delta^{j_0} * \cdots * \Delta^{j_n} \rightarrow \underline{K}_+$ and $\ell \in \{1, \dots, n\}$ such that $\Delta^{j_{n-\ell}} \neq \emptyset$. For any cochain $c_\sigma \in N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$, its restriction

$$(1) \quad c_{\sigma, n-\ell} \in N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(\Delta^{j_{n-\ell}} \times \{1\}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$$

can be written $c_{\sigma, n-\ell} = \sum_k c'_{\sigma, n-\ell}(k) \otimes c''_{\sigma, n-\ell}(k)$, with

- $c'_{\sigma, n-\ell}(k) \in N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\Delta^{j_{n-\ell-1}}) \otimes N^*(\Delta^{j_{n-\ell}} \times \{1\})$ and
- $c''_{\sigma, n-\ell}(k) \in N^*(c\Delta^{j_{n-\ell+1}}) \otimes \cdots \otimes N^*(\Delta^{j_n})$.

Observe that each term of the tensor product in Formula (1) has a finite canonical basis and the decomposition of $c_{\sigma, n-\ell}$ can be canonically chosen in function of the associated basis of the tensor product.

Definition 1.2. If $c_{\sigma, n-\ell} \neq 0$, the ℓ -perverse degree, $\|c_\sigma\|_\ell$, of c_σ is equal to

$$\|c_\sigma\|_\ell = \sup_k \{|c''_{\sigma, n-\ell}(k)| \text{ such that } c'_{\sigma, n-\ell}(k) \neq 0\},$$

where $|c''_{\sigma, n-\ell}(k)|$ denotes the usual degree of the cochain $c''_{\sigma, n-\ell}(k)$.

If $c_{\sigma, n-\ell} = 0$ or $\Delta^{j_{n-\ell}} = \emptyset$, we set $\|c_\sigma\|_\ell = -\infty$.

The *perverse degree* of a global section $c \in \tilde{N}^*(\underline{K})$ is the n -tuple

$$\|c\| = (\|c\|_1, \dots, \|c\|_n),$$

where $\|c\|_\ell$ is the supremum of the $\|c_\sigma\|_\ell$ for all $\sigma \in \underline{K}_+$.

Intersection cohomology requires a notion of perversity that we introduce now, following the convention of [15].

Definition 1.3. A *loose perversity* is a map $\bar{p}: \mathbb{N} \rightarrow \mathbb{Z}$, $i \mapsto \bar{p}(i)$, such that $\bar{p}(0) = 0$. A *perversity* is a loose perversity such that $\bar{p}(i) \leq \bar{p}(i+1) \leq \bar{p}(i) + 1$, for all $i \in \mathbb{N}$. A *Goresky-MacPherson perversity* (or *GM-perversity*) is a perversity such that $\bar{p}(1) = \bar{p}(2) = 0$.

If $\bar{p}_1$ and $\bar{p}_2$ are two loose perversities, we set $\bar{p}_1 \leq \bar{p}_2$ if we have $\bar{p}_1(i) \leq \bar{p}_2(i)$, for all $i \in \mathbb{N}$. The poset of all loose perversities is denoted $\mathcal{P}_{\text{loose}}^n$.

The lattice of GM-perversities, denoted $\mathcal{P}^n$, admits a maximal element, $\bar{t}$, called the *top perversity* and defined by $\bar{t}(i) = i - 2$, if $i \geq 2$, $\bar{t}(0) = \bar{t}(1) = 0$.

To these posets, we add an element, ∞ , which is the constant map to ∞ . We call it the *infinite perversity* despite the fact that it is not a perversity in the sense of the previous definition. Finally, we set $\hat{\mathcal{P}}^n = \mathcal{P}^n \cup \{\infty\}$ and $\hat{\mathcal{P}}_{\text{loose}}^n = \mathcal{P}_{\text{loose}}^n \cup \{\infty\}$.

Definition 1.4. Let $\bar{p}$ be a loose perversity. A global section $c \in \tilde{N}^*(\underline{K})$ is $\bar{p}$ -admissible if $\|c\|_i \leq \bar{p}(i)$, for any $i \in \{1, \dots, n\}$. A global section c is of $\bar{p}$ -intersection if c and its differential, δc , are $\bar{p}$ -admissible.

We denote by $\tilde{N}_{\bar{p}}^*(\underline{K})$ the complex of global sections of $\bar{p}$ -intersection and by $H_{\bar{p}}^*(\underline{K}; \tilde{N})$ its homology.

By using the same process with C^* in place of N^* , we obtain a second complex of global sections of $\bar{p}$ -intersection, $\tilde{C}_{\bar{p}}^*(\underline{K})$, of homology $H_{\bar{p}}^*(\underline{K}; \tilde{C})$. Directly from [4, Theorem A], we get an isomorphism between these two cohomologies.

Proposition 1.5. Let $\underline{K}$ be a filtered face set and $\bar{p}$ be a loose perversity. The canonical surjection, $C_*(-) \rightarrow N_*(-)$, induces a quasi-isomorphism, $\tilde{N}_{\bar{p}}^*(\underline{K}) \rightarrow \tilde{C}_{\bar{p}}^*(\underline{K})$, and therefore an isomorphism $H_{\bar{p}}^*(\underline{K}; \tilde{N}) \cong H_{\bar{p}}^*(\underline{K}; \tilde{C})$.

If there is no ambiguity, we denote by $H_{\text{TW}, \bar{p}}^*(\underline{K})$ this common value and called it the *Thom-Whitney cohomology* (henceforth TW-cohomology) of $\underline{K}$ with coefficients in $\mathbb{F}_2$ for the loose perversity $\bar{p}$.

The topological objects corresponding to the filtered face sets are locally conical, stratified topological spaces. We consider here only the case of pseudomanifolds defined as follows.

Definition 1.6. An n -dimensional topological pseudomanifold is a nonempty topological space with a filtration by closed subsets,

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \dots \subseteq X_{n-2} = X_{n-1} \subsetneq X_n = X,$$

such that, for all i , $X_i \setminus X_{i-1}$ is an i -dimensional metrizable topological manifold or the empty set. Moreover, for each point $x \in X_i \setminus X_{i-1}$, $i \neq n$, there exist,

- (a) an open neighborhood, V , of x in X , endowed with the induced filtration,
- (b) an open neighborhood, U , of x in $X_i \setminus X_{i-1}$,
- (c) a compact topological pseudomanifold, $L = (L_j)_{0 \leq j \leq n-i-1}$, of dimension $n - i - 1$, whose cone, $\mathring{c}L = (L \times [0, 1]) / (L \times \{0\})$, is endowed with the conic filtration, i.e., $(\mathring{c}L)_i = \mathring{c}L_{i-1}$, for $i \geq 0$,
- (d) a homeomorphism, $\varphi: U \times \mathring{c}L \rightarrow V$, such that
 - (1) $\varphi(u, \mathbf{v}) = u$, for any $u \in U$, with $\mathbf{v}$ the cone point,
 - (2) $\varphi(U \times \mathring{c}L_j) = V \cap X_{i+j+1}$, for any $j \in \{0, \dots, n - i - 1\}$.

The couple (V, φ) is called a *conic chart* of x and the filtered space, L , the *link* of x .

This definition makes sense with an induction on the dimension, starting from pseudomanifolds of dimension 0 which are discrete topological spaces, by definition. Also, one can prove that the subspace $X_n \setminus X_{n-2}$ is dense.

Remark 1.7. The set of filtered singular simplices is the bridge between pseudomanifolds and the more general notion of filtered face sets. More precisely, for any pseudomanifold, X , we define (see [4, Example 1.5]) the singular filtered face set by

$$\underline{\text{ISing}}^{\mathcal{F}}(X)_{j_0, \dots, j_n} = \{\sigma: \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow X \mid \sigma^{-1}X_i = \Delta^{j_0} * \dots * \Delta^{j_i}\}.$$

Such simplex is called *filtered*.

If X is a pseudomanifold and $\underline{K} = \underline{\text{ISing}}^{\mathcal{F}}(X)$, we use the notations $\tilde{N}_{\bar{p}}^*(X)$, $\tilde{C}_{\bar{p}}^*(X)$ and $H_{\text{TW}, \bar{p}}^*(X)$ for the Thom-Whitney complexes and their cohomology. (As $X_{n-1} = X_{n-2}$, the case $i = 1$ in Definition 1.4, is vacuous in this setting.)

We end this section with a reminder of Goresky-MacPherson cohomology (with coefficients in $\mathbb{F}_2$) and its link with the blow-up. Let $\bar{p}$ be a loose perversity. A filtered simplex, $\sigma: \Delta = \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow X$, has a perverse degree, $\|\sigma\| = (\|\sigma\|_0, \dots, \|\sigma\|_n)$, where $\|\sigma\|_\ell = \dim(\Delta^{j_0} * \dots * \Delta^{j_{n-\ell}})$, with the convention $\|\sigma\|_\ell = -\infty$ if $\sigma^{-1}X_{n-\ell} = \emptyset$.

A $\bar{p}$ -admissible simplex of X is a filtered simplex, $\sigma: \Delta = \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow X$, such that $\|\sigma\|_\ell \leq \dim \Delta - \ell + \bar{p}(\ell)$, for any $\ell \in \{1, \dots, n\}$. A $\bar{p}$ -admissible chain is a linear combination of $\bar{p}$ -admissible simplices. A chain, c , is of $\bar{p}$ -intersection if c and its boundary, ∂c , are $\bar{p}$ -admissible. Denote by $C_*^{\text{GM}, \bar{p}}(X)$, $N_*^{\text{GM}, \bar{p}}(X)$ the complexes of $\bar{p}$ -intersection chains, by $C_{\text{GM}, \bar{p}}^*(X) = \text{hom}(C_*^{\text{GM}, \bar{p}}(X), \mathbb{F}_2)$, $N_{\text{GM}, \bar{p}}^*(X) = \text{hom}(N_*^{\text{GM}, \bar{p}}(X), \mathbb{F}_2)$ their dual and by $H_{\text{GM}, \bar{p}}^*(X) = H(C_{\text{GM}, \bar{p}}^*(X)) = H(N_{\text{GM}, \bar{p}}^*(X))$ their homology, called the Goresky-MacPherson ([11]) intersection cohomology of X (henceforth GM-cohomology) with coefficients in $\mathbb{F}_2$. This cohomology is isomorphic to the original Goresky-MacPherson cohomology in the case of a pseudomanifold, X , and a GM-perversity, $\bar{p}$, see [4, Proposition A.29] and [15].

The GM and TW cohomologies are related in [4, Theorem B] that we recall here.

Proposition 1.8. *Let X be a pseudomanifold, $\bar{p}$ and $\bar{q}$ be two perversities such that $\bar{q} \geq 0$ and $\bar{p}(i) + \bar{q}(i) = i - 2$ for any $i \in \{2, \dots, n\}$. Then there is an isomorphism between the GM and the TW cohomologies, $H_{\text{TW}, \bar{q}}^*(X) \cong H_{\text{GM}, \bar{p}}^*(X)$.*

2. PERVERSE $\mathcal{E}(2)$ -ALGEBRAS AND FILTERED FACE SETS

Steenrod squares are built from an action of a normalized homogeneous Bar resolution, $\mathcal{E}(2)$, of the symmetric group Σ_2 , on the normalized singular cochains. This is the way the non-commutativity of the cup-product is controlled up to higher coherent homotopies. This action enriches the multiplicative structure given by the cup-product. We first review it in order to adapt this construction to the perverse setting.

Recall that the resolution $\mathcal{E}(2)$ of $\mathbb{F}_2$ as Σ_2 -module is defined by

$$\dots \rightarrow \mathcal{E}(2)_i \xrightarrow{d} \mathcal{E}(2)_{i-1} \rightarrow \dots$$

with $\mathcal{E}(2)_i = \mathbb{F}_2(e_i, \tau_i)$, $de_i = d\tau_i = e_{i-1} + \tau_{i-1}$. (As we are using cochain complexes, $\mathcal{E}(2)$ is negatively graded.) From the isomorphism $\Sigma_2 \cong \{e_i, \tau_i\}$ with τ_i the generator

of Σ_2 , the action (on the left) of Σ_2 defines a natural action on $\mathcal{E}(2)$. This action is extended to the tensor product $\mathcal{E}(2) \otimes \mathcal{E}(2)$ as a diagonal action. Moreover, the complex $\mathcal{E}(2)$ is equipped with a Σ_2 -equivariant diagonal, $\mathcal{D}: \mathcal{E}(2) \rightarrow \mathcal{E}(2) \otimes \mathcal{E}(2)$, defined by

$$\mathcal{D}(e_i) = \sum_{j=0}^i e_j \otimes \tau^j . e_{i-j},$$

with $\tau . e_k = \tau_k$, $\tau . \tau_k = e_k$. This diagonal is essential for the definition of the structure of $\mathcal{E}(2)$ -algebra on $\tilde{N}^*(\underline{K})$. Finally, observe that, for any vector space V , there is a Σ_2 -action on $\text{hom}_{\mathbb{F}_2}(V^{\otimes 2}, V)$, defined by $(\tau . f)(v_1 \otimes v_2) = f(v_2 \otimes v_1)$.

Definition 2.1. An $\mathcal{E}(2)$ -algebra structure on a cochain complex, A^* , is a cochain map, $\psi: \mathcal{E}(2) \otimes A^{\otimes 2} \rightarrow A$, which is Σ_2 -equivariant as map from $\mathcal{E}(2)$ to $\text{hom}_{\mathbb{F}_2}(A^{\otimes 2}, A)$.

If we denote $\psi(e_i \otimes x_1 \otimes x_2)$ by $x_1 \cup_i x_2$, the previous definition is equivalent to

- (1) $\psi(\tau_i \otimes x_1 \otimes x_2) = \psi(e_i \otimes x_2 \otimes x_1) = x_2 \cup_i x_1$,
- (2) together with the Leibniz condition:
 $\delta(x_1 \cup_i x_2) = x_1 \cup_{i-1} x_2 + x_2 \cup_{i-1} x_1 + \delta x_1 \cup_i x_2 + x_1 \cup_i \delta x_2$.

This means that an $\mathcal{E}(2)$ -algebra structure is given by a cochain map, called *cup_i-product*, $\cup_i: A^r \otimes A^s \rightarrow A^{r+s-i}$, satisfying the previous Leibniz condition.

Let L be a simplicial set. In [1], C. Berger et B. Fresse prove the existence of an $\mathcal{E}(2)$ -action on the normalized cochain complex of L , i.e., the existence of a cochain map

$$\psi_L: \mathcal{E}(2) \otimes N^*(L)^{\otimes 2} \rightarrow N^*(L), \quad e_i \otimes x_1 \otimes x_2 \mapsto x_1 \cup_i x_2,$$

which satisfies the requirements of Definition 2.1. As it is established by May in [18], classical properties of cup_i-products are a direct consequence of this $\mathcal{E}(2)$ -algebra structure, except two of them that we quote in the next definition. (Mention that $N^*(L)$ satisfies these two additional properties, see [1].)

Definition 2.2. An $\mathcal{E}(2)$ -algebra, A^* , is *nice* if it verifies the two next properties, for all $x, x' \in A$ of respective degrees $|x|$ and $|x'|$,

- (i) $x \cup_{|x|} x = x$,
- (ii) $x \cup_i x' = 0$ if $i > \min(|x|, |x'|)$.

Observe the useful next property of nice $\mathcal{E}(2)$ -algebras.

Lemma 2.3. Let A be a nice $\mathcal{E}(2)$ -algebra. If $a \in A^d$ and $b \in A^d$, we have

$$a \cup_d b = b \cup_d a.$$

Proof. Property (ii) of Definition 2.2 and Leibniz rule imply

$$\begin{aligned} \delta(a \cup_{d+1} b) &= 0 \\ &= a \cup_d b + b \cup_d a + \delta a \cup_{d+1} b + a \cup_{d+1} \delta b \\ &= a \cup_d b + b \cup_d a. \end{aligned}$$

□

We recall now from [1] the construction of *the tensor product of $\mathcal{E}(2)$ -algebras*. Let $\psi_i: \mathcal{E}(2) \otimes A_i^{\otimes 2} \rightarrow A_i$ be $\mathcal{E}(2)$ -algebras for $i = 0, 1$. We use the diagonal $\mathcal{D}$ of $\mathcal{E}(2)$ for the construction of an $\mathcal{E}(2)$ -action on the tensor product $A_0 \otimes A_1$, as the following composite, denoted by Φ ,

$$\begin{array}{ccc} \mathcal{E}(2) \otimes (A_0 \otimes A_1)^{\otimes 2} & \xrightarrow{Sh} & \mathcal{E}(2) \otimes A_0^{\otimes 2} \otimes A_1^{\otimes 2} \xrightarrow{\mathcal{D} \otimes \text{id} \otimes \text{id}} \mathcal{E}(2) \otimes \mathcal{E}(2) \otimes A_0^{\otimes 2} \otimes A_1^{\otimes 2} \\ & & \downarrow Sh \\ & & A_0 \otimes A_1 \xleftarrow{\psi_0 \otimes \psi_1} \mathcal{E}(2) \otimes A_0^{\otimes 2} \otimes \mathcal{E}(2) \otimes A_1^{\otimes 2}, \end{array}$$

where Sh are appropriate shuffle maps. We have to verify that the map Φ satisfies the two conditions stated after Definition 2.1. Assertion (2) is the compatibility with the differentials which is direct here, because Φ is the composite of maps that are compatible with the differentials. Thus, we are reduced to Assertion (1). Recall from the definition of the diagonal of $\mathcal{E}(2)$,

$$\mathcal{D}(e_i) = \sum_{j=0}^i e_j \otimes \tau^j . e_{i-j} \text{ and } \mathcal{D}(\tau_i) = \sum_{j=0}^i \tau . e_j \otimes \tau^{j+1} . e_{i-j} = \sum_{j=0}^i \tau_j \otimes \tau^j . \tau_{i-j}.$$

A computation from the definition of Φ gives,

$$\begin{aligned} \Phi(\tau_i \otimes a_0 \otimes a_1 \otimes b_0 \otimes b_1) &= \sum_{j=0}^i \psi_0(\tau_j \otimes a_0 \otimes b_0) \otimes \psi_1(\tau^j . \tau_{i-j} \otimes a_1 \otimes b_1), \\ \Phi(e_i \otimes b_0 \otimes b_1 \otimes a_0 \otimes a_1) &= \sum_{j=0}^i \psi_0(e_j \otimes b_0 \otimes a_0) \otimes \psi_1(\tau^j . e_{i-j} \otimes b_1 \otimes a_1). \end{aligned}$$

If j is even, we have

$$\psi_1(\tau^j . e_{i-j} \otimes b_1 \otimes a_1) = \psi_1(e_{i-j} \otimes b_1 \otimes a_1) = \psi_1(\tau_{i-j} \otimes a_1 \otimes b_1) = \psi_1(\tau^j . \tau_{i-j} \otimes a_1 \otimes b_1).$$

A similar computation in the case j odd gives

$$\Phi(\tau_i \otimes a_0 \otimes a_1 \otimes b_0 \otimes b_1) = \Phi(e_i \otimes b_0 \otimes b_1 \otimes a_0 \otimes a_1).$$

Consider now a family of $\mathcal{E}(2)$ -algebras, $\psi_i: \mathcal{E}(2) \otimes A_i^{\otimes 2} \rightarrow A_i$, with $i = 0, \dots, n$. As $\mathcal{D}: \mathcal{E}(2) \rightarrow \mathcal{E}(2) \otimes \mathcal{E}(2)$ is the diagonal of a Bar resolution, it is a cochain map, coassociative ([1]) and we may iterate it as,

$$\mathcal{D}^2(e_i) = \sum_{j=0}^i \mathcal{D}(e_j) \otimes \tau^j . e_{i-j} = \sum_{j=0}^i \sum_{k=0}^j e_k \otimes \tau^k . e_{j-k} \otimes \tau^j . e_{i-j}.$$

If we set $i_1 = k$, $i_2 = j - k$, $i_3 = i - j$, this last expression can be written as

$$\mathcal{D}^2(e_i) = \sum_{(i_1, i_2, i_3) \text{ with } i_1 + i_2 + i_3 = i} e_{i_1} \otimes \tau^{i_1} . e_{i_2} \otimes \tau^{i_1 + i_2} . e_{i_3}.$$

More generally, an induction gives,

$$\mathcal{D}^{n-1}(e_i) = \sum_{(i_1, \dots, i_n) \text{ with } i_1 + \dots + i_n = i} e_{i_1} \otimes \tau^{i_1}.e_{i_2} \otimes \dots \otimes \tau^{i_1 + \dots + i_{n-1}}.e_{i_n}.$$

As in the previous case of two $\mathcal{E}(2)$ -algebras, the action of $\mathcal{E}(2)$ on $\otimes_{i=0}^n A_i$ is obtained from appropriate shuffle maps and the iteration $\mathcal{D}^{n-1}$ of the diagonal. By using the notation in cup_i-products, this structure is defined as the map

$$(2) \quad \mathcal{E}(2) \otimes (\otimes_{i=0}^n A_i)^{\otimes 2} \xrightarrow{\Phi} \otimes_{i=0}^n A_i$$

which sends the element $e_i \otimes (\otimes_{i=0}^n x_i) \otimes (\otimes_{i=0}^n y_i)$ to

$$\sum_{(i_1, \dots, i_n) \text{ with } i_1 + \dots + i_n = i} (x_1 \cup_{i_1} y_1) \otimes (x_2 \cup_{i_2}^{i_1} y_2) \otimes \dots \otimes (x_n \cup_{i_n}^{i_1 + \dots + i_{n-1}} y_n),$$

where we set, for $j \geq 0$,

$$(3) \quad x \cup_i^j y = \begin{cases} x \cup_i y & \text{if } j \text{ is even,} \\ y \cup_i x & \text{if } j \text{ is odd.} \end{cases}$$

Up to shuffle maps, Φ is obtained as a composite and tensor product of equivariant cochain maps; thus it satisfies the requirements of Definition 2.1. Moreover, as we establish below, the tensor product of nice $\mathcal{E}(2)$ -algebras is a nice $\mathcal{E}(2)$ -algebra.

Lemma 2.4. *Any tensor product of nice $\mathcal{E}(2)$ -algebras is a nice $\mathcal{E}(2)$ -algebra for the product structure coming from the diagonal of $\mathcal{E}(2)$.*

Proof. By coassociativity of the diagonal of $\mathcal{E}(2)$, it is sufficient to reduce the proof to the case of the tensor product of two nice $\mathcal{E}(2)$ -algebras, A and B .

Let $x = \sum_k a_k \otimes b_k \in (A \otimes B)^d$ and $x' = \sum_\ell a'_\ell \otimes b'_\ell \in (A \otimes B)^{d'}$ with $d \leq d'$. We set $f = d + m$ with $m \geq 0$. One compute

$$x \cup_f x' = \sum_{f_1 + f_2 = f} \sum_{k, \ell} (a_k \cup_{f_1} a'_\ell) \otimes (b_k \cup_{f_2}^{f_1} b'_\ell).$$

If the element $(a_k \cup_{f_1} a'_\ell) \otimes (b_k \cup_{f_2}^{f_1} b'_\ell)$ of this sum is not equal to zero, we must have

$$f_1 \leq \min(|a_k|, |a'_\ell|) \text{ and } f_2 \leq \min(|b_k|, |b'_\ell|),$$

which implies $f = f_1 + f_2 = d + m \leq |a_k| + |b_k| = d$ and $m = 0$. We have established Property (ii) of Definition 2.2. As for Property (i), we consider

$$x \cup_d x = \sum_{f_1 + f_2 = d} \sum_{k, k'} (a_k \cup_{f_1} a_{k'}) \otimes (b_k \cup_{f_2}^{f_1} b_{k'}).$$

As above, if the element $(a_k \cup_{f_1} a_{k'}) \otimes (b_k \cup_{f_2}^{f_1} b_{k'})$ of this sum is not equal to zero, we must have

$$f_1 \leq \min(|a_k|, |a_{k'}|) \text{ and } f_2 \leq \min(|b_k|, |b_{k'}|).$$

Suppose $\min(|a_k|, |a_{k'}|) = |a_k|$, then we have $|b_{k'}| \leq |b_k|$, because $|a_k| + |b_k| = |a_{k'}| + |b_{k'}|$, and also $d = |a_k| + |b_k| = f_1 + f_2 \leq |a_k| + |b_{k'}|$, which imply $|b_k| = |b_{k'}|$. Therefore, the non-zero elements of this sum must be of the shape $(a_k \cup_{d-r} a_{k'}) \otimes (b_k \cup_r^{d-r} b_{k'})$ with $|a_k| = |a_{k'}| = d - r$, $|b_k| = |b_{k'}| = r$. With Lemma 2.3, if $a_k \neq a_{k'}$, the same term appears twice, as $(a_k \cup_{d-r} a_{k'}) \otimes (b_k \cup_r^{d-r} b_{k'})$ and as $(a_{k'} \cup_{d-r} a_k) \otimes (b_{k'} \cup_r^{d-r} b_k)$. Their sum is

equal to zero. With the same argument applied to the case $b_k \neq b_{k'}$, we have reduced the previous expression to

$$\begin{aligned} x \cup_d x &= \sum_k (a_k \cup_{d-r} a_k) \otimes (b_k \cup_r b_k) \\ &= \sum_k a_k \otimes b_k = x, \end{aligned}$$

and Property (i) of Definition 2.2 is established. $\square$

We come back to the intersection setting and recall ([4]) that a *perverse cochain complex* is a functor defined on $\hat{\mathcal{J}}^n$, with values in the category of cochain complexes. A functor from $\hat{\mathcal{J}}^n_{\text{loose}}$ with values in the category of cochain complexes is called a *generalized perverse cochain complex*. For instance, if $\underline{K}$ is a filtered face set, the association $\bar{p} \mapsto \tilde{N}^*_{\bar{p}}(\underline{K})$ is a (generalized) perverse cochain complex and this association is natural in $\underline{K}$.

Definition 2.5. Let A^* be a generalized perverse cochain complex. We denote by $\varphi_{\bar{p}}^{\bar{q}}: A^*_{\bar{p}} \rightarrow A^*_{\bar{q}}$ the morphism associated to $\bar{p} \leq \bar{q}$. A *perverse $\mathcal{E}(2)$ -algebra structure* on A^* is a family of cochain maps, $\psi_{\bar{p}, \bar{q}}: \mathcal{E}(2) \otimes A^*_{\bar{p}} \otimes A^*_{\bar{q}} \rightarrow A^*_{\bar{p}+\bar{q}}$, verifying

- (i) a *compatibility condition with perversities*: for any loose perversities, $\bar{p}_1, \bar{q}_1, \bar{p}_2, \bar{q}_2$, with $\bar{p}_1 \leq \bar{p}_2$ and $\bar{q}_1 \leq \bar{q}_2$, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{E}(2) \otimes A^*_{\bar{p}_1} \otimes A^*_{\bar{q}_1} & \xrightarrow{\psi_{\bar{p}_1, \bar{q}_1}} & A^*_{\bar{p}_1 + \bar{q}_1} \\ \text{id} \otimes \varphi_{\bar{p}_1}^{\bar{p}_2} \otimes \varphi_{\bar{q}_1}^{\bar{q}_2} \downarrow & & \downarrow \varphi_{\bar{p}_1 + \bar{q}_1}^{\bar{p}_2 + \bar{q}_2} \\ \mathcal{E}(2) \otimes A^*_{\bar{p}_2} \otimes A^*_{\bar{q}_2} & \xrightarrow{\psi_{\bar{p}_2, \bar{q}_2}} & A^*_{\bar{p}_2 + \bar{q}_2} \end{array}$$

- (ii) a Σ_2 -*equivariance* as map from $\mathcal{E}(2)$ to $(\text{hom}(A^*_{\bar{p}} \otimes A^*_{\bar{q}}, A^*_{\bar{p}+\bar{q}}))_{\bar{p}, \bar{q}}$ with the following Σ_2 -action on the codomain:

To any family $\eta_{\bar{p}, \bar{q}}: A^*_{\bar{p}} \otimes A^*_{\bar{q}} \rightarrow A^*_{\bar{p}+\bar{q}}$, we associate the family $(\tau\eta)_{\bar{p}, \bar{q}}: A^*_{\bar{p}} \otimes A^*_{\bar{q}} \rightarrow A^*_{\bar{p}+\bar{q}}$, defined by $(\tau\eta)_{\bar{p}, \bar{q}}(x_1 \otimes x_2) = \eta_{\bar{q}, \bar{p}}(x_2 \otimes x_1)$.

Equivalently, a *perverse $\mathcal{E}(2)$ -algebra structure* on A^* is entirely determined by maps, called *perverse cup_i-products*, $\cup_i: A^r_{\bar{p}} \otimes A^s_{\bar{q}} \rightarrow A^{r+s-i}_{\bar{p}+\bar{q}}$, satisfying the previous Leibniz condition and the compatibility conditions with the poset structure of perversities. (The two settings are related by $x \cup_i y = \psi_{\bar{p}, \bar{q}}(e_i \otimes x \otimes y)$.) *Nice perverse $\mathcal{E}(2)$ -algebras* are defined as in Definition 2.2.

When A^* is a perverse cochain complex and the sum $\bar{p} + \bar{q}$ replaced by the sum of GM-perversities, $\bar{p} \oplus \bar{q}$, in Definition 2.5, we say that A^* is a *GM-perverse $\mathcal{E}(2)$ -algebra*.

Let $\underline{K}$ be a filtered face set and $\sigma: \Delta = \Delta^{j_0} * \Delta^{j_1} * \dots * \Delta^{j_n} \rightarrow \underline{K}$. With the tensor product of $\mathcal{E}(2)$ -algebras recalled in (2) and the structure of nice $\mathcal{E}(2)$ -algebra defined on the normalized cochain complex in [1], we get a structure of nice $\mathcal{E}(2)$ -algebra on the tensor product $\tilde{N}^*(\underline{K})_{\sigma} = N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$. The next theorem establishes the compatibility of this structure with the perverse degrees.

Theorem A. *Let $\underline{K}$ be a filtered face set and $\bar{p}$ be a loose perversity. The generalized perverse cochain complex, $\bar{p} \mapsto \tilde{N}_{\bar{p}}^*(\underline{K})$, is a nice perverse $\mathcal{E}(2)$ -algebra, natural in $\underline{K}$, for the filtered face maps.*

Recall that a continuous map, $f: X = (X_j)_{0 \leq j \leq n} \rightarrow Y = (Y_j)_{0 \leq j \leq n}$, between pseudo-manifolds is a *stratum preserving stratified map* if, for any stratum S' of Y' , $f^{-1}(S')$ is a union of strata of X and $f^{-1}(Y_{n-\ell}) = X_{n-\ell}$, for any $\ell \geq 0$. As any stratum preserving, stratified map induces a filtered face set map, $\underline{\text{ISing}}^{\mathcal{F}}(X) \rightarrow \underline{\text{ISing}}^{\mathcal{F}}(Y)$, (see [4, Example 1.5]) the next result is a direct consequence of Theorem A.

Corollary 2.6. *Let X be a pseudomanifold and $\bar{p}$ be a loose perversity. The generalized perverse cochain complex, $\bar{p} \mapsto \tilde{N}_{\bar{p}}^*(\underline{\text{ISing}}^{\mathcal{F}}(X))$, is a nice perverse $\mathcal{E}(2)$ -algebra, natural in X by stratum preserving stratified maps.*

Proof of Theorem A. A cochain $c \in \tilde{N}^*(\underline{K})$ associates to any simplex, $\sigma: \Delta = \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow \underline{K}_+$, an element $c_\sigma \in N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$.

If we set $(c \cup_i c')_\sigma = c_\sigma \cup_i c'_\sigma$, by naturality of the structure of $\mathcal{E}(2)$ -algebra on $N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$, we get a global section $c \cup_i c' \in \tilde{N}^*(\underline{K})$. More precisely, we have a Σ_2 -equivariant cochain map,

$$\mathcal{E}(2) \otimes \tilde{N}^*(\underline{K})^{\otimes 2} \rightarrow \tilde{N}^*(\underline{K}),$$

entirely defined by $e_i \otimes c \otimes c' \mapsto c \cup_i c'$, which gives to $\tilde{N}^*(\underline{K})$ a structure of $\mathcal{E}(2)$ -algebra. The niceness of this structure is a direct consequence of Lemma 2.4.

The naturality in $\underline{K}$ comes from the naturality of the $\mathcal{E}(2)$ -algebra structure on a tensor product, already mentioned, and from the naturality of the association, $\underline{K} \mapsto \tilde{N}_{\bar{p}}^*(\underline{K})$, see [4, Proposition 1.36].

We study now the behavior of this structure with the perverse degree. The perversity degree being a local notion, we consider c and c' in $N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$, with $j_n \geq 0$, and $\ell \in \{1, \dots, n\}$ such that $\Delta^{j_{n-\ell}} \neq \emptyset$. We denote by $c_{n-\ell}$ and $c'_{n-\ell}$ the respective restrictions of c and c' to $N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(\Delta^{j_{n-\ell}} \times \{1\}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$.

We decompose c, c' in $c = \sum_{s=0}^m c_0^s \otimes \dots \otimes c_n^s$, $c' = \sum_{t=0}^{m'} c_0^{t'} \otimes \dots \otimes c_n^{t'}$ and their restriction in $c_{n-\ell} = \sum_{s=0}^m c_0^s \otimes \dots \otimes \iota_{n-\ell}^* c_{n-\ell}^s \otimes \dots \otimes c_n^s$, $c'_{n-\ell} = \sum_{t=0}^{m'} c_0^{t'} \otimes \dots \otimes \iota_{n-\ell}^* c_{n-\ell}^{t'} \otimes \dots \otimes c_n^{t'}$, where $\iota_{n-\ell}^*$ is induced by the inclusion $\Delta^{j_{n-\ell}} \times \{1\} \hookrightarrow c\Delta^{j_{n-\ell}}$. By definition, we have

$$\|c\|_\ell = \sup_s \{|c_{n-\ell+1}^s \otimes \dots \otimes c_n^s| \text{ such that } c_0^s \otimes \dots \otimes \iota_{n-\ell}^* c_{n-\ell}^s \neq 0\}.$$

Let

$$\begin{array}{c} N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(c\Delta^{j_{n-\ell}}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n}) \\ \downarrow \hat{\iota}_{n-\ell}^* = \text{id} \otimes \iota_{n-\ell}^* \otimes \text{id} \\ N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(\Delta^{j_{n-\ell}} \times \{1\}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n}). \end{array}$$

As the cup_i-product is natural, we have $\hat{\iota}_{n-\ell}^*(c \cup_i c') = \hat{\iota}_{n-\ell}^*(c) \cup_i \hat{\iota}_{n-\ell}^*(c')$.

- If $\hat{\iota}_{n-\ell}^*(c) = 0$ or $\hat{\iota}_{n-\ell}^*(c') = 0$, we have $\hat{\iota}_{n-\ell}^*(c) \cup_i \hat{\iota}_{n-\ell}^*(c') = 0$ and thus

$$\|c \cup_i c'\|_\ell = -\infty.$$

- Suppose now $\hat{i}_{n-\ell}^*(c) \neq 0$ and $\hat{i}_{n-\ell}^*(c') \neq 0$. By definition of the cup_i -product, $\hat{i}_{n-\ell}^*(c) \cup_i \hat{i}_{n-\ell}^*(c')$ is a sum of tensor products whose elements are of two kinds:
 - (1) $c_j^s \cup_{f_j} c_j^{tt}$, with $j \neq n - \ell$, or
 - (2) $\hat{i}_{n-\ell}^*(c_{n-\ell}^s) \cup_{f_{n-\ell}} \hat{i}_{n-\ell}^*(c_{n-\ell}^{tt})$.

As $|c_j^s \cup_{f_j} c_j^{tt}| \leq |c_j^s| + |c_j^{tt}| - f_j$, the cochain degree decreases and we obtain, for each ℓ ,

$$\|c \cup_i c'\|_\ell \leq \|c\|_\ell + \|c'\|_\ell,$$

by definition of the perverse degree, see Definition 1.2. Therefore, we have

$$\|c \cup_i c'\| \leq \|c\| + \|c'\|.$$

Now, the rule of Leibniz implies

$$\|\delta(c \cup_i c')\| \leq \max(\|\delta c\| + \|c'\|, \|\delta c'\| + \|c\|, \|c\| + \|c'\|).$$

Thus, if $\|c\| \leq \bar{p}$, $\|\delta c\| \leq \bar{p}$, $\|c'\| \leq \bar{q}$ and $\|\delta c'\| \leq \bar{q}$, we have $\|c \cup_i c'\| \leq \bar{p} + \bar{q}$ and $\|\delta(c \cup_i c')\| \leq \bar{p} + \bar{q}$. This implies that the $\mathcal{E}(2)$ -algebra structure on $\tilde{N}^*(\underline{K})$ induces equivariant cochain maps

$$\mathcal{E}(2) \otimes \tilde{N}_{\bar{p}}^*(\underline{K}) \otimes \tilde{N}_{\bar{q}}^*(\underline{K}) \rightarrow \tilde{N}_{\bar{p}+\bar{q}}^*(\underline{K}).$$

That means: $\tilde{N}_{\bullet}^*(\underline{K})$ is a perverse $\mathcal{E}(2)$ -algebra. □

3. STEENROD PERVERSE SQUARES

From the existence of perverse cup_i -products, we define Steenrod squares, as in the classical case. In the next statement, when $i > 0$, the fact that the loose perversity image of Sq^i is $\mathcal{L}(\bar{p}, i)$, defined by $\mathcal{L}(\bar{p}, i)(\ell) = \min(2\bar{p}(\ell), \bar{p}(\ell) + i)$, answers positively a conjecture of M. Goresky and W. Pardon, see [13, Conjecture 7.5]. More explicitly, we prove the existence of a dashed arrow which lifts the square Sq^i ,

$$\begin{array}{ccc} & & H_{\text{TW}, \mathcal{L}(\bar{p}, i)}^{r+i} \\ & \nearrow & \downarrow \\ H_{\text{TW}, \bar{p}}^r & \xrightarrow{\text{Sq}^i} & H_{\text{TW}, 2\bar{p}}^{r+i} \end{array}$$

We still denote by Sq^i this lifting.

Theorem B. *Let $\underline{K}$ be a filtered face set and $\bar{p}, \bar{q}$ be loose perversities. The perverse cup_i -products induce natural perverse squares, defined by $\text{Sq}^i(x) = x \cup_{|x|-i} x$, for $x \in H_{\text{TW}, \bar{p}}^{|x|}(\underline{K})$, which satisfy the following properties.*

- (1) *If $i < 0$, then $\text{Sq}^i(x) = 0$.*
- (2) *If $i \geq 0$, then we have*

$$\text{Sq}^i : H_{\text{TW}, \bar{p}}^r(\underline{K}) \rightarrow H_{\text{TW}, \mathcal{L}(\bar{p}, i)}^{r+i}(\underline{K}),$$

where $\mathcal{L}(\bar{p}, i) = \min(2\bar{p}, \bar{p} + i)$ and

- $\text{Sq}^i(x) = 0$ if $i > |x|$,
- $\text{Sq}^{|x|}(x) = x^2$,
- $\text{Sq}^0 = \text{id}$.

(iv) If $x \in H_{\text{TW}, \bar{p}}^{|x|}(\underline{K})$, $y \in H_{\text{TW}, \bar{q}}^{|y|}(\underline{K})$, one has the (internal) Cartan formula,

$$\text{Sq}^i(x \cup y) = \sum_{i_1+i_2=i} \text{Sq}^{i_1}(x) \cup \text{Sq}^{i_2}(y) \in H_{\text{TW}, \bar{r}}^{|x|+|y|+i}(\underline{K}),$$

with $\bar{r} = \min(2\bar{p} + 2\bar{q}, \bar{p} + \bar{q} + i)$ and $\cup = \cup_0$.

(v) For any pair (i, j) , with $i < 2j$, one has the Adem relation,

$$\text{Sq}^i \text{Sq}^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \text{Sq}^{i+j-k} \text{Sq}^k$$

and $\text{Sq}^i \text{Sq}^j$ sends $H_{\text{TW}, \bar{p}}^*$ into $H_{\text{TW}, \bar{r}}^{*+i+j}$, with $\bar{r} = \min(4\bar{p}, 2\bar{p} + i, \bar{p} + i + j)$.

Before proving this theorem, we establish a technical property on the tensor product of two nice $\mathcal{E}(2)$ -algebras, which is the keystone in the proof of Theorem B.

Lemma 3.1. *Let A and B be two nice $\mathcal{E}(2)$ -algebras and $A \otimes B$ their tensor product equipped with the $\mathcal{E}(2)$ -algebra structure coming from the diagonal of $\mathcal{E}(2)$. Let x, x' in A , y, y' in B such that $|x| + |y| = |x'| + |y'| = d$, $|y| \leq r$ and $|y'| \leq r$. Then, for any $k \in \{0, \dots, d-i\}$ such that $(x \cup_{d-k-i} x') \otimes (y \cup_k^{d-k-i} y') \neq 0$, we have $|y \cup_k^{d-k-i} y'| \leq r + i$.*

Proof. Suppose $d - k - i$ even. If $(x \cup_{d-k-i} x') \otimes (y \cup_k y') \neq 0$, we must have $k \leq \min(|y|, |y'|)$ and $d - k - i \leq \min(|x|, |x'|)$, which implies

$$d - i - \min(|x|, |x'|) \leq k.$$

Suppose $\min(|x|, |x'|) = |x|$. Then we have

$$\begin{aligned} |y| + |y'| - d + i + \min(|x|, |x'|) &= |y| + |y'| - (|x| + |y|) + i + |x| \\ &= |y'| + i, \end{aligned}$$

which implies

$$|y \cup_k y'| \leq |y| + |y'| - k \leq |y| + |y'| - d + i + \min(|x|, |x'|) \leq |y'| + i \leq r + i.$$

A similar argument gives the result in the case $\min(|x|, |x'|) = |x'|$. Also, the proof is analogous to the previous one if $d - k - i$ is odd, since $|y' \cup_k y| \leq |y'| + |y| - k$. $\square$

Directly from the definition of cup_k -products, the inequalities $|y| \leq r$ and $|y'| \leq r$ imply $|y \cup_k y'| \leq 2r$. Thus, the bound $|y \cup_k y'| \leq r + i$ obtained in Lemma 3.1 is exactly what is needed for the proof of the Goresky-Pardon conjecture, as we show in the beginning of the next proof.

Proof of Theorem B. Let $i \geq 0$. From their definition as particular cup_i -products, the Steenrod squares have their image in the intersection cohomology with loose perversity $2\bar{p}$. We prove first that the loose perversity $2\bar{p}$ can be replaced by $\mathcal{L}(\bar{p}, i)$. We take over the arguments and the method used at the end of the proof of Theorem A by considering a cocycle $c \in N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$, $\ell \in \{1, \dots, n\}$, such that $\Delta^{j_{n-\ell}} \neq \emptyset$, and the restriction $c_{n-\ell}$ of c to $N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(\Delta^{j_{n-\ell}} \times \{1\}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$. Observe first that, by naturality, we have $(c \cup_{|c|-i} c)_{n-\ell} = c_{n-\ell} \cup_{|c_{n-\ell}|-i} c_{n-\ell}$.

- If $c_{n-\ell} = 0$, we have $(c \cup_{|c|-i} c)_{n-\ell} = 0$ and $\|c \cup_{|c|-i} c\|_\ell = -\infty$.
- If $c_{n-\ell} \neq 0$, we decompose it in a canonical form, $c_{n-\ell} = \sum_s c_{n-\ell}^{\prime s} \otimes c_{n-\ell}^{\prime\prime s} \in A \otimes B$, with $A = N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(\Delta^{j_{n-\ell}} \times \{1\})$ and $B = N^*(c\Delta^{j_{n-\ell+1}}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes$

$N^*(\Delta^{j_n})$. Using Lemma 3.1, we know that $(c_{n-\ell}^{s'} \cup_{|c_{n-\ell}|-k-i} c_{n-\ell}^{t'}) \otimes (c_{n-\ell}^{s''} \cup_k^{c_{n-\ell}|-k-i} c_{n-\ell}^{t''}) \neq 0$ implies $|c_{n-\ell}^{s''} \cup_k^{c_{n-\ell}|-k-i} c_{n-\ell}^{t''}| \leq \bar{p}(\ell) + i$, for any pair of indices, (s, t) , in the writing of $c_{n-\ell}$. This implies $\|c \cup_{|c|-i} c\| \leq \bar{p} + i$, as announced.

The condition on the perversity of the differential of $c \cup_{|c|-i} c$ is immediate here because c is a cocycle, and the naturality follows from the fact that the lifting already exists at the level of the spaces of cocycles.

The list (1), (2)-(i), (2)-(ii), (2)-(iii) of properties is a direct consequence of Theorem A and [18, Section 5].

Let A and B be two nice $\mathcal{E}(2)$ -algebras. By definition of the diagonal action of $\mathcal{E}(2)$ on the tensor product, we have a Cartan external formula,

$$\mathrm{Sq}^i(a \otimes b) = \sum_{i_1+i_2=i} \mathrm{Sq}^{i_1}(a) \otimes \mathrm{Sq}^{i_2}(b),$$

for $a \in A$ and $b \in B$. In our case, each factor, A and B , satisfies the Cartan internal formula. Therefore, the Cartan internal formula on $A \otimes B$ is a direct consequence of the next equalities:

$$\begin{aligned} \mathrm{Sq}^i((a \otimes b) \cup (a' \otimes b')) &=_{(1)} \mathrm{Sq}^i((a \cup a') \otimes (b \cup b')) \\ &=_{(2)} \sum_{i_1+i_2=i} \mathrm{Sq}^{i_1}(a \cup a') \otimes \mathrm{Sq}^{i_2}(b \cup b') \\ &=_{(3)} \sum_{j_1+j_2+k_1+k_2=i} (\mathrm{Sq}^{j_1}(a) \cup \mathrm{Sq}^{j_2}(a')) \otimes (\mathrm{Sq}^{k_1}(b) \cup \mathrm{Sq}^{k_2}(b')) \end{aligned}$$

and

$$\begin{aligned} \sum_{i_1+i_2=i} \mathrm{Sq}^{i_1}(a \otimes b) \cup \mathrm{Sq}^{i_2}(a' \otimes b') &=_{(2)} \sum_{s_1+s_2+t_1+t_2=i} (\mathrm{Sq}^{s_1}(a) \otimes \mathrm{Sq}^{s_2}(b)) \cup (\mathrm{Sq}^{t_1}(a') \otimes \mathrm{Sq}^{t_2}(b')) \\ &=_{(1)} \sum_{s_1+s_2+t_1+t_2=i} (\mathrm{Sq}^{s_1}(a) \cup \mathrm{Sq}^{t_1}(a')) \otimes (\mathrm{Sq}^{s_2}(b) \cup \mathrm{Sq}^{t_2}(b')), \end{aligned}$$

where $=_{(1)}$ comes from the definition of the cup-product on a tensor product, $=_{(2)}$ from the application of the Cartan external formula and $=_{(3)}$ from the Cartan internal formula on each factor.

For the Adem's formula (2)-(v), we need to recall some properties in order to track the perversity conditions. The classical proof uses the Bar resolution, $\mathcal{E}(4)$, of $\mathbb{F}_2$ as a Σ_4 -module, and the existence of a Σ_4 -equivariant cochain map, $\mathcal{E}(4) \otimes N^*(L)^{\otimes 4} \rightarrow N^*(L)$, for any simplicial set L , called an $\mathcal{E}(4)$ -algebra. As these objects appear just in this part of proof, we do not recall them in detail, referring to [1, Section 1]. We mention only the points related to the control of perversities.

Denote by $\omega: \mathcal{E}(2) \otimes \mathcal{E}(2) \otimes \mathcal{E}(2) \rightarrow \mathcal{E}(4)$ the cochain map induced by the wreath product $\Sigma_2 \times \Sigma_2 \times \Sigma_2 \rightarrow \Sigma_4$. Let A be an $\mathcal{E}(2)$ and an $\mathcal{E}(4)$ -algebra whose structure maps are respectively denoted ψ_2 and ψ_4 . By definition, we say that A is an Adem-object

([18]) if there is a commutative diagram

$$\begin{array}{ccccc}
 \mathcal{E}(2) \otimes \mathcal{E}(2)^{\otimes 2} \otimes A^{\otimes 4} & \xrightarrow{\omega \otimes \text{id}} & \mathcal{E}(4) \otimes A^{\otimes 4} & \xrightarrow{\psi_4} & A \\
 \text{Sh} \downarrow & & & \nearrow \psi_2 & \\
 \mathcal{E}(2) \otimes (\mathcal{E}(2) \otimes A^{\otimes 2})^{\otimes 2} & \xrightarrow{\text{id} \otimes \psi_2^{\otimes 2}} & \mathcal{E}(2) \otimes A^{\otimes 2} & &
 \end{array}$$

where Sh is the appropriate shuffle map.

Let $\Delta = \Delta^{j_0} * \dots * \Delta^{j_n}$ and $A = N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(c\Delta^{j_{n-1}}) \otimes N^*(\Delta^{j_n})$. Because $N^*(L)$ is an Adem-object for any simplicial set L and because the tensor product of two nice $\mathcal{E}(2)$ -algebras which are Adem-objects is an Adem-object ([18, Lemma 4.2, Page 174]), A is an Adem-object.

In Theorem A, we prove that ψ_2 restricts to a map $\mathcal{E}(2) \otimes A_{\bar{p}} \otimes A_{\bar{q}} \rightarrow A_{\bar{p}+\bar{q}}$. Exactly the same argument can be used for ψ_4 , replacing $c \cup_i c'$ by $\psi_4(\alpha_i \otimes c_1 \otimes c_2 \otimes c_3 \otimes c_4)$ for each $\alpha_i \in \mathcal{E}(4)$, in the last part of the proof of Theorem A. Thus ψ_4 restricts to a map $\mathcal{E}(4) \otimes A_{\bar{p}_1} \otimes A_{\bar{p}_2} \otimes A_{\bar{p}_3} \otimes A_{\bar{p}_4} \rightarrow A_{\bar{p}_1+\bar{p}_2+\bar{p}_3+\bar{p}_4}$ and we get an Adem formula for intersection cohomology.

Successive applications of Lemma 3.1 show that the non-zero terms in the right-hand side of the Adem relation belong to intersection cohomology in perversities less than, or equal to, $\min(4\bar{p}, 2\bar{p} + 2j, 2\bar{p} + i, \bar{p} + i + j) \leq \min(4\bar{p}, 2\bar{p} + i, \bar{p} + i + j)$, since $i < 2j$. The same argument applied to the left-hand side implies that the non-zero terms belong also to intersection cohomology in the same range of perversities. $\square$

Remark 3.2. Previous definitions and results can be adapted to the context of GM-perversities. By restricting to GM-perversities $\bar{p}$ and $\bar{q}$ such that $\bar{p} + \bar{q} \leq \bar{t}$, the cup _{i} -products are defined by

$$\cup_i: A_{\bar{p}}^r \otimes A_{\bar{q}}^s \rightarrow A_{\bar{p} \oplus \bar{q}}^{r+s-i},$$

where the sum $\bar{p} \oplus \bar{q}$ is taken in the lattice $\mathcal{P}^n$, see [14] or [4, Section 2.1]. The Steenrod squares introduced in Section 3,

$$\text{Sq}^i: H_{\text{TW}, \bar{p}}^r \rightarrow H_{\text{TW}, \bar{r}}^{r+i},$$

are therefore defined for GM-perversities $\bar{p}, \bar{r}$ such that $\min(2\bar{p}, \bar{p} + i) \leq \bar{r}$.

4. COMPARISON WITH GORESKEY'S CONSTRUCTION

As this section is concerned with isomorphisms between different definitions of Steenrod squares in intersection cohomology, in some crucial points, we keep all the information in the notations of cohomology groups.

In [12] (see also [2, Chapter V]), the intersection cohomology on a pseudomanifold, X , is introduced by the use of a sheaf due to Deligne. The Deligne's sheaf, $\mathcal{P}_{\bar{p}}$, is defined by a sequence of truncations starting from the constant sheaf on $X_n \setminus X_{n-2}$. As we are not using this specific construction, we do not recall it, sending the reader to the previous references.

In [10], M. Goresky has already defined Steenrod squares, Sq_G^i , on the intersection cohomology, $H^*(X; \mathcal{P}_{\bar{p}})$, of a topological pseudomanifold, X , in the case of a GM-perversity $\bar{p}$. In this section, we prove that the two Steenrod squares, Sq^i and Sq_G^i , coincide.

Recall the filtered face set $\underline{\text{ISing}}^{\mathcal{F}}(X)$ introduced in Remark 1.7. The next result connects Goresky's definition of Steenrod squares on $H^*(X; \mathcal{P}_{\bar{p}})$ to our definition of Steenrod squares on the TW-cohomology of the filtered face set $\underline{\text{ISing}}^{\mathcal{F}}(X)$, denoted $H_{\text{TW}, \bar{p}}^*(X)$.

Theorem C. *Let X be an n -dimensional topological pseudomanifold. For any GM-perversity $\bar{q}$, there exists an isomorphism $\theta_{\bar{q}}^*: H_{\text{TW}, \bar{q}}^*(X) \rightarrow H^*(X; \mathcal{P}_{\bar{q}})$. Moreover, if $\bar{p}$ is a GM-perversity such that $2\bar{p} \leq \bar{t}$, then the following diagram commutes,*

$$\begin{array}{ccccc} H_{\text{TW}, \bar{p}}^r(X) & \xrightarrow{\text{Sq}^i} & H_{\text{TW}, \mathcal{L}(\bar{p}, i)}^{r+i}(X) & \longrightarrow & H_{\text{TW}, 2\bar{p}}^{r+i}(X) \\ \theta_{\bar{p}}^r \downarrow & & & & \downarrow \theta_{2\bar{p}}^{r+i} \\ H^r(X; \mathcal{P}_{\bar{p}}) & \xrightarrow{\text{Sq}_G^i} & & \longrightarrow & H^{r+i}(X; \mathcal{P}_{2\bar{p}}). \end{array}$$

The previous statement implies that $\theta_{\mathcal{L}(\bar{p}, i)}^{r+i} \circ \text{Sq}^i \circ (\theta_{\bar{p}}^r)^{-1}: H^r(X; \mathcal{P}_{\bar{p}}) \rightarrow H^{r+i}(X; \mathcal{P}_{\mathcal{L}(\bar{p}, i)})$ is a lift of the Steenrod squares defined by Goresky, $\text{Sq}_G^i: H^r(X; \mathcal{P}_{\bar{p}}) \rightarrow H^{r+i}(X; \mathcal{P}_{2\bar{p}})$. Therefore the Goresky-Pardon conjecture has a positive answer.

From the functor $\tilde{N}^*$, we define a presheaf on X by

$$IN_{\bar{p}}^*(U) = \tilde{N}_{\bar{p}}^*(\underline{\text{ISing}}^{\mathcal{F}}(U)),$$

for any open set U of X . Denote by $\text{Cov}(U)$ the directed set of open covers of U , ordered by inclusions. For any $\mathcal{U} \in \text{Cov}(U)$, $\underline{\text{ISing}}^{\mathcal{F}, \mathcal{U}}(U)$ is the sub-filtered face set of $\underline{\text{ISing}}^{\mathcal{F}}(U)$ whose elements have a support included in an element of $\mathcal{U}$. The sheafification of $IN_{\bar{p}}^*$ is given by

$$\mathbf{IN}_{\bar{p}}^*(U) = \lim_{\mathcal{U} \in \text{Cov}(U)} \tilde{N}_{\bar{p}}^*(\underline{\text{ISing}}^{\mathcal{F}, \mathcal{U}}(U)),$$

see [9, Exemple 3.9.1.] in the case of singular cochains. The cup_i -products introduced in Section 3 on $\tilde{N}_{\bullet}^*(\underline{\text{ISing}}^{\mathcal{F}, \mathcal{U}}(U))$ induce cup_i -products on $\mathbf{IN}_{\bullet}^*(U)$, by definition of the last one as a direct limit.

Theorem C is a direct consequence of Lemmas 4.1 and 4.3. First, we connect the definition of Steenrod squares on $\underline{\text{ISing}}^{\mathcal{F}}(X)$ with a definition involving the sheaf $\mathbf{IN}_{\bullet}^*$ on X .

Lemma 4.1. *For any n -dimensional topological pseudomanifold, X , and any GM-perversity $\bar{p}$, we have a commutative diagram,*

$$\begin{array}{ccc} H_{\text{TW}, \bar{p}}^r(X) & \xrightarrow{\text{Sq}^i} & H_{\text{TW}, \mathcal{L}(\bar{p}, i)}^{r+i}(X) \\ \cong \downarrow & & \downarrow \cong \\ H^r(X; \mathbf{IN}_{\bar{p}}) & \xrightarrow{\text{Sq}^i} & H^{r+i}(X; \mathbf{IN}_{\mathcal{L}(\bar{p}, i)}), \end{array}$$

in which vertical maps are quasi-isomorphisms induced by the canonical map $IN_{\bullet}^* \rightarrow \mathbf{IN}_{\bullet}^*$.

Proof. For any $\mathcal{U} \in \text{Cov}(X)$, there is a restriction map, $r_{\mathcal{U}}: IN_{\bar{p}}^*(X) \rightarrow \tilde{N}_{\bar{p}}^*(\underline{\text{ISing}}^{\mathcal{F}, \mathcal{U}}(X))$, compatible with the inclusions of open covers. This gives the morphism,

$$IN_{\bar{p}}^*(X) \rightarrow \Gamma(X, \mathbf{IN}_{\bar{p}}^*) := \lim_{\mathcal{U} \in \text{Cov}(X)} \tilde{N}_{\bar{p}}^*(\underline{\text{ISing}}^{\mathcal{F}, \mathcal{U}}(X)),$$

induced by the canonical map $IN_{\bullet}^* \rightarrow \mathbf{IN}_{\bullet}^*$. By taking the direct limit of the quasi-isomorphisms of Lemma 4.2, we get an isomorphism

$$H^* \left(\lim_{\mathcal{U} \in \text{Cov}(U)} \tilde{N}_{\bar{p}}^*(\underline{\text{ISing}}^{\mathcal{F}, \mathcal{U}}(U)) \right) \cong H^* \left(\tilde{N}_{\bar{p}}^*(\underline{\text{ISing}}^{\mathcal{F}}(U)) \right) = H_{\text{TW}, \bar{p}}^*(U).$$

In a second step, by following the lines of [9, Exemple 3.9.1.], we prove that the sheaf, $\mathbf{IN}_{\bar{p}}^*$, is soft. The elements of $IN_{\bar{0}}^0(U)$ are $\bar{0}$ -admissible vertices; they are the vertices of the regular part and the map $N^0(U) \rightarrow IN_{\bar{0}}^0(U)$ can be considered as the restriction to the regular part. Also, in this degree 0, the presheaves N^0 and $IN_{\bar{0}}^0$ are clearly sheaves and $N^0(U) \rightarrow IN_{\bar{0}}^0(U)$, is a morphism of sheaves of rings. Observe also that $IN_{\bar{p}}^*(U)$ is an $IN_{\bar{0}}^0(U)$ -module for the cup-product. As the sheaf N^0 is soft, and as (see [9, Théorème 3.7.1.]) any sheaf of modules over a soft sheaf of rings is soft, we deduce the softness of $\mathbf{IN}_{\bar{p}}^*$. Thus, the hypercohomology is the cohomology of the space of sections of the sheaf and we get a series of isomorphisms,

$$H^*(X; \mathbf{IN}_{\bar{p}}^*) \cong H^*(\Gamma(X, \mathbf{IN}_{\bar{p}}^*)) \cong H^*(\tilde{N}_{\bar{p}}^*(\underline{\text{ISing}}^{\mathcal{F}}(X))) = H_{\text{TW}, \bar{p}}^*(X).$$

By definition of the cup_i -products on $\mathbf{IN}_{\bullet}^*$, the following diagram commutes,

$$\begin{array}{ccc} IN_{\bar{p}}^r(X) \otimes IN_{\bar{q}}^s(X) & \xrightarrow{\cup_i} & IN_{\bar{p} \oplus \bar{q}}^{r+s-i}(X) \\ \simeq \downarrow & & \downarrow \simeq \\ \Gamma(X, \mathbf{IN}_{\bar{p}}^r) \otimes \Gamma(X, \mathbf{IN}_{\bar{q}}^s) & \xrightarrow{\cup_i} & \Gamma(X, \mathbf{IN}_{\bar{p} \oplus \bar{q}}^{r+s-i}). \end{array}$$

With the properties already established, the vertical maps are quasi-isomorphisms induced by the canonical map $IN_{\bullet}^* \rightarrow \mathbf{IN}_{\bullet}^*$. The stated result is now a consequence of the definition of Steenrod squares from cup_i -products. $\square$

Lemma 4.2. *Let X be an n -dimensional pseudomanifold and $\mathcal{U}$ be an open cover of X . The canonical inclusion, $\iota: \underline{\text{ISing}}^{\mathcal{F}, \mathcal{U}}(X) \rightarrow \underline{\text{ISing}}^{\mathcal{F}}(X)$, induces an isomorphism in intersection cohomology, for any GM-perversity $\bar{p}$.*

Proof. With Proposition 1.5, we can replace $\tilde{N}^*(-)$ by the blow-up $\tilde{C}^*(-)$, already studied in [4]. Let $\bar{q}$ be the GM-perversity defined by $\bar{p}(k) + \bar{q}(k) = k - 2$. Recall from [4, Theorem B], the existence of a quasi-isomorphism, $\text{eval}: \tilde{C}_{\bar{p}}^*(\underline{K}) \rightarrow \text{hom}(C_{*}^{\text{GM}, \bar{q}}(\underline{K}), \mathbb{F}_2)$, defined for any filtered face set, $\underline{K}$, as follows:

for any $\Phi \in \tilde{C}_{\bar{p}}^*(\underline{K})$, $\sigma: \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow \underline{K}$, we have $\Phi_{\sigma} = \sum_j \Phi_{0, \sigma, j} \otimes \dots \otimes \Phi_{n, \sigma, j} \in C^*(c\Delta^{j_0}) \otimes \dots \otimes C^*(\Delta^{j_n})$ and we set

$$\text{eval}(\Phi)(\sigma) = \sum_j \Phi_{0, \sigma, j}([c\Delta^{j_0}]) \cdot \dots \cdot \Phi_{n, \sigma, j}([\Delta^{j_n}]),$$

where $[-]$ is the maximal simplex. By using it for $\underline{K} = \underline{\mathrm{ISing}}^{\mathcal{F}}(X)$ and $\underline{K} = \underline{\mathrm{ISing}}^{\mathcal{F}, \mathcal{U}}(X)$, we get the following diagram, whose commutativity follows directly from the definitions of maps,

$$\begin{array}{ccc} H_{\mathrm{TW}, \bar{p}}^*(\underline{\mathrm{ISing}}^{\mathcal{F}}(X)) & \xrightarrow{\mathrm{eval}^*} & H_{\mathrm{GM}, \bar{q}}^*(\underline{\mathrm{ISing}}^{\mathcal{F}}(X)) \\ \downarrow \iota_{\mathrm{TW}}^* & & \downarrow \iota_{\mathrm{GM}}^* \\ H_{\mathrm{TW}, \bar{p}}^*(\underline{\mathrm{ISing}}^{\mathcal{F}, \mathcal{U}}(X)) & \xrightarrow{\mathrm{eval}^*} & H_{\mathrm{GM}, \bar{q}}^*(\underline{\mathrm{ISing}}^{\mathcal{F}, \mathcal{U}}(X)). \end{array}$$

We know that the two evaluation maps, eval^* , are quasi-isomorphisms and we have to prove that the map, ι_{TW}^* , induced by the inclusion, ι , is an isomorphism. With the commutativity of the previous diagram, and the fact that the homology is over a field, it is sufficient to prove that

$$\iota_{\mathrm{GM}, *}: H_*^{\mathrm{GM}, \bar{q}}(\underline{\mathrm{ISing}}^{\mathcal{F}, \mathcal{U}}(X)) \rightarrow H_*^{\mathrm{GM}, \bar{q}}(\underline{\mathrm{ISing}}^{\mathcal{F}}(X))$$

is an isomorphism. Set $C_*^{\bar{q}}(X) = C_*^{\mathrm{GM}, \bar{q}}(\underline{\mathrm{ISing}}^{\mathcal{F}}(X))$. Recall from [4, Lemma A.16], the existence of a chain map, which is the classical subdivision, $\mathrm{sd}: C_*^{\bar{q}}(X) \rightarrow C_*^{\bar{q}}(X)$, and, for any integer m , the existence of a homomorphism, $T: C_*^{\bar{q}}(X) \rightarrow C_{*+1}^{\bar{q}}(X)$, such that $\partial T + T\partial = \mathrm{id} - \mathrm{sd}^m$. By construction, for any element $c \in C_*^{\bar{q}}(X)$, there is an integer m such that $\mathrm{sd}^m c \in C_*^{\mathrm{GM}, \bar{q}}(\underline{\mathrm{ISing}}^{\mathcal{F}, \mathcal{U}}(X))$. Moreover, if $c \in C_*^{\mathrm{GM}, \bar{q}}(\underline{\mathrm{ISing}}^{\mathcal{F}, \mathcal{U}}(X))$ then $Tc \in C_*^{\mathrm{GM}, \bar{q}}(\underline{\mathrm{ISing}}^{\mathcal{F}, \mathcal{U}}(X))$. Also, if c is a cycle, $\mathrm{sd}^m c$ is a cycle also and the two homology classes $[c]$ and $[\mathrm{sd}^m c]$ are equal. This implies the surjectivity and the injectivity of $\iota_{\mathrm{GM}, *}$ through a classical argument. $\square$

The second step in the proof of Theorem C is the comparison of the two definitions of Steenrod squares, respectively associated to the sheaf $\mathbf{IN}_\bullet^*$ and to the Deligne sheaf $\mathcal{P}_\bullet^*$. This is a consequence of the comparison of the two associated cup $_i$ -products, done in the next lemma.

Lemma 4.3. *Let X be an n -dimensional topological pseudomanifold and let $\bar{p}, \bar{q}$ be two GM-perversities, such that $\bar{p} \oplus \bar{q} \leq \bar{t}$, where $\bar{p} \oplus \bar{q}$ is the smallest GM-perversity, $\bar{r}$, such that $\bar{p} + \bar{q} \leq \bar{r}$. Then, for any i , there is a commutative square in the derived category of sheaves on X , linking the two cup $_i$ -products,*

$$\begin{array}{ccc} \mathbf{IN}_{\bar{p}}^*(X) \otimes \mathbf{IN}_{\bar{q}}^*(X) & \xrightarrow{\cup_i} & \mathbf{IN}_{\bar{p} \oplus \bar{q}}^*(X) \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{P}_{\bar{p}}^*(X) \otimes \mathcal{P}_{\bar{q}}^*(X) & \xrightarrow{\cup_i} & \mathcal{P}_{\bar{p} \oplus \bar{q}}^*(X), \end{array}$$

and such that vertical arrows are isomorphisms.

Proof. Let $\mathbf{S}^*$ be a differential graded sheaf on the pseudomanifold X . We denote by $\mathbf{S}_k^*$ the restriction of $\mathbf{S}^*$ to the open set $X \setminus X_{n-k}$, for $k \in \{2, \dots, n+1\}$. Recall the conditions (AX1) of [2, V.2.3]:

- (a) $\mathbf{S}^*$ is bounded, $\mathbf{S}^i = 0$ for $i < 0$ and $\mathbf{S}_2^*$ is quasi-isomorphic to the ordinary singular cohomology.
- (b) For any $k \in \{2, \dots, n\}$ and any $x \in X_{n-k} \setminus X_{n-k-1}$, we have $\mathcal{H}^i(\mathbf{S})_x = 0$ if $i > \bar{p}(k)$.

(c) The attachment map, $\alpha_k: \mathbf{S}_{k+1}^* \rightarrow Ri_k^* \mathbf{S}_k^*$, induced by the canonical inclusion $X \setminus X_{n-k} \rightarrow X \setminus X_{n-k-1}$, is a quasi-isomorphism up to $\bar{p}(k)$.

If $\mathbf{S}^*$ is soft, from [20, Remark 2.3.], we may replace condition (c) by the following equivalent one:

(c') for any $k \in \{2, \dots, n\}$, $j \leq \bar{p}(k)$ and $x \in X_{n-k} \setminus X_{n-k-1}$, the restriction map induces an isomorphism,

$$\varinjlim_{U_x} H^j(\Gamma(U_x; \mathbf{S}^*)) \xrightarrow{\cong} \varinjlim_{U_x} H^j(\Gamma(U_x \setminus X_{n-k}; \mathbf{S}^*)),$$

where U_x varies into a cofinal family of neighborhoods of x in $X \setminus X_{n-k-1}$.

On the regular part, the sheaf $\mathbf{IN}_\bullet^*$ is the sheafification of N^* and thus computes the singular cohomology. Therefore, condition (a) is satisfied for $\mathbf{IN}_\bullet^*$. In order to prove that the sheaf $\mathbf{IN}_\bullet^*$ satisfies the axioms (b) and (c'), we use the isomorphism established in Lemma 4.1,

$$H^*(X; \mathbf{IN}_\bullet) \cong H_{\text{TW}, \bullet}^*(X).$$

Let $x \in X_{n-k} \setminus X_{n-k-1}$. The cohomology $\mathcal{H}^*(\mathbf{IN}_\bullet)_x$ is determined by the following isomorphisms,

$$\mathcal{H}^*(\mathbf{IN}_\bullet)_x = \varinjlim_{U_x} H^*(\Gamma(U_x; \mathbf{IN}_\bullet)) \cong \varinjlim_{U_x} H_{\text{TW}, \bullet}^*(U_x),$$

where the direct limits are taken over the open neighborhoods U_x of x . (The first equality is the definition of the stalk at a point.) Moreover, these limits can also be obtained from a restriction to a cofinal family of trivializing open neighborhoods, $U_x \cong \mathbb{R}^{n-k} \times cL$, where L is the link of x . Axiom (b) follows now from $H_{\text{TW}, \bar{p}}^*(\mathbb{R}^{n-k} \times cL) = H_{\text{TW}, \bar{p}}^*(cL) = 0$, if $* > \bar{p}(k)$, see [4, Corollary 1.47].

The verification of (c') is quite similar. As noticed in [20, Proof of Theorem 7.1.], we are reduce to analyze the map,

$$\varinjlim_{U_x} H^*(U_x; \mathbf{IN}_\bullet) \rightarrow \varinjlim_{U_x} H^*(U_x \setminus X_{n-k}; \mathbf{IN}_\bullet),$$

where the direct limit is taken over a cofinal family of trivializing open neighborhoods of x , $U_x \cong \mathbb{R}^{n-k} \times cL$. We consider the following commutative diagram, whose horizontal maps are induced by the canonical inclusions and vertical maps are isomorphisms,

$$\begin{array}{ccccc} H^*(U_x; \mathbf{IN}_\bullet) & \longrightarrow & H^*(U_x \setminus X_{n-k}; \mathbf{IN}_\bullet) & & \\ \cong \downarrow & & \cong \downarrow & & \\ H^*(\mathbb{R}^{n-k} \times cL; \mathbf{IN}_\bullet) & \longrightarrow & H^*(\mathbb{R}^{n-k} \times (cL - \{\mathbf{v}\}); \mathbf{IN}_\bullet) & & \\ \cong \downarrow & & \cong \downarrow & & \\ H^*(cL; \mathbf{IN}_\bullet) & \longrightarrow & H^*(cL - \{\mathbf{v}\}; \mathbf{IN}_\bullet) & \xrightarrow{\cong} & H^*(L; \mathbf{IN}_\bullet). \end{array}$$

Finally, we note that the composite at the bottom is an isomorphism when $* \leq \bar{p}(k)$, as shows the classical computation of the intersection cohomology of a cone. Modulo the vertical isomorphisms, this is exactly the axiom (c').

Therefore, the sheaf $\mathbf{IN}_\bullet^*$ satisfies conditions (AX1) and, by [2, Theorem 2.5], there exists a quasi-isomorphism between $\mathbf{IN}_\bullet^*$ and $\mathcal{P}_\bullet^*$ (see also [12]). As a consequence, these

two sheaves have a common injective resolution and we may apply to it the uniqueness of cup_i-products established by M. Goresky in [10, Proposition 3.6]. $\square$

From the previous results on cup_i-products, we get an isomorphism of algebras of cohomology, with coefficients in $\mathbb{F}_2$.

Corollary 4.4. *If X is an n -dimensional pseudomanifold, there are isomorphisms of perverse algebras,*

$$H_{\text{TW},\bullet}^*(X) \cong H^*(X; \mathbf{IN}_\bullet) \cong H^*(X; \mathcal{P}_\bullet).$$

Moreover, if X is compact and PL, one has also an isomorphism of algebras,

$$H^*(X; \mathcal{P}_\bullet) \cong H_{n-*}^{\bar{t}-\bullet}(X; \mathbb{F}_2),$$

with the intersection product on the last term.

Proof. The two first isomorphisms are consequences of the previous results on cup_i-products. The last one is established by G. Friedman in [6]. $\square$

If we are interested only by the cup-product, $\cup_0$, we may consider versions of the sheaves, $\mathbf{IN}$ and $\mathcal{P}$, over any field. In this case, the previous corollary is still true for any field and not only for $\mathbb{F}_2$. With more work of this type, one should be able also to show the existence of an isomorphism between our definition of cup-product and the definition of G. Friedman and J. E. McClure ([8]).

5. PSEUDOMANIFOLDS WITH ISOLATED SINGULARITIES

In this section, we determine Steenrod squares on the intersection cohomology of pseudomanifolds with isolated singularities. In this case, if the pseudomanifold is of dimension n , the perversity $\bar{p}$ is determined by one number, $\bar{p}(n)$. Recall now that the intersection cohomology of a cone cY on a space Y is given by $H_{\text{TW},\bar{p}}^r(cY) = H^r(Y)$, if $r \leq \bar{p}(n)$ and 0 otherwise.

Proposition 5.1. *Let $\bar{p}$ be a GM-perversity and X be an n -dimensional pseudomanifold obtained from a triangulated manifold with boundary, $(W, \partial W)$, by attaching cones on the connected components, $(\partial_u W)_{u \in I}$, of ∂W , i.e., X is the push out*

$$\begin{array}{ccc} \partial W = \sqcup_{u \in I} \partial_u W & \xrightarrow{\iota} & W \\ \downarrow & & \downarrow \\ \sqcup_{u \in I} c(\partial_u W) & \longrightarrow & X. \end{array}$$

We filter the pseudomanifold X by $\emptyset \subset \{\mathbf{v}_u \mid u \in I\} \subset X$, where $\mathbf{v}_u$ is the cone point of $c(\partial_u W)$. Then, the following properties are satisfied.

- (i) *The cochain complex, $\tilde{N}_{\bar{p}}^*(X)$, is quasi-isomorphic to the pullback in the category of cochain complexes, $N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq \bar{p}(n)} N^*(\partial W)$, where $\tau_{\leq \bar{p}(n)} N^*(\partial W)$ is the usual truncation (see [2, Page 52]),*

$$(\tau_{\leq \bar{p}(n)} N^*(\partial W))^r = \begin{cases} N^r(\partial W) & \text{if } r < \bar{p}(n), \\ \mathcal{Z}N^{\bar{p}(n)}(\partial W) & \text{if } r = \bar{p}(n), \\ 0 & \text{if } r > \bar{p}(n), \end{cases}$$

in which $\mathcal{Z}$ denotes the vector space of cocycles. Moreover, the GM-perverse $\mathcal{E}(2)$ -algebra, $\bar{p} \mapsto \tilde{N}_{\bar{p}}^*(X)$, is quasi-isomorphic to the pullback in the category of GM-perverse $\mathcal{E}(2)$ -algebras, defined by $\bar{p} \mapsto N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq \bar{p}(n)} N^*(\partial W)$, with the $\mathcal{E}(2)$ -algebra structure on $N^*(-)$ defined in [1].

(ii) The intersection cohomology of X is determined by

$$H_{\text{TW}, \bar{p}}^k(X) = \begin{cases} H^k(W) & \text{if } k \leq \bar{p}(n), \\ \text{Ker}(H^k(W) \rightarrow H^k(\partial W)) & \text{if } k = \bar{p}(n) + 1, \\ H^k(W, \partial W) & \text{if } k > \bar{p}(n) + 1. \end{cases}$$

(iii) If $(\alpha, \iota^* \alpha) \in N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq \bar{p}(n)} N^*(\partial W)$ is a cocycle of $\bar{p}$ -intersection and i is a positive integer, we have

$$\text{Sq}^i(\alpha, \iota^* \alpha) = (\text{Sq}^i \alpha, \iota^* \text{Sq}^i \alpha) \in H_{\text{TW}, \mathcal{L}(\bar{p}, i)}^{*+i}(X).$$

Proof. (i) Starting from a triangulation of $(W, \partial W)$, we may suppose that X , W , ∂W and $\sqcup_{u \in I} c(\partial_u W)$ are triangulated in such a way that any simplex of the triangulation of X is filtered, for the filtration $\emptyset \subset \{\mathbf{v}_u \mid u \in I\} \subset X$.

Let Y be one of the spaces above and Y^τ be the associated triangulated space. In [7, Chapter 3 and Chapter 5], G. Friedman proves that the cochains $C_{\text{GM}, \bar{p}}^*(Y)$ and $C_{\text{GM}, \bar{p}}^*(Y^\tau)$ are quasi-isomorphic for any GM-perversity $\bar{p}$. Let $\bar{p}$ and $\bar{q}$ be two GM-perversities such that $\bar{p}(k) + \bar{q}(k) = k - 2$. There exists a quasi-isomorphism between $C_{\text{GM}, \bar{q}}^*(Y)$ and $\tilde{C}_{\bar{p}}^*(Y)$, see [4, Theorem B] or Proposition 1.8. Recall also from Proposition 1.5 the existence of a quasi-isomorphism between $\tilde{C}_{\bar{p}}^*(Y)$ and $\tilde{N}_{\bar{p}}^*(Y)$. Thus, the isomorphism,

$$C_{\text{GM}, \bar{q}}^*(X^\tau) \cong C^*(W^\tau) \oplus_{(\oplus_{u \in I} C^*(\partial_u W^\tau))} (\oplus_{u \in I} C_{\text{GM}, \bar{q}}^*(c(\partial_u W)^\tau)),$$

obtained by construction of the triangulations, gives quasi-isomorphisms,

$$\begin{aligned} C_{\text{GM}, \bar{q}}^*(X) &\simeq C^*(W) \oplus_{C^*(\partial W)} (\oplus_{u \in I} C_{\text{GM}, \bar{q}}^*(c(\partial_u W))) \\ &\simeq C^*(W) \oplus_{C^*(\partial W)} (\oplus_{u \in I} \tau_{\leq \bar{t}(n) - \bar{q}(n)} C^*(\partial_u W)) \\ &\simeq C^*(W) \oplus_{C^*(\partial W)} \tau_{\leq \bar{t}(n) - \bar{q}(n)} C^*(\partial W) \\ &\simeq N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq \bar{t}(n) - \bar{q}(n)} N^*(\partial W). \end{aligned}$$

Therefore we have obtained a quasi-isomorphism,

$$\tilde{N}_{\bar{p}}^*(X) \simeq N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq \bar{p}(n)} N^*(\partial W).$$

We investigate now the structure of $\mathcal{E}(2)$ -algebra. In [1], C. Berger and B. Fresse prove that a restriction map, $N^*(Y) \rightarrow N^*(Z)$, induced by an inclusion $Z \hookrightarrow Y$, is a morphism of $\mathcal{E}(2)$ -algebras. Therefore, we obtain functors from the lattice of GM-perversities (and ∞) to GM-perverse $\mathcal{E}(2)$ -algebras, defined by $\bar{p} \mapsto \tilde{N}_{\bar{p}}^*(X)$, $\bar{p} \mapsto N^*(W)$, $\bar{p} \mapsto N^*(\partial W)$, $\bar{p} \mapsto \tau_{\leq \bar{p}(n)} N^*(\partial W)$. Restriction maps define GM-perverse $\mathcal{E}(2)$ -algebra maps between $\tilde{N}_{\bar{p}}^*(X)$ and the three other GM-perverse $\mathcal{E}(2)$ -algebras. From them, we obtain a GM-perverse $\mathcal{E}(2)$ -algebra map

$$(4) \quad \tilde{N}_{\bar{p}}^*(X) \rightarrow N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq \bar{p}(n)} N^*(\partial W),$$

whose codomain is a pullback in the category of GM-perverse $\mathcal{E}(2)$ -algebras, see [1]. We have proved above that this last map is a quasi-isomorphism for each $\bar{p}$ and the first item of the statement is established.

(ii) An element of the previous sum is of the type $(\alpha, \iota^* \alpha)$, with $\iota^* \alpha$ of degree less than, or equal to, $\bar{p}(n)$. This means that, if α is of degree k , we must have

$$\begin{cases} \iota^* \alpha = 0 & \text{if } k > \bar{p}(n), \\ \iota^* \alpha \text{ is a cocycle} & \text{if } k = \bar{p}(n), \\ \text{no condition} & \text{if } k < \bar{p}(n). \end{cases}$$

This implies immediately $H_{\text{TW}, \bar{p}}^k(X) = H^k(W)$ if $k \leq \bar{p}(n)$ and that $H_{\text{TW}, \bar{p}}^k(X) = H^k(W, \partial W)$ if $k > \bar{p}(n) + 1$. In degree $k = \bar{p}(n) + 1$, the $\bar{p}$ -intersection cohomology of X is formed of the elements of $H^k(W)$ which are in the image of $H^k(W, \partial W)$, i.e., the kernel of $H^k(W) \rightarrow H^k(\partial W)$.

(iii) The quasi-isomorphisms between $\tilde{N}_{\bar{p}}^*(X)$ and $N^*(W) \oplus_{N^*(\partial W)} \tau_{\leq \bar{p}(n)} N^*(\partial W)$ defining a map of GM-perverse $\mathcal{E}(2)$ -algebras, they are compatible with the cup_i -products, and the right-hand complex of (4) can be used for the determination of cup_i -products, i.e., we have

$$(\alpha, \iota^* \alpha) \cup_i (\beta, \iota^* \beta) = (\alpha \cup_i \beta, \iota^* \alpha \cup_i \iota^* \beta),$$

from which we deduce the announced formula for Steenrod squares. $\square$

Remark 5.2. This remark gives a direct proof of the Goresky-Pardon conjecture in the case of isolated singularities. Let $(\alpha, \iota^* \alpha)$ be a cocycle in $N^k(W) \oplus_{N^k(\partial W)} \tau_{\leq \bar{p}(n)} N^k(\partial W)$. The perverse degree of the Steenrod square, $\text{Sq}^j(\alpha, \iota^* \alpha) = (\alpha, \iota^* \alpha) \cup_{k-j} (\alpha, \iota^* \alpha)$, verifies

$$\|(\alpha, \iota^* \alpha) \cup_{k-j} (\alpha, \iota^* \alpha)\| \leq_{(1)} |\iota^* \alpha \cup_{k-j} \iota^* \alpha| \leq_{(2)} k + j \leq_{(3)} \bar{p}(n) + j,$$

where

- $\leq_{(1)}$ comes from the fact that the perverse degree of a cochain is less than, or equal, to its usual degree,
- $\leq_{(2)}$ is a consequence of $|a \cup_i b| \leq |a| + |b| - i$,
- $\leq_{(3)}$ uses $\iota^* \alpha = 0$ if $k > \bar{p}(n)$.

Remark 5.3. The fact that the image of $H_{\text{TW}, \bar{p}}^*(X)$ by Sq^i is in perversity $\mathcal{L}(\bar{p}, i) = \min(2\bar{p}, \bar{p} + i)$ is perfectly in phase with the characterization of the intersection cohomology of X , written in Proposition 5.1.(ii). This remark follows from the next observations for a cocycle $(\alpha, \iota^* \alpha) \in N^k(W) \oplus_{N^k(\partial W)} \tau_{\leq \bar{p}(n)} N^k(\partial W)$.

- If $k \leq \bar{p}(n)$, then, by definition of the Steenrod squares in $H^*(W)$, we have $|\text{Sq}^i(\alpha)| = k + i \leq \bar{p}(n) + i$.
 - If $i \leq \bar{p}(n)$, this implies $|\text{Sq}^i(\alpha, \iota^* \alpha)| \leq \mathcal{L}(\bar{p}, i)(n)$.
 - If $i > \bar{p}(n)$, we have $\text{Sq}^i(\alpha, \iota^* \alpha) = (\text{Sq}^i \alpha, \iota^* \text{Sq}^i \alpha) = 0$.
- If $k > \bar{p}(n)$, then $\iota^* \alpha = 0$ and $|\text{Sq}^i(\alpha, \iota^* \alpha)| = k + i > \bar{p}(n) + i \geq \mathcal{L}(\bar{p}, i)(n)$.

In conclusion, Sq^i respects the cæsurae in the determination of the perverse cohomologies, $H_{\text{TW}, \bar{p}}^*(X)$ and $H_{\text{TW}, \mathcal{L}(\bar{p}, i)}^*(X)$. Moreover, in degrees $k \leq \bar{p}(n)$, the Steenrod squares on $H_{\text{TW}, \bar{p}}^k(X)$ coincide with the Steenrod squares on $H^k(W)$.

Example 5.4 (*Steenrod squares on the intersection cohomology of the suspension of a manifold*). Let X be an $(n-1)$ -dimensional manifold and $\bar{p}$ be a GM-perversity. The following pushout defines ΣX , as in Proposition 5.1,

$$\begin{array}{ccc} X_{-1} \sqcup X_1 & \xrightarrow{\iota_{-1} \sqcup \iota_1} & X \times [-1, 1] \\ \downarrow & & \downarrow \\ cX_{-1} \sqcup cX_1 & \longrightarrow & \Sigma X, \end{array}$$

where $\iota_1: X_1 = X \times \{1\} \rightarrow X \times [-1, 1]$ and $\iota_{-1}: X_{-1} = X \times \{-1\} \rightarrow X \times [-1, 1]$ are the canonical injections. From (i) of Proposition 5.1, we know that $\tilde{N}_{\bar{p}}^*(\Sigma X)$ is quasi-isomorphic to the cochain complex

$$(5) \quad \begin{aligned} N^*(X \times [-1, 1])^{<\bar{p}(n)} &\oplus \{ \alpha \in N^{\bar{p}(n)}(X \times [-1, 1]) \mid d\iota_1^*(\alpha) = d\iota_{-1}^*(\alpha) = 0 \} \\ &\oplus (\text{Ker } \iota_1^* \cap \text{Ker } \iota_{-1}^*)^{>\bar{p}(n)}, \end{aligned}$$

in which the superscript refers to the degree. For instance, $(A)^{<k}$ is the set of elements of A of degree less than k .

The suspension, ΣX , can also be obtained as a cofiber, $X_1 \sqcup X_{-1} \rightarrow X \times [-1, 1] \rightarrow \Sigma X$, which gives a short exact sequence,

$$0 \rightarrow (\text{Ker } \iota_1^* \cap \text{Ker } \iota_{-1}^*) \hookrightarrow N^*(X \times [-1, 1]) \xrightarrow{(\iota_1^*, \iota_{-1}^*)} N^*(X_1) \oplus N^*(X_{-1}) \rightarrow 0.$$

The morphism of $\mathcal{E}(2)$ -algebras $([1]), N^*(\Sigma X) \rightarrow N^*(X \times [-1, 1])$, lifts as a quasi-isomorphism of cochain complexes, $N^*(\Sigma X) \rightarrow (\text{Ker } \iota_1^* \cap \text{Ker } \iota_{-1}^*)$. From (5) and the previous observation, we deduce the intersection cohomology of the suspension ΣX , as

$$H_{\text{TW}, \bar{p}}^k(\Sigma X) = \begin{cases} H^k(X) & \text{if } k \leq \bar{p}(n), \\ 0 & \text{if } k = \bar{p}(n) + 1, \\ H^k(\Sigma X) = H^{k-1}(X) & \text{if } k > \bar{p}(n) + 1. \end{cases}$$

With Remark 5.3, we know that, in degrees $k \leq \bar{p}(n)$, the Steenrod squares on $H_{\text{TW}, \bar{p}}^k(\Sigma X)$ coincide with the Steenrod squares on $H^k(X)$. Moreover, the intersection of kernels being endowed with the induced structure of $\mathcal{E}(2)$ -algebra of $N^*(X \times [-1, 1])$, the quasi-isomorphism $N^*(\Sigma X) \rightarrow (\text{Ker } \iota_1^* \cap \text{Ker } \iota_{-1}^*)$ is a morphism of $\mathcal{E}(2)$ -algebras, see [1]. Thus, in degrees $k > \bar{p}(n) + 1$, the Steenrod squares on $H_{\text{TW}, \bar{p}}^k(\Sigma X)$ coincide with the Steenrod squares on $H^k(\Sigma X)$, which are the suspensions of the Steenrod squares on X .

We consider now the case of the Thom space of a vector bundle, $\mathbb{R}^m \rightarrow E \rightarrow B$.

Example 5.5 (*Steenrod squares on the intersection cohomology of a Thom space*). Let $\mathbb{R}^m \rightarrow D_E \xrightarrow{g} B$ be the disk-bundle of associated sphere-bundle $S^{m-1} \rightarrow S_E \xrightarrow{f} B$. The Thom space, $\text{Th}(E)$, is built from the disk-bundle along the process described in Proposition 5.1. We filter $\text{Th}(E)$ by the point of compactification. Let $\bar{p}$ be a GM-perversity entirely determined in this case by the number $\bar{p}(n)$ with $n = \dim E$. In this example, we prove that the Steenrod squares on $H_{\text{TW}, \bar{p}}^*(\text{Th}(E))$ are entirely determined by the Steenrod squares on the base space and the Stiefel-Whitney classes of the bundle.

Denote by $c \in H^m(B)$ the Euler class and by $\theta \in H^m(\text{Th}(E))$ the Thom class. Let $j: D_E \rightarrow \text{Th}(E)$ be the canonical map and recall that the Thom isomorphism,

$\mathcal{T}h: H^{k-m}(B) \rightarrow H^k(\text{Th}(E)) \cong H^k(D_E, S_E)$, is defined by $\mathcal{T}h(\gamma) = g^*(\gamma) \cup \theta$. The Euler and the Thom classes are connected by the two exact sequences,

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^k(\text{Th}(E)) & \xrightarrow{j^*} & H^k(D_E) & \longrightarrow & H^k(S_E) \longrightarrow H^{k+1}(\text{Th}(E)) \longrightarrow \dots \\ & & \uparrow \mathcal{T}h & & \uparrow g^* & & \parallel \\ \dots & \longrightarrow & H^{k-m}(B) & \xrightarrow{-\cup c} & H^k(B) & \xrightarrow{f^*} & H^k(S_E) \longrightarrow H^{k+1-m}(B) \longrightarrow \dots \\ & & & & & & \uparrow \mathcal{T}h \end{array}$$

and $j^*(\theta) = g^*(c)$. From Proposition 5.1, we know that the complex, $\tilde{N}_{\bar{p}}^*(\text{Th}(E))$, is quasi-isomorphic to

$$\begin{aligned} \mathcal{N}^* &= N^*(B) \oplus_{N^*(S_E)} \tau_{\leq \bar{p}(n)} N^*(S_E) \\ &\cong N^{< \bar{p}(n)}(B) \oplus \{\alpha \in N^{\bar{p}(n)}(B) \mid df^*(\alpha) = 0\} \\ &\quad \oplus (\text{Ker}(N^k(B) \xrightarrow{f^*} N^k(S_E)))^{> \bar{p}(n)}. \end{aligned}$$

Thus, we recover (see [16, Page 77]) the intersection cohomology of the Thom space,

$$H_{\text{TW}, \bar{p}}^k(\text{Th}(E)) = H^k(\mathcal{N}) = \begin{cases} H^k(B) & \text{if } k \leq \bar{p}(n), \\ (\text{Im}(-\cup c))^k & \text{if } k = \bar{p}(n) + 1, \\ H^{k-m}(B) \cong_{\mathcal{T}h} H^k(\text{Th}(E)) & \text{if } k > \bar{p}(n) + 1. \end{cases}$$

- In the case $k \leq \bar{p}(n) + 1$, the Steenrod squares, $\text{Sq}^i: H_{\bar{p}}^k(\text{Th}(E)) \rightarrow H_{\mathcal{L}(\bar{p}, i)}^{k+i}(\text{Th}(E))$, coincide with the Steenrod squares, $\text{Sq}^i: H^k(B) \rightarrow H^{k+i}(B)$, cf. Remark 5.3.
- Let $k > \bar{p}(n) + 1$ and $\gamma \in H^{k-m}(B)$. The (classical) internal Cartan formula gives,

$$\begin{aligned} \text{Sq}^j(g^*(\gamma) \cup \theta) &= \sum_{\ell=0}^j \text{Sq}^{j-\ell}(g^*(\gamma)) \cup \text{Sq}^\ell(\theta) \\ &= \sum_{\ell=0}^j g^*(\text{Sq}^{j-\ell}(\gamma)) \cup g^*(\omega_\ell) \cup \theta, \end{aligned}$$

where the ω_ℓ 's are the Stiefel-Whitney classes of the fibration f , see [19, Page 91]. Set $\mu = g^*(\gamma) \cup \theta = \mathcal{T}h(\gamma) \in H^k(\text{Th}(E))$. In this range of degrees, the Steenrod squares on $\text{Th}(E)$, denoted by Sq_{Th} , and the Steenrod squares on B , denoted by Sq_B , are related by

$$\text{Sq}_{\text{Th}}^j(\mu) = \sum_{\ell=0}^j g^*(\text{Sq}_B^{j-\ell}(\gamma) \cup \omega_\ell) \cup \theta.$$

With the Thom isomorphism, $\mathcal{T}h: H^{k-m}(B) \rightarrow H^k(\text{Th}(E))$, the previous formula can be written as,

$$\text{Sq}_{\text{Th}}^j(\mu) = \mathcal{T}h \left(\sum_{\ell=0}^j \text{Sq}_B^{j-\ell}(\mathcal{T}h^{-1}(\mu)) \cup \omega_\ell \right).$$

6. EXAMPLE OF A FIBRATION WITH FIBER A CONE

In this section, we construct an example showing the interest of the lifting of the image of Sq^i to the perversity $\mathcal{L}(\bar{p}, i)$ instead of $2\bar{p}$. As the case of Sq^1 was analyzed in [13], we choose an example with Sq^2 .

Proposition 6.1. *There exists a pseudomanifold X and a GM-perversity $\bar{p}$, with an explicit non-trivial perverse square,*

$$\text{Sq}^2 \neq 0: H_{\text{TW}, \bar{p}}^6(X) \rightarrow H_{\text{TW}, \mathcal{L}(\bar{p}, 2)}^8(X),$$

whose composition with the canonical map $H_{\text{TW}, \mathcal{L}(\bar{p}, 2)}^8(X) \rightarrow H_{\text{TW}, 2\bar{p}}^8(X)$ is zero.

Proof. To begin with, we describe the general strategy of the proof. The first step is the construction of a fibration, $S^7 \times S^4 \rightarrow E \xrightarrow{\varphi} \mathbb{C}P(2)$, with a non-trivial differential on a generator a_7 of $H^7(S^7 \times S^4)$, in the Serre spectral sequence. Secondly, we consider the fiberwise conification, $c(S^7 \times S^4) \rightarrow X \xrightarrow{\psi} \mathbb{C}P(2)$, of the fibration φ . The space X is a pseudomanifold. A GM-perversity, $\bar{p}$, on X is determined by the value $\bar{p}(12) = k$ and we denote it by $\bar{k}$. (As $\bar{p}$ is a GM-perversity, we have $k \leq 10$.) In our fibration, depending on the value of k , the element a_7 is a class of $\bar{p}$ -intersection or not; more precisely, we get $H_{\text{TW}, \bar{k}}^8(X) \neq 0$ if $k = 6$ and $H_{\text{TW}, \bar{k}}^8(X) = 0$ if $k = 8$. This property generates a non-trivial Steenrod square, $\text{Sq}^2: H_{\text{TW}, \bar{p}}^6(X) \rightarrow H_{\text{TW}, \mathcal{L}(\bar{p}, 2)}^8(X)$, such that the composite with the canonical map, $H_{\text{TW}, \bar{p}}^6(X) \xrightarrow{\text{Sq}^2} H_{\text{TW}, \mathcal{L}(\bar{p}, 2)}^8(X) \rightarrow H_{\text{TW}, 2\bar{p}}^8(X)$, is the zero map. Details are as follows.

- First, we observe, from the cellular approximation theorem and the construction of $K(\mathbb{Z}, 8)$, that the classifying map of the top class, $\mathbb{C}P(2) \times S^4 \rightarrow K(\mathbb{Z}, 8)$, lifts as a map $f: \mathbb{C}P(2) \times S^4 \rightarrow S^8$. We denote by $p_1: E \rightarrow \mathbb{C}P(2) \times S^4$ the pullback of the Hopf fibration, $S^{15} \rightarrow S^8$, along f . We compose p_1 with the trivial fibration, $p_2: \mathbb{C}P(2) \times S^4 \rightarrow \mathbb{C}P(2)$ and obtain a fibration

$$\varphi: E \rightarrow \mathbb{C}P(2),$$

whose fiber, F , is $S^7 \times S^4$. To show this last point, consider the next commutative diagram:

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & S^{15} \\ \downarrow & \boxed{2} & \downarrow p_1 & \boxed{3} & \downarrow \\ S^4 & \longrightarrow & \mathbb{C}P(2) \times S^4 & \xrightarrow{f} & S^8 \\ \downarrow & \boxed{1} & \downarrow p_2 & & \\ * & \longrightarrow & \mathbb{C}P(2) & & \end{array}$$

The rectangle formed of $\boxed{1}$ and $\boxed{2}$ is a pullback. As $\boxed{1}$ is a pullback, we deduce ([17, Section III.4]) that $\boxed{2}$ is a pullback. Therefore, the rectangle formed of $\boxed{2}$ and $\boxed{3}$ is a pullback and the triviality of the map $S^4 \rightarrow S^8$ implies that F is $S^7 \times S^4$.

We study now the Serre spectral sequence of the fibration φ . We denote by a_4 , a_7 and $a_7 \times a_4$ the generators of the reduced cohomology of $S^7 \times S^4$ and by x and x^2 the generators of the reduced cohomology of $\mathbb{C}P(2)$. An inspection of the degrees in the differentials, $d_r: E_r^{s,t} \rightarrow E_r^{s+r,t-r+1}$, shows that the only differential which can be potentially non-trivial is

$$d_4: E_4^{0,7} = E_2^{0,7} = \mathbb{F}_2 a_7 \rightarrow E_4^{4,4} = E_2^{4,4} = \mathbb{F}_2(x^2 \otimes a_4).$$

By definition of $S^7 \rightarrow E \rightarrow \mathbb{C}P(2) \times S^4$ as a pullback of the Hopf fibration, we already know ([21, Section III.4]) that the top class a_7 of S^7 transgresses on the product $x^2 \otimes a_4$. This gives $d_4(a_7) = x^2 \otimes a_4$ in the Serre spectral sequence of the fibration $\varphi: E \rightarrow \mathbb{C}P(2)$.

We continue with the determination of the image of the cohomology class $x \otimes a_4$ by Sq^2 in $H^*(\mathbb{C}P(2)) \otimes H^*(S^7 \times S^4)$. From the external Cartan formula, we have

$$\text{Sq}^2(x \otimes a_4) = \text{Sq}^2(x) \otimes a_4 + \text{Sq}^1(x) \otimes \text{Sq}^1(a_4) + x \otimes \text{Sq}^2(a_4).$$

The last two terms are zero, for degree reasons. The equality $\text{Sq}^2(x) = x^2$ gives

$$\text{Sq}^2(x \otimes a_4) = x^2 \otimes a_4.$$

• The second step is the fiberwise conification, $c(S^7 \times S^4) \rightarrow X \xrightarrow{\psi} \mathbb{C}P(2)$, of the fibration φ . If $x \in \mathbb{C}P(2)$, we denote by $(S^7 \times S^4)_x$ the fiber over x and by $\mathbf{v}_x$ the cone point of the cone $c((S^7 \times S^4)_x)$. A continuous section μ of ψ , defined by $\mu(x) = \mathbf{v}_x$, identifies $\mathbb{C}P^2$ to a closed subspace of X . We filter X by $\emptyset \subset X_0 = \mathbb{C}P(2) \subset X$. Observe that the singular set in X is $\mathbb{C}P(2)$ and that the link of a singular point is $S^7 \times S^4$.

Let $\bar{k}$ be a GM-perversity. The intersection cohomology, $H_{\text{TW},\bar{k}}^*(X)$, is the abutment (see [5, Theorem 3.5]) of a Serre spectral sequence with

$$\bar{k}E_2^{r,s} = H^r(\mathbb{C}P(2)) \otimes H_{\text{TW},\bar{k}}^s(c(S^7 \times S^4)).$$

We may replace the right-hand term of this tensor product by its value and obtain

$$\bar{k}E_2^{r,s} = H^r(\mathbb{C}P(2)) \otimes H^s(S^7 \times S^4),$$

if $s \leq k$ and 0 otherwise. The existence of a morphism, $E \rightarrow X$, over the identity on $\mathbb{C}P(2)$, gives a morphism of spectral sequences, $(\bar{k}E_*^{r,s}, d_*) \rightarrow (E_*^{r,s}, d_*)$. From our previous determination of the Serre spectral sequence, $(E_*^{r,s}, d_*)$, associated to the fibration $\varphi: E \rightarrow \mathbb{C}P(2)$, we deduce that the differentials d_* of $\bar{k}E_*^{r,s}$ are zero, except $d_4(a_7) = x^2 \otimes a_4$, if $7 \leq k$. Thus, in perversity $k < 7$, as the class a_7 is not of $\bar{k}$ -intersection, the class $x^2 \otimes a_4$ survives and $H_{\text{TW},\bar{k}}^8(X) \neq 0$. But, if $k = 8$, the class a_7 is of $\bar{k}$ -intersection and kills the element $x^2 \otimes a_4$ (which is the only element of degree 8 in the E_2 -term). Thus $H_{\text{TW},\bar{8}}^8(X) = 0$.

The square Sq^2 , that we have previously determined, arises in the GM-perversity $\bar{4}$ and we have

$$\text{Sq}^2: H_{\text{TW},\bar{4}}^6(X) = \mathbb{F}_2(x \otimes a_4) \rightarrow H_{\text{TW},\bar{4}}^8(X) = \mathbb{F}_2(x^2 \otimes a_4).$$

Observe that $\bar{6} = \mathcal{L}(\bar{4}, \bar{4} + 2)$ is a GM-perversity and thus, with the argument above, Sq^2 still survives as map from $H_{\text{TW},\bar{4}}^6$ to $H_{\text{TW},\bar{6}}^8 = H_{\text{TW},\bar{4}}^8$. But, for the GM-perversity

$\bar{8} = 2 \times \bar{4}$, as $H_{\text{TW},\bar{8}}^8(X) = 0$, this square Sq^2 disappears if we express it as a map from $H_{\text{TW},\bar{4}}^6$ to $H_{\text{TW},2 \times \bar{4}}^8$. $\square$

7. TOPOLOGICAL INVARIANCE OF THE STEENROD SQUARES IN INTERSECTION COHOMOLOGY

In the case of PL-pseudomanifolds, we know from [10] that the Steenrod squares are topological invariants, as homomorphisms $H_{\text{TW},\bar{p}}^r(X) \rightarrow H_{\text{TW},2\bar{p}}^{r+i}(X)$. In this section, we prove that the lifting we have introduced before, $\text{Sq}^i: H_{\text{TW},\bar{p}}^r(X) \rightarrow H_{\text{TW},\mathcal{L}(\bar{p},i)}^{r+i}(X)$, is also a topological invariant. The proof is based on the original combinatorial description of Steenrod squares made in [22].

Theorem D. *Let X be an n -dimensional PL-pseudomanifold and $\bar{p}$ be a GM-perversity. Then, the Steenrod squares, $\text{Sq}^i: H_{\text{TW},\bar{p}}^*(X) \rightarrow H_{\text{TW},\mathcal{L}(\bar{p},i)}^{*+i}(X)$, do not depend on the stratification of X .*

Theorem D is a direct consequence of Proposition 7.1 and Proposition 7.9. Before stating and proving these two results, we need to introduce some material. First, recall from [15, Page 150] and [7, Chapter 2], the existence of a PL-pseudomanifold, X^* , which is an intrinsic coarsest stratification of X , together with a stratified map, $\nu: X \rightarrow X^*$, defined by the identity map, see [4, Definition A.18]. In [15], H. King proves that ν induces a quasi-isomorphism between the Goresky-MacPherson chain (and cochain) complexes. Here we consider the map χ , induced by ν between the Thom-Whitney complexes.

Proposition 7.1. *Let X be an n -dimensional PL-pseudomanifold and $\bar{p}$ be a GM-perversity. Then the canonical map, $\nu: X \rightarrow X^*$, induces a quasi-isomorphism, $\chi: \tilde{N}_{\bar{p}}^*(X^*) \rightarrow \tilde{N}_{\bar{p}}^*(X)$.*

Construction of χ , the local step. Before giving the proof, we detail the construction of χ , based on the effect of $\nu: X \rightarrow X^*$ on filtered simplices of X . Let $\sigma: \Delta = \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow X$ be a filtered simplex of X . Suppose that,

- for some integer $0 \leq i \leq n-1$, the set $\sigma(\Delta^{j_0} * \dots * \Delta^{j_i}) \setminus \sigma(\Delta^{j_0} * \dots * \Delta^{j_{i-1}})$ is included in an i -stratum of X which “disappears” inside an $(i+1)$ -stratum of X^* ,
- for the other indices, $\ell \neq i$, the corresponding strata of $\sigma(\Delta)$ stay unmodified.

Then, the filtered simplex $\sigma: \Delta = \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow X$ becomes a filtered simplex of X^* , $\nu \circ \sigma: \Delta(i) = \Delta^{k_0} * \dots * \Delta^{k_n} \rightarrow X^*$, with

$$(6) \quad \begin{cases} k_\ell = j_\ell & \text{if } \ell < i \text{ or } \ell > i+1, \\ k_i = -1 & \text{and } k_{i+1} = j_i + j_{i+1} + 1. \end{cases}$$

This process is called an *elementary amalgamation*. In general, the simplex $\nu \circ \sigma: \Delta \rightarrow X^*$ can be written as a filtered simplex after a finite number of elementary amalgamations. As we work with blow-ups, we need to consider two cases, depending if $i+1 = n$ or not. We write

$$\tilde{N}^*(\Delta) = N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(c\Delta^{j_i}) \otimes N^*(c\Delta^{j_{i+1}}) \otimes \dots \otimes N^*(\Delta^{j_n})$$

and

$$\tilde{N}^*(\Delta(i)) = \begin{cases} N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\emptyset) \otimes N^*(c\Delta^{j_i+j_{i+1}+1}) \otimes \cdots \otimes N^*(\Delta^{j_n}), & \text{if } i \neq n-1, \\ N^*(c\Delta^{j_0}) \otimes \cdots \otimes N^*(c\emptyset) \otimes N^*(\Delta^{j_{n-1}+j_n+1}), & \text{if } i = n-1. \end{cases}$$

We define below two morphisms,

$$\alpha: N^*(c\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(c\Delta^b) \text{ and } \beta: N^*(c\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(\Delta^b),$$

which correspond to the cases $i \neq n-1$ and $i = n-1$.

Let $\mathbf{v}$ be the cone point of $c\emptyset$. We use α and β for the definition of a morphism $\xi_i: \tilde{N}^*(\Delta(i)) \rightarrow \tilde{N}^*(\Delta)$ as follows. If $\Phi = \sum_j \Phi_{0,j} \otimes \cdots \otimes \Phi_{n,j} \in \tilde{N}^*(\Delta(i))$, we set

- for $i \neq n-1$,

$$(7) \quad \xi_i(\Phi) = \sum_j \Phi_{i,j}([\mathbf{v}]) \cdot \Phi_{0,j} \otimes \cdots \otimes \Phi_{i-1,j} \otimes \alpha(\Phi_{i+1,j}) \otimes \Phi_{i+2,j} \otimes \cdots \otimes \Phi_{n,j},$$

- for $i = n-1$,

$$(8) \quad \xi_i(\Phi) = \sum_j \Phi_{n-1,j}([\mathbf{v}]) \cdot \Phi_{0,j} \otimes \cdots \otimes \Phi_{n-2,j} \otimes \beta(\Phi_{n,j}).$$

These ξ_i 's are the local ingredients used in the (global) definition of χ , stated below.

Construction of α : $N^*(c\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(c\Delta^b)$. We define α by its values on the elements of a basis. If L is one of the simplicial complexes, $c\Delta^a$, $c\Delta^b$ or $c\Delta^{a+b+1}$, we denote by $\{1_F\}$ the dual basis of $N^*(L)$ obtained from the basis of faces, F , of L .

If we represent by F_a the faces of Δ^a and by F_b the faces of Δ^b , a face of $c\Delta^{a+b+1}$ is of the type $c(F_a * F_b)$ or $F_a * F_b$, where F_a and F_b can also be the emptyset. A linear map α is entirely determined by

$$\begin{cases} \alpha(1_{c(F_a * F_b)}) &= 1_{cF_a} \otimes 1_{cF_b}, & \text{the cases } F_a = \emptyset, F_b = \emptyset \text{ being included,} \\ \alpha(1_{F_a * F_b}) &= 1_{cF_a} \otimes 1_{F_b}, & \text{if } F_b \neq \emptyset, \text{ the case } F_a = \emptyset \text{ being included,} \\ \alpha(1_{F_a}) &= 1_{F_a} \otimes 1_{\mathbf{v}_b} + 1_{F_a} \otimes 1_{\mathbf{v}_b}, \end{cases}$$

where $1_{\mathbf{v}_b}$ is the sum of $1_{\mathbf{p}}$ when $\mathbf{p}$ runs in the set of vertices of Δ^b and $\mathbf{v}_b$ is the cone point of $c\Delta^b$.

Construction of β : $N^*(\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(\Delta^b)$. With the previous notation, the linear map β is defined by

$$\begin{cases} \beta(1_{F_a * F_b}) &= 1_{cF_a} \otimes 1_{F_b}, & \text{if } F_b \neq \emptyset, \text{ the case } F_a = \emptyset \text{ being included,} \\ \beta(1_{F_a}) &= 1_{F_a} \otimes 1_{\mathbf{v}_b}. \end{cases}$$

These maps verify the next properties whose proofs are postpone after the proof of Proposition 7.1.

Lemma 7.2. *The two morphisms, $\alpha: N^*(c\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(c\Delta^b)$ and $\beta: N^*(\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(\Delta^b)$, are compatible with the differentials and the restrictions to faces of Δ^a and Δ^b .*

Lemma 7.3. *The morphism $\xi_i: \tilde{N}^*(\Delta(i)) \rightarrow \tilde{N}^*(\Delta)$ is compatible with the differentials and the restrictions to faces of the Δ^{j_ℓ} 's. Moreover, it respects the perverse degree, i.e., $\xi_i(\tilde{N}_{\bar{p}}^*(\Delta(i))) \subset \tilde{N}_{\bar{p}}^*(\Delta)$, for any GM-*perverse*, $\bar{p}$.*

Construction of $\chi: \tilde{N}^*(X^*) \rightarrow \tilde{N}^*(X)$, the global step. Let $\sigma: \Delta_\sigma = \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow X$ be a filtered simplex of X , of blow-up $\tilde{\Delta}_\sigma = c\Delta^{j_0} \times \dots \times \Delta^{j_n}$. As we have noted before, the domain of the *filtered* simplex, $\nu \circ \sigma: \Delta_{\nu \circ \sigma} = \Delta^{k_0} * \dots * \Delta^{k_n} \rightarrow X^*$, has a different decomposition, obtained by a succession of elementary amalgamations. We denote by $\tilde{\Delta}_{\nu \circ \sigma}$ the associated blow-up.

These elementary amalgamations give a finite sequence of decompositions, $\Delta(i_\ell)_{0 \leq \ell \leq m}$, such that $\Delta(0) = \Delta^{j_0} * \dots * \Delta^{j_n} = \Delta_\sigma$ and $\Delta(m) = \Delta^{k_0} * \dots * \Delta^{k_n} = \Delta_{\nu \circ \sigma}$. Two consecutive terms correspond to an elementary amalgamation, i.e., $\Delta(i_\ell) = \Delta^{x_0} * \dots * \Delta^{x_n}$ and $\Delta(i_{\ell+1}) = \Delta^{y_0} * \dots * \Delta^{y_n}$, with

$$\begin{cases} y_u = x_u & \text{if } u < i_\ell \text{ or } u > i_\ell + 1, \\ y_{i_\ell} = -1 & \text{and } y_{i_\ell+1} = x_{i_\ell} + x_{i_\ell+1} + 1. \end{cases}$$

Recall the map $\xi_{i_\ell}: \tilde{N}^*(\Delta(i_{\ell+1})) \rightarrow \tilde{N}^*(\Delta(i_\ell))$ defined in (7) and (8). We set,

$$\chi_\sigma = \xi_{i_0} \circ \dots \circ \xi_{i_{m-1}}.$$

Finally, with Lemma 7.2, we have a map, $\chi: \tilde{N}^*(X^*) \rightarrow \tilde{N}^*(X)$, defined on $\sigma: \Delta \rightarrow X$ and $\Phi \in \tilde{N}^*(X^*)$, by

$$\chi(\Phi)_\sigma = \chi_\sigma(\Phi_{\nu \circ \sigma}).$$

Proof of Proposition 7.1. With Lemma 7.3, the previous map, $\chi: \tilde{N}^*(X^*) \rightarrow \tilde{N}^*(X)$, is a cochain map which restricts as $\chi: \tilde{N}_P^*(X^*) \rightarrow \tilde{N}_P^*(X)$.

There exists also a map, $\text{eval}_N: \tilde{N}^*(X) \rightarrow \text{hom}(N_*^{\text{GM}}(X), \mathbb{F}_2)$, defined as follows. For any $\Phi \in \tilde{N}^*(X)$ and $\sigma: \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow X$, with $\Phi_\sigma = \sum_j \Phi_{0,j} \otimes \dots \otimes \Phi_{n,j} \in N^*(c\Delta^{j_0}) \otimes \dots \otimes N^*(\Delta^{j_n})$, we set

$$\text{eval}_N(\Phi)(\sigma) = \sum_j \Phi_{0,j}([c\Delta^{j_0}]) \cdot \dots \cdot \Phi_{n,j}([\Delta^{j_n}]).$$

Let $\bar{q}$ be the GM-perversity such that $\bar{p} + \bar{q} = \bar{t}$. The canonical morphism, $\rho_*: C_*(-) \rightarrow N_*(-)$, induces $\rho^*: \text{hom}(N_*^{\text{GM}, \bar{q}}(-), \mathbb{F}_2) \rightarrow \text{hom}(C_*^{\text{GM}, \bar{q}}(-), \mathbb{F}_2)$ and $\tilde{\rho}: \tilde{N}_P^*(-) \rightarrow \tilde{C}_P^*(-)$. The previous map, eval_N is connected with the morphism eval introduced in the proof of Lemma 4.2 by $\rho^* \circ \text{eval}_N = \text{eval} \circ \tilde{\rho}$. As $\tilde{\rho}$ and eval are quasi-isomorphisms (cf. Proposition 1.5 and Proposition 1.8), we know that the composite $\rho^* \circ \text{eval}_N$ is a quasi-isomorphism. Consider now the following diagram,

$$(9) \quad \begin{array}{ccccc} \tilde{N}_P^*(X^*) & \xrightarrow{\text{eval}_N} & \text{hom}(N_*^{\text{GM}, \bar{q}}(X^*), \mathbb{F}_2) & \xrightarrow{\rho^*} & \text{hom}(C_*^{\text{GM}, \bar{q}}(X^*), \mathbb{F}_2) \\ \chi \downarrow & & \downarrow N^*(\nu) & & \downarrow C^*(\nu) \\ \tilde{N}_P^*(X) & \xrightarrow{\text{eval}_N} & \text{hom}(N_*^{\text{GM}, \bar{q}}(X), \mathbb{F}_2) & \xrightarrow{\rho^*} & \text{hom}(C_*^{\text{GM}, \bar{q}}(X), \mathbb{F}_2). \end{array}$$

The right-hand square is commutative by naturality of ρ_* . We prove now the commutativity of the left-hand one. Let $\Phi \in \tilde{N}_P^*(X^*)$ and $\sigma: \Delta_\sigma = \Delta^{j_0} * \dots * \Delta^{j_n} \rightarrow X$ be a filtered simplex, of associated filtered simplex $\nu \circ \sigma: \Delta_{\nu \circ \sigma} \rightarrow X^*$. We have to check

$$(10) \quad (N^*(\nu) \circ \text{eval}_N(\Phi))(\sigma) = \text{eval}_N(\chi(\Phi))(\sigma).$$

For a given σ , we can decompose ν in a finite number of elementary amalgamations and thus replace $\chi_\sigma: \tilde{N}^*(\Delta_{\nu \circ \sigma}) \rightarrow \tilde{N}^*(\Delta_\sigma)$ by $\xi_i: \tilde{N}^*(\Delta(i)) \rightarrow \tilde{N}^*(\Delta)$, as defined in (7) and (8). Set $\Phi_{\nu \circ \sigma} = \sum_j \Phi_{0,j} \otimes \cdots \otimes \Phi_{n,j} \in \tilde{N}^*(\Delta(i))$ and suppose $i \neq n-1$. By definition (7), we have

$$\begin{aligned} \chi(\Phi)_\sigma &= \chi_\sigma(\Phi_{\nu \circ \sigma}) = \xi_i(\Phi_{\nu \circ \sigma}) \\ &= \sum_j \Phi_{i,j}([v]) \cdot \Phi_{0,j} \otimes \cdots \otimes \Phi_{i-1,j} \otimes \alpha(\Phi_{i+1,j}) \otimes \Phi_{i+2,j} \otimes \cdots \otimes \Phi_{n,j} \end{aligned}$$

and the right-hand side of (10) is equal to

$$\begin{aligned} \text{eval}_N(\chi(\Phi))(\sigma) &= \sum_j \Phi_{0,j}([c\Delta^{j_0}]) \cdots \Phi_{i-1,j}([c\Delta^{j_i}]) \cdot \Phi_{i,j}([v]) \cdot \\ &\quad \alpha(\Phi_{i+1,j})([c\Delta^{j_i}]) \otimes [c\Delta^{j_{i+1}}] \cdots \Phi_{n,j}([c\Delta^{j_n}]). \end{aligned}$$

We determine now the left-hand side of (10),

$$\begin{aligned} (N^*(\nu) \circ \text{eval}_N(\Phi))(\sigma) &= \text{eval}_N(\Phi)(\nu \circ \sigma) \\ &= \sum_j \Phi_{0,j}([c\Delta^{j_0}]) \cdots \Phi_{i-1,j}([c\Delta^{j_{i-1}}]) \cdot \Phi_{i,j}([v]) \cdot \\ &\quad \Phi_{i+1,j}([c\Delta^{j_i+j_{i+1}+1}]) \cdots \Phi_{n,j}([c\Delta^{j_n}]). \end{aligned}$$

Thus, the left-hand and the right-hand sides coincide by definition of α . A similar argument gives the result when $i = n-1$.

We have established above that the two horizontal lines of the commutative square (9) are quasi-isomorphisms. The right-hand vertical map is a quasi-isomorphism also (cf. [15]). Thus, $\chi: \tilde{N}_{\mathcal{P}}^*(X^*) \rightarrow \tilde{N}_{\mathcal{P}}^*(X)$ is a quasi-isomorphism. $\square$

Proof of Lemma 7.2. We consider first the map α and its behavior with restriction maps. Let ∇_a and ∇_b be faces of Δ^a and Δ^b , respectively, including the cases $\nabla_a = \emptyset$ or $\nabla_b = \emptyset$. Then the following diagram commutes,

$$\begin{array}{ccc} N^*(c\Delta^{a+b+1}) & \xrightarrow{\alpha} & N^*(c\Delta^a) \otimes N^*(c\Delta^b) \\ \text{Res} \downarrow & & \downarrow \text{Res} \\ N^*(c\nabla^{a+b+1}) & \xrightarrow{\alpha} & N^*(c\nabla^a) \otimes N^*(c\nabla^b), \end{array}$$

where the two vertical maps are given by the restriction map. To verify this assertion, we consider two faces, F_a of ∇_a and F_b of ∇_b , and check the commutativity for the cochain $1_{c(F_a * F_b)}$, the other cases being similar,

$$\text{Res}(\alpha(1_{c(F_a * F_b)})) = \text{Res}(1_{cF_a} \otimes 1_{cF_b}) = 1_{cF_a} \otimes 1_{cF_b} = \alpha(1_{c(F_a * F_b)}) = \alpha(\text{Res}(1_{c(F_a * F_b)})).$$

Now, comes the differential. Let L be a finite simplicial complex, endowed with a partial order of its vertices such that the vertices of any simplex are simply ordered. In the cone, cL , the cone point is the greatest element. In the sequel, we adopt (see [22, Page 292]) the

Steenrod's convention: A symbol, as F , G , ∇ will denote, ambiguously, either (1) a simplex of L or (2) the array of vertices of the simplex ordered as in L , or (3) the orientation of the simplex determined by this order, or (4) the elementary cochain which attaches $+1$ to this oriented simplex and 0 to all others. The ambiguity can usually be resolved by examining the context in which the symbol is used.

With this convention, the definition of $\alpha: N^*(c\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(c\Delta^b)$ can be written as

$$\begin{cases} \alpha(c(F_a * F_b)) &= cF_a \otimes cF_b, & \text{the cases } F_a = \emptyset, F_b = \emptyset \text{ being included,} \\ \alpha(F_a * F_b) &= cF_a \otimes F_b, & \text{if } F_b \neq \emptyset, \text{ the case } F_a = \emptyset \text{ being included,} \\ \alpha(F_a) &= F_a \otimes \mathbf{v}_b + F_a \otimes \mathbb{V}_b, \end{cases}$$

where $\mathbb{V}_b$ denotes the sum of the vertices of Δ^b . The definition of the coboundary with this convention is also specified in [22, Page 296]. Let $F_a = (a_0, \dots, a_k)$ be a nonempty face of a simplex Δ^a , we denote by $cF_a = (a_0, \dots, a_k, \mathbf{v}_a)$ the face obtained from the adjunction of the cone point $\mathbf{v}_a$. It is important to observe that, in this setting, the differential of a face F (view as a cochain) depends on the simplicial complex in which we do the computation. For instance, the differentials δ^a in Δ^a and δ^{ca} in $c\Delta^a$ are linked by

$$\begin{cases} \delta^{ca}(cF_a) &= c(\delta^a F_a), \\ \delta^{ca} F_a &= \delta^a F_a + cF_a, \\ \delta^{ca} \mathbf{v}_a &= c\mathbb{V}_a, \end{cases}$$

where $\mathbb{V}_a$ is the sum of the vertices of Δ^a . If F_b is a nonempty face in Δ^b , the differential δ^{a*b} in $\Delta^a * \Delta^b$ is defined by:

$$\begin{cases} \delta^{a*b}(F_a * F_b) &= (\delta^a F_a) * F_b + F_a * (\delta^b F_b), \\ \delta^{a*b} F_a &= \delta^a F_a + F_a * \mathbb{V}_b, \\ \delta^{a*b} F_b &= \mathbb{V}_a * F_b + \delta^b F_b. \end{cases}$$

The differential on $c(\Delta^a * \Delta^b) = (c\Delta^a) * \Delta^b$ can be deduced from the combination of the previous equalities; we denote it by $\delta^{c(a*b)}$. We make uniform the notations by setting $\delta^{ca \otimes cb}$ and $\delta^{ca \otimes b}$ for the product differentials on $N^*(c\Delta^a) \otimes N^*(c\Delta^b)$ and $N^*(c\Delta^a) \otimes N^*(\Delta^b)$, respectively. We verify now the compatibility of α with the differentials, by considering the various cases.

- Suppose $F_a \neq \emptyset$ and $F_b \neq \emptyset$.

$$\begin{aligned} (\delta^{ca \otimes cb})(\alpha(c(F_a * F_b))) &= (\delta^{ca \otimes cb})(cF_a \otimes cF_b) = \delta^{ca}(cF_a) \otimes cF_b + cF_a \otimes \delta^{cb}(cF_b) \\ &= c\delta^a F_a \otimes cF_b + cF_a \otimes c\delta^b F_b \\ &= \alpha(c((\delta^a F_a) * F_b)) + \alpha(c(F_a * (\delta^b F_b))) = \alpha(c(\delta^{a*b}(F_a * F_b))) \\ &= \alpha(\delta^{c(a*b)} c(F_a * F_b)). \end{aligned}$$

$$\begin{aligned} (\delta^{ca \otimes cb})\alpha(F_a * F_b) &= (\delta^{ca \otimes cb})(cF_a \otimes F_b) = (\delta^{ca} cF_a) \otimes F_b + cF_a \otimes (\delta^{cb} F_b) \\ &= (c\delta^a F_a) \otimes F_b + cF_a \otimes (\delta^b F_b) + cF_a \otimes cF_b \\ &= \alpha((\delta^a F_a) * F_b) + \alpha(F_a * (\delta^b F_b)) + \alpha(c(F_a * F_b)) \\ &= \alpha(\delta^{a*b}(F_a * F_b) + c(F_a * F_b)) \\ &= \alpha(\delta^{c(a*b)}(F_a * F_b)). \end{aligned}$$

- Suppose $F_a \neq \emptyset$ and $F_b = \emptyset$.

$$\begin{aligned}
(\delta^{ca \otimes cb})\alpha(cF_a) &= (\delta^{ca \otimes cb})(cF_a \otimes \mathbf{v}_b) = (\delta^{ca} cF_a) \otimes \mathbf{v}_b + cF_a \otimes \delta^{cb} \mathbf{v}_b \\
&= (c\delta^a F_a) \otimes \mathbf{v}_b + (cF_a) \otimes c\mathbb{V}_b = \alpha(c(\delta^a F_a + F_a * \mathbb{V}_b)) \\
&= \alpha(c(\delta^{a*b} F_a)) = \alpha(\delta^{c(a*b)} cF_a).
\end{aligned}$$

$$\begin{aligned}
(\delta^{ca \otimes cb})\alpha(F_a) &= (\delta^{ca \otimes cb})(F_a \otimes \mathbf{v}_b + F_a \otimes \mathbb{V}_b) = \delta^{ca} F_a \otimes (\mathbf{v}_b + \mathbb{V}_b) + F_a \otimes \delta^{cb} (\mathbf{v}_b + \mathbb{V}_b) \\
&= (\delta^a F_a) \otimes (\mathbf{v}_b + \mathbb{V}_b) + cF_a \otimes \mathbf{v}_b + cF_a \otimes \mathbb{V}_b + 0 \\
&= \alpha(\delta^a F_a) + \alpha(cF_a) + \alpha(F_a * \mathbb{V}_b) = \alpha(\delta^{ca} F_a + cF_a) \\
&= \alpha(\delta^{c(a*b)} F_a).
\end{aligned}$$

(We have used $\delta^{ca} F_a = \delta^a F_a + cF_a$ and $\delta^{cb} (\mathbf{v}_b + \mathbb{V}_b) = 0$. For the last one, observe that $\mathbf{v}_b + \mathbb{V}_b$ is the non trivial cocycle in degree 0.)

- Suppose $F_a = \emptyset$ and $F_b \neq \emptyset$.

$$\begin{aligned}
(\delta^{ca \otimes cb})\alpha(cF_b) &= (\delta^{ca \otimes cb})(\mathbf{v}_a \otimes cF_b) = c\mathbb{V}_a \otimes cF_b + \mathbf{v}_a \otimes c\delta^b F_b \\
&= \alpha(c(\mathbb{V}_a * F_b)) + \alpha(c\delta^b F_b) = \alpha(c\delta^{a*b} F_b) \\
&= \alpha(\delta^{c(a*b)} cF_b).
\end{aligned}$$

$$\begin{aligned}
(\delta^{ca \otimes cb})\alpha(F_b) &= (\delta^{ca \otimes cb})(\mathbf{v}_a \otimes F_b) = c\mathbb{V}_a \otimes F_b + \mathbf{v}_a \otimes \delta^{cb} F_b \\
&= c\mathbb{V}_a \otimes F_b + \mathbf{v}_a \otimes \delta^b F_b + \mathbf{v}_a \otimes cF_b \\
&= \alpha(\mathbb{V}_a * F_b) + \alpha(\delta^b F_b) + \alpha(cF_b) \\
&= \alpha(\delta^{c(a*b)} F_b).
\end{aligned}$$

- Suppose $F_a = F_b = \emptyset$.

$$\begin{aligned}
(\delta^{ca \otimes cb})\alpha(c\emptyset) &= (\delta^{ca \otimes cb})(\mathbf{v}_a \otimes \mathbf{v}_b) = c\mathbb{V}_a \otimes \mathbf{v}_b + \mathbf{v}_a \otimes c\mathbb{V}_b \\
&= \alpha(c\mathbb{V}_a + c\mathbb{V}_b) = \alpha(c\mathbb{V}_{a+b+1}) \\
&= \alpha(\delta^{c(a*b)} c\emptyset).
\end{aligned}$$

As for the map $\beta: N^*(\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(\Delta^b)$, its description with Steenrod's convention writes,

$$\begin{cases} \beta(F_a * F_b) &= cF_a \otimes F_b, \quad \text{if } F_b \neq \emptyset, \text{ the case } F_a = \emptyset \text{ being included,} \\ \beta(F_a) &= F_a \otimes \mathbb{V}_b. \end{cases}$$

The proof of its compatibility with restriction maps is totally similar to the proof done for α . Therefore we are reduced to check the compatibility of β with the differentials. As before, we list the different cases.

- Suppose $F_a \neq \emptyset$ and $F_b \neq \emptyset$.

$$\begin{aligned}
(\delta^{ca \otimes b})(\beta(F_a * F_b)) &= (\delta^{ca \otimes b})(cF_a \otimes F_b) = (\delta^{ca} cF_a) \otimes F_b + cF_a \otimes (\delta^b F_b) \\
&= (c\delta^a F_a) \otimes F_b + cF_a \otimes (\delta^b F_b) \\
&= \beta((\delta^a F_a) * F_b) + \beta(F_a * (\delta^b F_b)) \\
&= \beta(\delta^{a*b}(F_a * F_b)).
\end{aligned}$$

- Suppose $F_a \neq \emptyset$ and $F_b = \emptyset$.

$$\begin{aligned} (\delta^{ca \otimes b})(\beta(F_a)) &= (\delta^{ca \otimes b})(F_a \otimes \mathbb{V}_b) = (\delta^{ca} F_a) \otimes \mathbb{V}_b + F_a \otimes (\delta^b \mathbb{V}_b) \\ &= (\delta^a F_a) \otimes \mathbb{V}_b + (cF_a) \otimes \mathbb{V}_b + 0 = \beta(\delta^a F_a + cF_a) \\ &= \beta(\delta^{a * b} F_a). \end{aligned}$$

- Suppose $F_a = \emptyset$ and $F_b \neq \emptyset$.

$$\begin{aligned} (\delta^{ca \otimes b})(\beta(F_b)) &= (\delta^{ca \otimes b})(\mathbb{V}_a \otimes F_b) = (c\mathbb{V}_a) \otimes F_b + \mathbb{V}_a \otimes (\delta^b F_b) \\ &= \beta(\mathbb{V}_a * F_b) + \beta(\delta^b F_b) = \beta(\delta^{a * b} F_b). \end{aligned}$$

□

Proof of Lemma 7.3. The compatibilities with restriction maps and differentials being local, they are direct consequences of Lemma 7.2. We study now the behavior of ξ_i with the perverse degrees.

We continue with the Steenrod's convention and begin with the expression of the perverse degree in this context. Let $F = F_0 \otimes \cdots \otimes F_n$ be a tensor product of nonempty faces in $\widehat{\Delta} = c\Delta^{j_0} \times \cdots \times c\Delta^{j_{n-1}} \times \Delta^{j_n}$. In Steenrod's convention, we do not distinguish between F and the tensor product of cochains, $1_{F_0} \otimes \cdots \otimes 1_{F_n}$. We observe that, if a face F_k of $c\Delta^{j_k}$ is not included in Δ^{j_k} , then the cochain 1_{F_k} restricts to 0 on the subcomplex $\Delta^{j_k} \times \{1\}$ of $c\Delta^{j_k}$. Therefore, by Definition 1.2, the perverse degree of F is given by

$$\|F_0 \otimes \cdots \otimes F_n\|_\ell = \begin{cases} -\infty & \text{if } F_{n-\ell} \not\subset \Delta^{j_{n-\ell}}, \\ |F_{n-\ell+1}| + \cdots + |F_n| & \text{if } F_{n-\ell} \subset \Delta^{j_{n-\ell}}, \end{cases}$$

for any $\ell \in \{1, \dots, n\}$. A similar definition occurs for the blow-up $\widetilde{\Delta}(i)$ of $\Delta(i)$.

As ξ_i is compatible with the differentials, it is sufficient to prove that the image of a $\overline{p}$ -admissible cochain is $\overline{p}$ -admissible. Let $\nabla = \nabla_0 \otimes \cdots \otimes \nabla_n$ be a tensor product of faces of $\widetilde{\Delta}(i)$ such that

$$\|\nabla_0 \otimes \cdots \otimes \nabla_n\|_\ell \leq \overline{p}(\ell), \text{ for any } \ell \in \{1, \dots, n\}.$$

As we are dealing with $\Delta(i)$ (cf. (6)), we have $\nabla_i = c\emptyset$ and, with the notations of (7), $\Phi_{ij}([v]) = 1$. Thus,

$$\xi_i(\nabla_0 \otimes \cdots \otimes \nabla_n) = \begin{cases} \nabla_0 \otimes \cdots \otimes \nabla_{i-1} \otimes \alpha(\nabla_{i+1}) \otimes \nabla_{i+2} \otimes \cdots \otimes \nabla_n, & \text{if } i \neq n-1, \\ \nabla_0 \otimes \cdots \otimes \nabla_{n-2} \otimes \beta(\nabla_n), & \text{if } i = n-1. \end{cases}$$

The morphisms α and β preserving the dimension of faces, it is sufficient to consider the following cases.

- Suppose $i \neq n-1$ and $n-\ell = i$. Then $\ell \neq 1$ and we have,

$$\begin{aligned} \|\xi_i(\nabla_0 \otimes \cdots \otimes \nabla_n)\|_\ell &= \begin{cases} -\infty, & \text{if } \nabla_{i+1} = \nabla_a * \nabla_b \text{ with } \nabla_b \neq \emptyset, \\ & \text{or if } \nabla_{i+1} = c(\nabla_a * \nabla_b), \\ \max(|v|, |\mathbb{V}|) + |\nabla_{n-\ell+2}| + \cdots + |\nabla_n|, & \text{if } \nabla_{i+1} = \nabla_a, \end{cases} \\ &\leq \|\nabla_0 \otimes \cdots \otimes \nabla_n\|_{\ell-1} \leq \overline{p}(\ell-1) \leq \overline{p}(\ell). \end{aligned}$$

We have used here that the perversity $\overline{p}$ is order-preserving and $|v| = |\mathbb{V}| = 0$ in $c\Delta^{j_{n-\ell+1}}$.

- Suppose $i \neq n - 1$ and $n - \ell = i + 1$. We have,

$$\begin{aligned} \|\xi_i(\nabla_0 \otimes \cdots \otimes \nabla_n)\|_\ell &= \begin{cases} -\infty & \text{if } \nabla_{i+1} = c(\nabla_a * \nabla_b), \\ |\nabla_{n-\ell+1}| + \cdots + |\nabla_n| & \text{if } \nabla_{i+1} = \nabla_a * \nabla_b \text{ or } \nabla_{i+1} = \nabla_a, \end{cases} \\ &\leq \|\nabla_0 \otimes \cdots \otimes \nabla_n\|_\ell \leq \bar{p}(\ell). \end{aligned}$$

- Suppose $i = n - 1$ and $\ell = 1$. We have,

$$\begin{aligned} \|\xi_i(\nabla_0 \otimes \cdots \otimes \nabla_n)\|_1 &= \begin{cases} -\infty & \text{if } \nabla_n = \nabla_a * \nabla_b \text{ with } \nabla_b \neq \emptyset, \\ |\mathbb{V}| & \text{if } \nabla_n = \nabla_a, \end{cases} \\ &\leq 0 = \bar{p}(1). \end{aligned}$$

□

Steenrod's definition of cup_i-products in [22]. Let L be a finite simplicial complex, endowed with a partial order of its vertices such that the vertices of any simplex are simply ordered. Let $F = (a_0, \dots, a_k)$ and $G = (b_0, \dots, b_\ell)$ be two (ordered) simplices of L and let $i \geq 0$ be an integer. The ordered pair, (F, G) , is called *i-regular* if F and G have *exactly* $(i + 1)$ vertices in common, $(c_0, \dots, c_i)$, such that

- $c_0 = b_0$,
- c_0 and c_1 are adjacent vertices in F ,
- $\dots$,
- c_j and c_{j+1} are adjacent in F if j is even and adjacent in G if j is odd,
- $\dots$,
- c_i is the last vertex of F if i is even and the last vertex of G if i is odd.

Denote by F_0 the face of F spanned by its vertices lower than or equal to c_0 and by F_{2j} the face of F spanned by its vertices greater than or equal to c_{2j-1} and lower than or equal to c_{2j} , ($0 < 2j \leq i$). If i is odd, let F_{i+1} be the face of F spanned by its vertices greater than or equal to c_i . We do a similar decomposition for G , denoting by G_{2j+1} ($1 \leq 2j+1 < i+1$) the face of G spanned by its vertices greater than or equal to c_{2j} and lower than or equal to c_{2j+1} . If i is even, let G_{i+1} be the face spanned by the vertices greater than or equal to c_i . This gives the decompositions

$$F = F_0 * F_2 * \cdots * F_{2s} \text{ and } G = G_1 * G_3 * \cdots * G_{2s+(-1)^i},$$

with $2s = i$ if i even and $2s = i + 1$ if i is odd.

Now, we denote by G'_{2j+1} the face of G_{2j+1} obtained by deleting the vertices c_{2j} and c_{2j+1} . Moreover, if i is even, let G'_{i+1} be the face of G_{i+1} obtained by deleting c_i .

Definition 7.4. We define $F \cup_i G = 0$ in the group of $(k + \ell - i)$ -cochains, if the couple (F, G) is not *i-regular* and, otherwise, by

$$F \cup_i G = F_0 * G'_1 * F_2 * G'_3 * \cdots * \begin{cases} G'_{i+1} & \text{if } i \text{ even,} \\ F_{i+1} & \text{if } i \text{ odd.} \end{cases}$$

Example 7.5. We give an illustration of the cases i even and i odd in low dimensions.

1) The pair $(F = (a_0, \dots, a_k), G = (b_0, \dots, b_\ell))$ is 0-regular, if $a_k = b_0$. We write their vertices as follows,

$$\begin{array}{ccc} F & : & a_0 - - - a_k \\ & & \parallel \\ G & : & b_0 - - - b_\ell \end{array}$$

By definition, $F \cup_0 G = (a_0, \dots, a_k, b_1, \dots, b_\ell)$ is the (classical) cup-product.

2) The pair (F, G) is 1-regular, if they have two common vertices (c_0, c_1) such that the vertices of F and G can be put in two lines, as follows

$$\begin{array}{ccc} F & : & a_0 - - - a_{k_0} = c_0 \quad a_{k_0+1} = c_1 - - - a_k \\ & & \parallel \quad \parallel \\ G & : & b_0 = c_0 - - - b_{\ell_1} = c_1 \end{array}$$

By definition, $F \cup_1 G = (a_0, \dots, a_{k_0}, b_1, \dots, b_{\ell_1-1}, a_{k_0+1}, \dots, a_k)$.

3) The pair (F, G) is 2-regular, if they have three common vertices (c_0, c_1, c_2) such that the vertices of F and G can be put in two lines as,

$$\begin{array}{ccc} F & : & a_0 - - - a_{k_0} = c_0 \quad a_{k_0+1} = c_1 - - - a_k = c_2 \\ & & \parallel \quad \parallel \quad \parallel \\ G & : & b_0 = c_0 - - - b_{\ell_1} = c_1 \quad b_{\ell_1+1} = c_2 - - - b_\ell \end{array}$$

By definition, $F \cup_2 G = (a_0, \dots, a_{k_0}, b_1, \dots, b_{\ell_1-1}, a_{k_0+1}, \dots, a_k, b_{\ell_1+2}, \dots, b_\ell)$.

For the convenience of the reader, we recall the next statement of Steenrod, written with the notations of this paper.

Proposition 7.6 ([22, Theorem Page 295]). *The Steenrod squares verify the next properties,*

- (1) $F \cup_i (cG) = \begin{cases} c(F \cup_i G), & \text{if } i \text{ even,} \\ 0, & \text{if } i \text{ odd,} \end{cases}$
- (2) $(cF) \cup_i G = \begin{cases} 0, & \text{if } i \text{ even,} \\ c(F \cup_i G), & \text{if } i \text{ odd,} \end{cases}$
- (3) $(cF) \cup_i (cG) = c(F \cup_{i-1} G)$.

The domains of the applications α and β are the euclidean simplices $\Delta^{a+b+1} = \Delta^a * \Delta^b$ and $c(\Delta^a * \Delta^b)$. Therefore, we need to study the cup_i-products in these complexes. The case of the cone, $c(\Delta^a * \Delta^b)$, can be deduced from the first one, Δ^{a+b+1} , with Proposition 7.6. We order the vertices of Δ^{a+b+1} such that any vertex of Δ^a is lower to any vertex of Δ^b . Also, the cone point, v , is the greatest element of the set of vertices. Recall from (3) the notation,

$$F \cup_i^j G = \begin{cases} F \cup_i G & \text{if } j \text{ is even,} \\ G \cup_i F & \text{if } j \text{ is odd.} \end{cases}$$

Lemma 7.7. *Let F_a, G_a be (nonempty) faces of Δ^a , F_b, G_b be (nonempty) faces of Δ^b . For any $i > 0$, we have, in $\Delta^a * \Delta^b$,*

$$(11) \quad (F_a * F_b) \cup_i (G_a * G_b) = \sum_{i_1+i_2=i-1} (F_a \cup_{i_1} G_a) * (F_b \cup_{i_2}^{i_1+1} G_b).$$

Note that the right-hand side of the equality (11) has at most one non-zero term.

If we set $-\cup_{-1}- = 0$, the right hand side of (11) is equal to zero in the case $i = 0$. Note also that F_a and G_b cannot have a common vertex, neither F_b and G_a . Therefore, with the hypotheses of Lemma 7.7, the simplices $F_a * F_b$ and $G_a * G_b$ cannot have exactly one vertex in common and respect the convention on the order of the vertices. As a consequence, the equality (11) is also true for $i = 0$, with the two sides equal to zero.

Proof. The cup $_i$ -product of $F = F_a * F_b$ and $G = G_a * G_b$ is not zero only if F and G have $(i + 1)$ vertices in common. Denote by $(x + 1)$, with $0 \leq x \leq i$, the number of vertices in common for F_a and G_a . Thus F_b and G_b have $(i - x)$ vertices in common. We observe also that the only non-zero term of the right-hand side of (11) corresponds to $i_1 = x$.

Suppose i even and x odd. With the previous notations, we decompose,

$$\begin{aligned} F_a &= F_{a,0} * \cdots * F_{a,x+1} \quad \text{and} \quad G_a = G_{a,1} * \cdots * G_{a,x}, \\ F_b &= F_{b,0} * \cdots * F_{b,i-x-1} \quad \text{and} \quad G_b = G_{b,1} * \cdots * G_{b,i-x}. \end{aligned}$$

Thus, we have

$$\begin{aligned} (F_a * F_b) \cup_i (G_a * G_b) &= F_{a,0} * G'_{a,1} * \cdots * G'_{a,x} * F_{a,x+1} * F_{b,0} * G'_{b,1} * \cdots * G'_{b,i-x} \\ &= (F_a \cup_x G_a) * (F_b \cup_{i-x-1} G_b) \\ &= (F_a \cup_x G_a) * (F_b \cup_{i-x-1}^{x+1} G_b). \end{aligned}$$

Suppose now i even and x even. We decompose

$$F_a = F_{a,0} * \cdots * F_{a,x} \quad \text{and} \quad G_a = G_{a,1} * \cdots * G_{a,x+1}.$$

Thus, we have

$$F_a \cup_x G_a = F_{a,0} * G'_{a,1} * \cdots * F_{a,x} * G'_{a,x+1}.$$

Note that $F_a \cup_x G_a$ contains all the vertices of $F \cup_i G$ belonging to Δ^a . The first vertex in common between F_a and G_a is the first vertex of G_a . The number of common points in Δ^a being the odd number $x + 1$, the last vertex of Δ^a in common must be the last vertex of F_a . Therefore, the first vertex of Δ^b in common is the first vertex of F_b . (See Example 7.8 for an illustration of this argument.) Thus, for writing this final part of vertices in $F \cup_i G$ as a cup $_{i-x-1}$ -product, we have to decompose F_b and G_b as follows,

$$G_b = G_{b,0} * \cdots * G_{b,i-x} \quad \text{and} \quad F_b = F_{b,1} * \cdots * F_{b,i-x-1}.$$

We deduce

$$\begin{aligned} (F_a * F_b) \cup_i (G_a * G_b) &= F_{a,0} * G'_{a,1} * \cdots * G'_{a,x+1} * G_{b,0} * F'_{b,1} * \cdots * F'_{b,i-x-1} * G_{b,i-x} \\ &= (F_a \cup_x G_a) * (G_b \cup_{i-x-1} F_b) \\ &= (F_a \cup_x G_a) * (F_b \cup_{i-x-1}^{x+1} G_b). \end{aligned}$$

In the case i odd, the conclusion is obtained with totally similar arguments. $\square$

Example 7.8. We particularize with $x = 2$ the argument done in the previous proof. Let $F_a * F_b = (f_0^a, \dots, f_\ell^a) * (f_0^b, \dots, f_k^b)$ and $G_a * G_b = (g_0^a, \dots, g_u^a) * (g_0^b, \dots, g_v^b)$. The following diagram represents $(F_a * F_b) \cup_i (G_a * G_b) = (F_a \cup_2 G_a) * (F_b \cup_{i-3}^3 G_b) = (F_a \cup_2 G_a) * (G_b \cup_{i-3} F_b)$.

$$\begin{array}{ccc}
 F_a * F_b & : & \begin{array}{c} f_0^a \text{ --- } \parallel \text{ --- } f_\ell^a \qquad \qquad \qquad f_0^b \text{ --- } \parallel \text{ --- } \\ \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \end{array} \\
 G_a * G_b & : & \begin{array}{c} \text{---} \parallel \text{---} \text{---} g_u^a g_0^b \text{---} \parallel \text{---} \parallel \text{---} \\ \parallel \qquad \qquad \qquad \parallel \end{array}
 \end{array}$$

Proposition 7.9. Let X be an n -dimensional PL-pseudomanifold, $\bar{p}$ and $\bar{q}$ be GM-perversities. Then, the quasi-isomorphism, $\chi: \tilde{N}_\bullet^*(X^*) \rightarrow \tilde{N}_\bullet^*(X)$, induced by $\nu: X \rightarrow X^*$ is compatible with the cup $_i$ -products, i.e.,

$$\chi(\Phi \cup_i \Psi) = \chi(\Phi) \cup_i \chi(\Psi),$$

for any $i \geq 0$, $\Phi \in \tilde{N}_{\bar{p}}^r(X^*)$, $\Psi \in \tilde{N}_{\bar{q}}^s(X^*)$ and $\Phi \cup_i \Psi \in \tilde{N}_{\bar{p} \oplus \bar{q}}^{r+s-i}(X^*)$.

As cup $_i$ -products on $\tilde{N}^*(-)$ are defined locally, it is sufficient to do the proof for an elementary amalgamation. Thus Proposition 7.9 is a direct consequence of the next lemma.

Lemma 7.10. The two morphisms, $\alpha: N^*(c\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(c\Delta^b)$ and $\beta: N^*(\Delta^{a+b+1}) \rightarrow N^*(c\Delta^a) \otimes N^*(\Delta^b)$, are compatible with the cup $_i$ -products.

Proof. Consider the faces $F = F_a * F_b$ and $G = G_a * G_b$ of Δ^{a+b+1} .

• Suppose first $F_a \neq \emptyset$, $F_b \neq \emptyset$, $G_a \neq \emptyset$, $G_b \neq \emptyset$ and begin with the map β . We have to prove,

$$(12) \quad \beta((F_a * F_b) \cup_i (G_a * G_b)) = \beta(F_a * F_b) \cup_i \beta(G_a * G_b).$$

From Lemma 7.7 and the definition of β , we get

$$\beta((F_a * F_b) \cup_i (G_a * G_b)) = \sum_{i_1+i_2=i-1} (c(F_a \cup_{i_1} G_a) \otimes (F_b \cup_{i_2}^{i_1+1} G_b)).$$

On the other side, we have

$$\begin{aligned}
 \beta(F_a * F_b) \cup_i \beta(G_a * G_b) &=_{(1)} (cF_a \otimes F_b) \cup_i (cG_a \otimes G_b) \\
 &=_{(2)} \sum_{k=0}^i (cF_a \cup_k cG_a) \otimes (F_b \cup_{i-k}^k G_b) \\
 &=_{(3)} \sum_{k=1}^i c(F_a \cup_{k-1} G_a) \otimes (F_b \cup_{i-k}^k G_b) \\
 &= \sum_{i_1+i_2=i-1} (c(F_a \cup_{i_1} G_a)) \otimes (F_b \cup_{i_2}^{i_1+1} G_b),
 \end{aligned}$$

where $=_{(1)}$ is the definition of β , $=_{(2)}$ comes from the structure of $\mathcal{E}(2)$ -algebra on a tensor product of $\mathcal{E}(2)$ -algebras, recalled in Section 2, and $=_{(3)}$ is [22, Formula (4.3)], recalled in Proposition 7.6.

• With the same restriction, $F_a \neq \emptyset$, $F_b \neq \emptyset$, $G_a \neq \emptyset$, $G_b \neq \emptyset$, we study now the map α . The arguments are coming from Lemma 7.7, from the structure of $\mathcal{E}(2)$ -algebra on a tensor product and from Proposition 7.6, as before. In the sequel, we use them without an explicit recall. We have only to study the cases where one of the faces contains the cone point, the other cases being already verified when we have considered the map β .

(i) Let $c(F_a * F_b)$ and $c(G_a * G_b)$ be faces of $c\Delta^{a+b+1}$. Then we have,

$$\begin{aligned} \alpha(c(F_a * F_b) \cup_i c(G_a * G_b)) &= \alpha(c((F_a * F_b) \cup_{i-1} (G_a * G_b))) \\ &= \alpha\left(c\left(\sum_{i_1+i_2=i-2} (F_a \cup_{i_1} G_a) * (F_b \cup_{i_2}^{i_1+1} G_b)\right)\right) \\ &= \sum_{i_1+i_2=i-2} c(F_a \cup_{i_1} G_a) \otimes c(F_b \cup_{i_2}^{i_1+1} G_b), \end{aligned}$$

and

$$\begin{aligned} \alpha(c(F_a * F_b)) \cup_i \alpha(c(G_a * G_b)) &= (cF_a \otimes cF_b) \cup_i (cG_a \otimes cG_b) \\ &= \sum_{k=0}^i (cF_a \cup_k cG_a) \otimes (cF_b \cup_{i-k}^k cG_b) \\ &= \sum_{k=1}^{i-1} c(F_a \cup_{k-1} G_a) \otimes c(F_b \cup_{i-k-1}^k G_b) \\ &= \sum_{i_1+i_2=i-2} c(F_a \cup_{i_1} G_a) \otimes c(F_b \cup_{i_2}^{i_1+1} G_b). \end{aligned}$$

The compatibility with cup_i -products is proved for these faces.

(ii) Let $c(F_a * F_b)$ and $(G_a * G_b)$ be faces of $c\Delta^{a+b+1}$. We have to prove that,

$$(13) \quad \alpha(c(F_a * F_b) \cup_i (G_a * G_b)) = \alpha(c(F_a * F_b)) \cup_i \alpha(G_a * G_b).$$

Observe first, $\alpha(c(F_a * F_b) \cup_i (G_a * G_b)) = 0$ if i even, cf. Proposition 7.6. We study now the cup_i -product of the images by α .

$$\begin{aligned} \alpha(c(F_a * F_b)) \cup_i \alpha(G_a * G_b) &= (cF_a \otimes cF_b) \cup_i (cG_a \otimes G_b) \\ &= \sum_{k=0}^i (cF_a \cup_k cG_a) \otimes (cF_b \cup_{i-k}^k G_b). \end{aligned}$$

We study the last right-hand side term in the case i even.

- If k is even, then $(cF_b \cup_{i-k}^k G_b) = cF_b \cup_{i-k} G_b = 0$, since $i - k$ is even.
- If k is odd, then $(cF_b \cup_{i-k}^k G_b) = G_b \cup_{i-k} cF_b = 0$, since $i - k$ is odd.

Thus the equation (13) is satisfied for i even.

Suppose now that i is *odd*. The left-hand side of (13) can be developed as,

$$\begin{aligned}
 \alpha(c(F_a * F_b) \cup_i (G_a * G_b)) &= \alpha(c((F_a * F_b) \cup_i (G_a * G_b))) \\
 &= \alpha\left(c\left(\sum_{i_1+i_2=i-1} (F_a \cup_{i_1} G_a) * (F_b \cup_{i_2}^{i_1+1} G_b)\right)\right) \\
 &= \sum_{i_1+i_2=i-1} c(F_a \cup_{i_1} G_a) \otimes c(F_b \cup_{i_2}^{i_1+1} G_b).
 \end{aligned}$$

We consider now the expression of the right-hand side of (13) already obtained,

$$\sum_{k=0}^i (cF_a \cup_k cG_a) \otimes (cF_b \cup_{i-k}^k G_b).$$

- If k is even, then $(cF_b \cup_{i-k}^k G_b) = cF_b \cup_{i-k} G_b = c(F_b \cup_{i-k} G_b)$, since $i - k$ is odd.
- If k is odd, then $(cF_b \cup_{i-k}^k G_b) = G_b \cup_{i-k} cF_b = c(G_b \cup_{i-k} F_b)$, since $i - k$ is even.

In conclusion, we have proved, $cF_b \cup_{i-k}^k G_b = c(F_b \cup_{i-k}^k G_b)$ and

$$\begin{aligned}
 \alpha(c(F_a * F_b)) \cup_i \alpha(G_a * G_b) &= \sum_{k=1}^i c(F_a \cup_{k-1} G_a) \otimes c(F_b \cup_{i-k}^k G_b) \\
 &= \sum_{i_1+i_2=i-1} c(F_a \cup_{i_1} G_a) \otimes c(F_b \cup_{i_2}^{i_1+1} G_b).
 \end{aligned}$$

We have established the compatibility with cup_i -products in this case.

(iii) Let $(F_a * F_b)$ and $c(G_a * G_b)$ be faces of $c\Delta^{a+b+1}$. This situation is similar to the previous one.

• We consider now the case where at least one of the subsets, F_a, F_b, G_a, G_b , is the empty set and begin with the map β . The verification follows the same routine than above but we cannot apply Lemma 7.7 in this situation. Therefore, we prove the compatibility with a direct computation of the two sides of the equality (12). We list the different possibilities with the values of the left-hand side (LHS) and of the right-hand side (RHS). If $F_a = F_b = \emptyset$ or $G_a = G_b = \emptyset$, the expressions become trivial and we may focus on the cases below.

Before doing these verifications, we note that $\mathbb{V}_a$ is a chain of vertices and, if $F_a \subset \Delta^a$ is given, one (and only one) of these vertices, say a_t , is the first vertex of F_a . This implies $\mathbb{V}_a \cup_0 F_a = (a_t) \cup_0 F_a = F_a$. Similarly, we have $F_a \cup_0 \mathbb{V}_a = F_a$ and $\mathbb{V}_a$ acts as a neutral element for $-\cup_0-$. Also, as $\mathbf{v}_a \notin F_a$, we have $\mathbf{v}_a \cup_0 F_a = F_a \cup_0 \mathbf{v}_a = 0$.

(1) $F_a = \emptyset, F_b \neq \emptyset, G_a = \emptyset, G_b \neq \emptyset$.

$$\text{LHS} = \beta(F_b \cup_i G_b) = \mathbf{v}_a \otimes (F_b \cup_i G_b).$$

$$\text{RHS} = (\mathbf{v}_a \otimes F_b) \cup_i (\mathbf{v}_a \otimes G_b) = (\mathbf{v}_a \cup_0 \mathbf{v}_a) \otimes (F_b \cup_i^0 G_b) = \mathbf{v}_a \otimes (F_b \cup_i G_b).$$

(2) $F_a \neq \emptyset, F_b = \emptyset, G_a \neq \emptyset, G_b = \emptyset$.

$$\text{LHS} = \beta(F_a \cup_i G_a) = (F_a \cup_i G_a) \otimes (\mathbf{v}_b + \mathbb{V}_b).$$

$$\text{RHS} = (F_a \otimes (\mathbf{v}_b + \mathbb{V}_b)) \cup_i (G_a \otimes (\mathbf{v}_b + \mathbb{V}_b)) = (F_a \cup_i G_a) \otimes ((\mathbf{v}_b + \mathbb{V}_b) \cup_0^i (\mathbf{v}_b + \mathbb{V}_b)) = (F_a \cup_i G_a) \otimes (\mathbf{v}_b + \mathbb{V}_b).$$

- (3) $F_a \neq \emptyset, F_b = \emptyset, G_a = \emptyset, G_b \neq \emptyset$.
 LHS = $\beta(F_a \cup_i G_b) = 0$.
 RHS = $(F_a \otimes (\mathbf{v}_b + \mathbb{V}_b)) \cup_i (\mathbf{v}_a \otimes G_b) = 0$, because $F_a \cup_k \mathbf{v}_a = 0$ for any k .
- (4) $F_a = \emptyset, F_b \neq \emptyset, G_a \neq \emptyset, G_b = \emptyset$.
 LHS = $\beta(F_b \cup_i G_a) = 0$.
 RHS = $\beta(F_b) \cup_i \beta(G_a) = (\mathbf{v}_a \otimes F_b) \cup_i (G_a \otimes (\mathbf{v}_b + \mathbb{V}_b)) = 0$, because $\mathbf{v}_a \cup_k G_a = 0$ for any k .
- (5) $F_a = \emptyset, F_b \neq \emptyset, G_a \neq \emptyset, G_b \neq \emptyset$.
 LHS = $\beta(F_b \cup_i (G_a * G_b)) = 0$, because $F_b \cap G_a = \emptyset$ and $G_a \neq \emptyset$.
 RHS = $\beta(F_b) \cup_i \beta(G_a * G_b) = (\mathbf{v}_a \otimes F_b) \cup_i (cG_a \otimes G_b) = (\mathbf{v}_a \cup_0 cG_a) \otimes (F_b \cup_i G_b) = 0$, because the cone point, $\mathbf{v}_a$, is the greatest vertex.
- (6) $F_a \neq \emptyset, F_b = \emptyset, G_a \neq \emptyset, G_b \neq \emptyset$.
 LHS = $\beta(F_a \cup_i (G_a * G_b)) = \begin{cases} \beta((F_a \cup_i G_a) * G_b) = c(F_a \cup_i G_a) \otimes G_b, & \text{if } i \text{ even,} \\ 0, & \text{if } i \text{ odd.} \end{cases}$
 RHS = $\beta(F_a) \cup_i \beta(G_a * G_b) = (F_a \otimes (\mathbf{v}_b + \mathbb{V}_b)) \cup_i (cG_a \otimes G_b) = (F_a \cup_i cG_a) \otimes (\mathbb{V}_b \cup_0^i G_b) = \begin{cases} c(F_a \cup_i G_a) \otimes G_b, & \text{if } i \text{ even,} \\ 0, & \text{if } i \text{ odd.} \end{cases}$
 The nullity when i is odd comes from Proposition 7.6.
- (7) $F_a \neq \emptyset, F_b \neq \emptyset, G_a = \emptyset, G_b \neq \emptyset$.
 LHS = $\beta((F_a * F_b) \cup_i G_b) = \beta(F_a * (F_b \cup_i G_b)) = cF_a \otimes (F_b \cup_i G_b)$.
 RHS = $\beta(F_a * F_b) \cup_i \beta(G_b) = (cF_a \otimes F_b) \cup_i (\mathbf{v}_a \otimes G_b) = (cF_a \cup_0 \mathbf{v}_a) \otimes (F_b \cup_i G_b) = cF_a \otimes (F_b \cup_i G_b)$.
- (8) $F_a \neq \emptyset, F_b \neq \emptyset, G_a \neq \emptyset, G_b = \emptyset$.
 LHS = $\beta((F_a * F_b) \cup_i G_a) = \begin{cases} 0, & \text{if } i \text{ even,} \\ \beta((F_a \cup_i G_a) * F_b) = c(F_a \cup_i G_a) \otimes F_b, & \text{if } i \text{ odd.} \end{cases}$
 RHS = $\beta(F_a * F_b) \cup_i \beta(G_a) = (cF_a \otimes F_b) \cup_i (G_a \otimes (\mathbf{v}_b + \mathbb{V}_b)) = (cF_a \cup_i G_a) \otimes (F_b \cup_0^i \mathbb{V}_b) = \begin{cases} 0, & \text{if } i \text{ even,} \\ c(F_a \cup_i G_a) \otimes F_b, & \text{if } i \text{ odd,} \end{cases}$
 with the argument already used in the case (6).

• The end of the proof is concerned with the map α when at least one of the subsets, F_a, F_b, G_a, G_b is the empty set. Computations are similar to the previous ones. $\square$

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