On approximation numbers of composition operators

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Abstract. *We show that the approximation numbers of a compact composition operator on the weighted Bergman spaces $\mathfrak{B}_\alpha$ of the unit disk can tend to 0 arbitrarily slowly, but that they never tend quickly to 0: they grow at least exponentially, and this speed of convergence is only obtained for symbols which do not approach the unit circle. We also give an upper bounds and explicit an example.*

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1 Introduction

Let $\mathbb{D}$ be the open unit disk of the complex plane, equipped with its normalized area measure $dA(z) = \frac{dxdy}{\pi}$. For $\alpha > -1$, let $\mathfrak{B}_\alpha$ be the weighted Bergman space of analytic functions $f(z) = \sum_{n=0}^\infty a_n z^n$ on $\mathbb{D}$ such that

$$\|f\|_\alpha^2 = (\alpha+1)\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha \, dA(z) = \sum_{n=0}^\infty \frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)} |a_n|^2 < \infty.$$

The limiting case, as $\alpha \xrightarrow{>} -1$, of those spaces is the usual Hardy space H^2 (indeed, if f is a polynomial, we have $\lim_{\alpha \xrightarrow{>} -1} \|f\|_\alpha^2 = \sum_{n=0}^\infty |a_n|^2 = \|f\|_{H^2}^2$), which we shall treat as $\mathfrak{B}_{-1}$. Note that $\|f\|_\alpha^2 \approx \sum_{n=0}^\infty \frac{|a_n|^2}{(n+1)^{\alpha+1}}$ and that

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, dA(z)$$

is a probability measure on $\mathbb{D}$.

Bergman spaces ([46] page 75, page 78) are Hilbert spaces of analytic functions on $\mathbb{D}$ with reproducing kernel $K_a \in \mathfrak{B}_\alpha$, given by $K_a(z) = (\frac{1}{1-\overline{a}z})^{\alpha+2}$, namely, for every $a \in \mathbb{D}$:

$$f(a) = \langle f, K_a \rangle \,,\ \forall f \in \mathfrak{B}_\alpha; \quad \text{and} \quad \|K_a\|^2 = K_a(a) = \Big(\frac{1}{1-|a|^2}\Big)^{\alpha+2}. \tag{1.1}$$

An important common feature of those spaces is that the multipliers of $\mathfrak{B}_\alpha$ can be (isometrically) identified with the space H^∞ of bounded analytic functions on $\mathbb{D}$, that is:

$$\forall g \in H^\infty, \qquad \|g\|_\infty = \sup_{f\in\mathfrak{B}_\alpha, \|f\|_\alpha \leq 1} \|fg\|_\alpha. \tag{1.2}$$

Indeed, $\|fg\|_\alpha \leq \|g\|_\infty \|f\|_\alpha$ is obvious, and if $\|fg\|_\alpha \leq C\|f\|_\alpha$ for all $f \in \mathfrak{B}_\alpha$, testing this inequality successively on $f = 1, g, \ldots, g^n, \ldots$ easily gives $g \in H^\infty$ and $\|g\|_\infty \leq C$.

Let now φ be a *non-constant* analytic self-map (a so-called *Schur function*) of $\mathbb{D}$ and let $C_\varphi \colon \mathfrak{B}_\alpha \to \mathscr{H}(\mathbb{D})$ the associated composition operator:

$$C_\varphi(f) = f \circ \varphi.$$

It is well-known ([9] page 30) that such an operator is always bounded from $\mathfrak{B}_\alpha$ into itself, and we are interested in its approximation numbers.

Also recall that the *approximation* (or *singular*) *numbers* $a_n(T)$ of an operator $T \in \mathcal{L}(H_1, H_2)$, between two Hilbert spaces H_1 and H_2, are defined, for $n = 1, 2, \ldots$, by:

$$a_n(T) = \inf\{\|T - R\|\,;\ \mathrm{rank}\,(R) < n\}.$$

We have:

$$a_n(T) = c_n(T) = d_n(T)\,,$$

where the numbers c_n (resp. d_n) are the *Gelfand* (resp. *Kolmogorov*) *numbers* of T ([6], page 59 and page 51 respectively).

In the sequel we shall need the following quantity:

$$\beta(T) = \liminf_{n\to\infty} \big[a_n(T)\big]^{1/n}. \tag{1.3}$$

Those approximation numbers form a non-increasing sequence such that

$$a_1(T) = \|T\|, \qquad a_n(T) = a_n(T^*) = \sqrt{a_n(T^*T)}$$

and verify the so-called "ideal" and "subadditivity" properties ([17] page 57 and page 68):

$$a_n(ATB) \leq \|A\|\, a_n(T)\, \|B\|\,; \quad a_{n+m-1}(S+T) \leq a_n(S) + a_m(T). \tag{1.4}$$

Moreover, the sequence $(a_n(T))$ tends to 0 iff T is compact. If $(a_n(T)) \in \ell_p$, we say that T belongs to the Schatten class S_p of index p, $0 < p < \infty$. Taking for T a compact diagonal operator, we see that this sequence is non-increasing with limit 0, but otherwise arbitrary. But if we restrict ourselves to a *specified class* of operators, the answer is far from being so simple, although in some cases the situation is completely elucidated. For example, for the class of Hankel operators on H^2 (those operators $\mathcal{H}_\phi$ whose matrix $(a_{i,j})$ on the canonical basis of H^2 is of the form $a_{i,j} = \widehat{\phi}(i+j)$ for some function $\phi \in L^\infty$), it is known that $\mathcal{H}_\phi$ is compact if and only if the conjugate $\bar{\phi}$ of the symbol ϕ belongs to $H^\infty + \mathcal{C}$, where $\mathcal{C}$ denotes the space of continuous, 2π-periodic functions (Hartman's theorem, [32] page 214). For those Hankel operators, the following theorem, due to A. V. Megretskii, V. V. Peller, and S. R. Treil ([31] and [37], Theorem 0.1, page 490), shows that the approximation numbers are absolutely arbitrary, under the following form.

Theorem 1.1 (Megretskii-Peller-Treil) *Let $(\varepsilon_n)_{n\geq 1}$ be a non-increasing sequence of positive numbers. Then there exists a Hankel operator $\mathcal{H}_\phi$ satisfying:*

$$a_n(\mathcal{H}_\phi) = \varepsilon_n, \qquad \forall n \geq 1.$$

Indeed, if we take a positive self-adjoint operator A whose eigenvalues s_n coincide with the ε_n's and whose kernel is infinite-dimensional, it is easily checked that this operator A verifies the three necessary and sufficient conditions of Theorem 0.1, page 490 in [37] and is therefore unitarily equivalent to a Hankel operator $\mathcal{H}_\phi$ which will verify, in view of (1.4):

$$a_n(\mathcal{H}_\phi) = a_n(A) = \varepsilon_n, \qquad n = 1, 2, \ldots$$

In particular, if $\varepsilon_n \to 0$, the above Hankel operator will be compact, and in no Schatten class if $\varepsilon_n = 1/\log(n+1)$ for example. We also refer to [16] for the following slightly weaker form due to S. V. Khruscëv and V. Peller, but with a more elementary proof based on interpolation sequences in the Carleson sense: for any $\delta > 0$, there exists a Hankel operator $\mathcal{H}_\phi$ such that

$$\frac{1}{1+\delta}\,\varepsilon_n \leq a_n(\mathcal{H}_\phi) \leq (1+\delta)\,\varepsilon_n, \qquad n = 1, 2, \ldots$$

Now, the aim of this work is to prove analogous theorems for the class of composition operators (whose compactness was characterized in [29] and [42]). But if we are able to obtain the Khruscëv-Peller analogue for the *lower bounds*, we will only obtain subexponential estimates for the upper bounds, a fact which is explained by our second result: the speed of convergence to 0 of the approximation numbers of a composition operator cannot be greater than geometric (and *is* geometric for symbols φ verifying $\|\varphi\|_\infty < 1$). Our first result involves a constant < 1 and is not as precise as the result of Megretskii-Peller-Treil or even that of Khruscëv-Peller; this is apparently due to the non-linearity of the dependence with respect to the symbol for the class of composition operators, contrary to the case of the Hankel class.This latter lower bound improves several previously known results on "non-Schattenness" of those operators (see Corollary 4.2 below) and also answers in the positive to a question which was first asked to us by C. Le Merdy ([26]) in the OT Conference 2008 of Timisoara, concerning the bad rate of approximation of compact composition operators. Those theorems are, to our knowledge, the first individual results on approximation numbers a_n of composition operators (in the work of Parfenov [35], some good estimates are given for the approximation numbers of the Carleson embedding operator in the case of the space $H^2 = \mathfrak{B}_{-1}$, but they remain fairly implicit, and are not connected with composition operators), whereas all previous results where in terms of symmetric norms of the sequence (a_n), not on the behaviour of each a_n.

Before describing our results, let us recall two definitions. For every ξ with $|\xi| = 1$ and $0 < h < 1$, the Carleson window $W(\xi, h)$ centered at ξ and of size h is the set

$$W(\xi, h) = \{z \in \overline{\mathbb{D}}\,;\ |z| \geq 1 - h \text{ and } |\arg(z\overline{\xi})| \leq \pi h\}.$$

Let μ be a positive, finite, measure on $\overline{\mathbb{D}}$; the associated maximal function ρ_μ is defined by:

$$\rho_\mu(h) = \sup_{|\xi|=1} \mu\big(W(\xi, h)\big). \tag{1.5}$$

The measure μ is called a *Carleson measure* for the Bergman space $\mathfrak{B}_\alpha$, or an $(\alpha+2)$-*Carleson measure* (including the case $\mathfrak{B}_{-1} = H^2$), if $\rho_\mu(h) = O\left(h^{2+\alpha}\right)$ as $h \to 0$. For any Schur function φ, we shall denote by m_φ the image $\varphi^*(m)$ of the Haar measure m of the unit circle under the radial limits function $\varphi^*(u) = \lim_{r\to 1^-} \varphi(ru)$ of φ, $|u| = 1$, and by $A_{\varphi,\alpha+2}$ the image of the probability measure $(\alpha+1)(1-|z|^2)^\alpha dA(z)$ under φ. The corresponding maximal function will be denoted by $\rho_{\varphi,\alpha+2}$. This notation is justified by the fact that $m_\varphi \overset{def}{=} A_{\varphi,1}$ is a 1-Carleson measure and $A_{\varphi,\alpha}$ an $(\alpha+2)$-Carleson measure for $\alpha > -1$, in view of the famous Carleson embedding theorem which, expressed under a quantitative and generalized form, states the following, implicit as concerns $\|j\|$ and with different notations, but fully proved in [44], Theorem 1.2, for the case $\alpha > -1$ (see [32], page 153).

Theorem 1.2 (Carleson's theorem) *For any $(\alpha+2)$-Carleson measure μ, the canonical inclusion mapping $j\colon \mathfrak{B}_\alpha \to L^2(\mu)$ is defined and continuous, and its norm satisfies*

$$C^{-1} \sup_{0<h<1} \sqrt{\frac{\rho_\mu(h)}{h^{2+\alpha}}} \leq \|j\| \leq C \sup_{0<h<1} \sqrt{\frac{\rho_\mu(h)}{h^{2+\alpha}}}. \tag{1.6}$$

The paper is organized as follows. Section 1 is this introduction. In Section 2, we prove some preliminary lemmas. Our first theorems concern lower bounds. In Section 3, we prove (Theorem 3.1) that the convergence of the approximation numbers $a_n(C_\varphi)$ of a composition operator $C_\varphi\colon \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ cannot exceed an exponential speed: for some $r \in (0,1)$ and some constant $c > 0$, one has $a_n(C_\varphi) \geq c\, r^n$. More precisely, with the notations (1.3) and (3.1), one has $\beta(C_\varphi) \geq [\varphi]^2$. Moreover, this speed of convergence is only attained if the values of φ do not approach the boundary of the unit disk: $\|\varphi\|_\infty < 1$ (Theorem 3.4). On the other hand, the speed of convergence to 0 of $a_n(C_\varphi)$ can be arbitrarily slow; this is proved in Section 4. The proof is mainly an adaptation of the one in [7], but is fairly technical at some points, and will require several additional explanations. In Section 5, we prove an upper estimate (Theorem 5.1), and give three applications of this theorem. In the final Section 6, we test our general results against the example of lens maps, which are known to generate composition operators belonging to all Schatten classes.

2 Preliminary lemmas

In this Section, we shall state several lemmas, which are either already known or quite elementary, but turn out to be necessary for the proofs of our Theorem 3.1 and Theorem 4.1.

For the proof of Theorem 3.1, we shall need the Weyl lemma ([6] Proposition 4.4.2, page 157).

Lemma 2.1 (Weyl lemma) *Let $T\colon H \to H$ be a compact operator. Suppose that $(\lambda_n)_{n\geq 1}$ is the sequence of eigenvalues of T rearranged in non-increasing order. Then, we have:*

$$\prod_{k=1}^{n} a_k(T) \geq \prod_{k=1}^{n} |\lambda_k|.$$

We recall ([3], [13] pages 194–195, [33] pages 302–303) that an *interpolation sequence* (z_n) with (best) interpolation constant C is a sequence (z_n) (necessarily Blaschke, i.e. $\sum_{n=1}^{\infty}(1-|z_n|)<\infty$) in the unit disk such that, for any bounded sequence (w_n) of scalars, there exists a bounded analytic function f (i.e. $f \in H^\infty$) such that:

$$f(z_n) = w_n\,,\ \forall n \geq 1, \quad \text{and} \quad \|f\|_\infty \leq C \sup\nolimits_{n\geq 1} |w_n|.$$

The *Carleson constant* δ of a Blaschke sequence (z_n) is defined as follows:

$$\delta_n = \prod_{j\neq n} \rho(z_n, z_j)\,; \qquad \delta = \inf \delta_n = \inf_{n\geq 1} (1-|z_n|^2)|B'(z_n)|, \tag{2.1}$$

where B is the Blaschke product with zeroes z_n, $n \geq 1$. The interpolation constant C is related to the Carleson constant δ by the following inequality ([10] page 278), in which λ is a positive numerical constant:

$$\frac{1}{\delta} \leq C \leq \frac{\lambda}{\delta}\left(1+\log\frac{1}{\delta}\right)\cdot \tag{2.2}$$

This latter inequality can be viewed as a quantitative form of the Carleson interpolation theorem. Interpolation sequences and reproducing kernels of $\mathfrak{B}_\alpha$ are related as follows ([33] pages 302–303).

Lemma 2.2 *Let $(z_n)_{n\geq 1}$ be an H^∞-interpolation sequence of the unit disk, with interpolation constant C. Then, the sequence $(f_n) = (K_{z_n}/\|K_{z_n}\|)$ of* *normalized reproducing kernels at z_n is C-equivalent to an orthonormal basis in $\mathfrak{B}_\alpha$, namely we have for any finite sequence (λ_n) of scalars:*

$$C^{-1}\Big(\sum\nolimits_n |\lambda_n|^2\Big)^{1/2} \leq \Big\|\sum\nolimits_n \lambda_n f_n\Big\|_\alpha \leq C\Big(\sum\nolimits_n |\lambda_n|^2\Big)^{1/2}. \tag{2.3}$$

The proof in [33] is only for H^2, therefore we indicate a simple proof valid for Bergman spaces $\mathfrak{B}_\alpha$ as well. Let $S = \sum \lambda_n K_{z_n}$ be a finite linear combination of the kernels K_{z_n}, $\omega = (\omega_n)$ be a sequence of complex signs, $S_\omega = \sum \omega_n \lambda_n K_{z_n}$ and $g \in H^\infty$ an interpolating function for the sequence $(\overline{\omega}_n)$, i.e. $g(z_n) = \overline{\omega}_n$ and $\|g\|_\infty \leq C$. If $f \in \mathfrak{B}_\alpha$ and $\|f\|_\alpha \leq 1$, we see that:

$$\langle S_\omega, f\rangle = \sum \omega_n \lambda_n \overline{f(z_n)} = \sum \lambda_n \overline{(fg)(z_n)} = \sum \lambda_n \langle K_{z_n}, fg\rangle = \langle S, fg\rangle,$$

so that using (1.2):

$$|\langle S_\omega, f\rangle| \leq \|S\|_\alpha \|fg\|_\alpha \leq \|S\|_\alpha \|g\|_\infty \|f\|_\alpha \leq C\|S\|_\alpha$$

and passing to the supremum on f, we get $\|S_\omega\|_\alpha \leq C\|S\|_\alpha$. Since the coefficients λ_n are arbitrary, this implies that (f_n) is C-unconditional, namely:

$$C^{-1}\Big\|\sum \omega_n \lambda_n f_n\Big\|_\alpha \leq \Big\|\sum \lambda_n f_n\Big\|_\alpha \leq C\,\Big\|\sum \omega_n \lambda_n f_n\Big\|_\alpha.$$

Now, squaring and integrating with respect to random, independent, choices of signs ω_n's, we get (2.3). □

We also recall ([13] pages 203–204) that an increasing sequence (r_n) of numbers such that $0 < r_n < 1$ and $\frac{1-r_{n+1}}{1-r_n} \leq \rho < 1$ (i.e. verifying the so-called *Hayman-Newman condition*) is an interpolation sequence (see also [32]). In the following, let (r_n) be such a sequence verifying moreover the backward induction relation:

$$\varphi(r_{n+1}) = r_n. \tag{2.4}$$

Set $f_n = K_{r_n}/\|K_{r_n}\|$ and $W = \overline{\mathrm{span}}(f_n)$. Let $(e_n)_{n\geq 1}$ be the canonical basis of ℓ^2, φ a Schur function and $h \in H^\infty$ a function vanishing at r_1. Denote by $M_h \colon \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ the operator of multiplication by h. Then, we have the following basic lemma, which shows that some compression of C_φ^* is a backward shift with controlled weights ([7]).

Lemma 2.3 *Let $J \colon \ell^2 \to W$ be the isomorphism given by $J(e_n) = f_n$. Then, the operator $\mathbf{B} = J^{-1}C_\varphi^* M_h^* J \colon \ell^2 \to \ell^2$ is the weighted backward shift given by:*

$$\mathbf{B}(e_{n+1}) = w_n e_n \quad \textit{and} \quad \mathbf{B}(e_1) = 0, \quad \textit{where} \quad w_n = \overline{h(r_{n+1})}\frac{\|K_{r_n}\|}{\|K_{r_{n+1}}\|}\,. \tag{2.5}$$

To exploit Lemma 2.3, we shall need the following simple fact on approximation numbers of weighted backward shifts.

Lemma 2.4 *Let $(e_n)_{n\geq 1}$ be an orthonormal basis of the Hilbert space H and $\mathbf{B} \in \mathcal{L}(H)$ the weighted backward shift defined by*

$$\mathbf{B}(e_1) = 0 \quad \textit{and} \quad \mathbf{B}(e_{n+1}) = w_n e_n, \qquad \textit{where } w_n \to 0.$$

Assume that $|w_n| \geq \varepsilon_n$ for all $n \geq 1$, where (ε_n) is a non-increasing sequence of positive numbers. Then $\mathbf{B}$ is compact, and satisfies:

$$a_n(\mathbf{B}) \geq \varepsilon_n, \quad \forall n \geq 1. \tag{2.6}$$

Proof. The compactness of $\mathbf{B}$ is obvious. Let R be an operator of rank $< n$. Then $\ker R$ is of codimension $< n$, and therefore intersects the n-dimensional space generated by $e_2, \ldots, e_{n+1}$ in a vector $x = \sum_{j=1}^n x_j e_{j+1}$ of norm one. We then have:

$$\|\mathbf{B} - R\|^2 \geq \|\mathbf{B}x - Rx\|^2 = \|\mathbf{B}x\|^2 = \sum\nolimits_{j=1}^n |w_j|^2 |x_j|^2$$
$$\geq \sum\nolimits_{j=1}^n \varepsilon_j^2 |x_j|^2 \geq \varepsilon_n^2 \sum\nolimits_{j=1}^n |x_j|^2 = \varepsilon_n^2.$$

This ends the proof of Lemma 2.4. □

Now, in view of (1.1) and (2.5), the weight w_n roughly behaves as $\sqrt{\frac{1-r_{n+1}}{1-r_n}}$, so we shall need good estimates on that quotient, before defining the sequence (r_n) explicitly.

We first connect this estimate with the hyperbolic distance d in $\mathbb{D}$. We denote (see [12] or [15] for the definition) by $d(z, w; U)$ the hyperbolic distance of two points z, w of a simply connected domain U. It follows from the generalized Schwarz-Pick lemma ([15] Theorem 7.3.1, page 130) applied to the canonical injection $U \to V$ that the bigger the domain the smaller the hyperbolic distance, namely:

$$U \subset V \text{ and } z, w \in U \quad \Longrightarrow \quad d(z, w; V) \leq d(z, w; U). \tag{2.7}$$

Moreover, as is well-known,

$$0 \leq r < 1 \quad \Longrightarrow \quad d(0, r; \mathbb{D}) = \frac{1}{2} \log \frac{1+r}{1-r} \cdot$$

Recall that the pseudo-hyperbolic and hyperbolic distances ρ and d on $\mathbb{D}$ are defined by:

$$\rho(a, b) = \Big| \frac{a-b}{1-\overline{a}b} \Big|, \qquad d(a, b) = \frac{1}{2} \log \frac{1+\rho(a,b)}{1-\rho(a,b)}, \quad a, b \in \mathbb{D},$$

In the sequel, we shall omit the symbol $\mathbb{D}$ as far as the open unit disk is concerned. For this unit disk, we have the following simple inequality ([7]) .

Lemma 2.5 *Let $a, b \in \mathbb{D}$ with $0 < a < b < 1$. Then:*

$$\mathrm{e}^{-2d(a,b)} \leq \frac{1-b}{1-a} \leq 2\, \mathrm{e}^{-2d(a,b)}. \tag{2.8}$$

Finally, before proceeding to the construction of our Schur function φ in Section 4, it will be useful to note the following simple technical lemma.

Lemma 2.6 *Let (ε_n) be a non-increasing sequence of positive numbers of limit 0. Then there exists a decreasing and logarithmically convex sequence (δ_n) of positive numbers, with limit 0, such that $\delta_n \geq \varepsilon_n$ for all $n \geq 1$.*

Proof. Provided that we replace ε_n by $\varepsilon_n + \frac{1}{n}$, we may assume that (ε_n) is decreasing. Let us define our new sequence by the inductive relation:

$$\delta_1 = \varepsilon_1\,; \quad \delta_2 = \varepsilon_2\,; \quad \delta_{n+1} = \max\big(\varepsilon_{n+1}, \delta_n^2/\delta_{n-1}\big).$$

This sequence is log-convex by definition, i.e. $\delta_n^2 \leq \delta_{n+1}\delta_{n-1}$. By induction, it is seen to be decreasing. Therefore, it has a limit $l \geq 0$. If $\delta_n = \varepsilon_n$ for infinitely many indices, $l = 0$. Otherwise, for n large enough, we have the inductive relation $\delta_{n+1} = \delta_n^2/\delta_{n-1}$, which implies that $\delta_n = \exp(\lambda n + \mu)$ for some constants λ, μ. Since (δ_n) is decreasing, we must have $\lambda < 0$ and again we get $l = 0$. □

In the sequel, we may and will thus assume, without loss of generality, that (ε_n) is decreasing and logarithmically convex.

3 Lower bounds

We first introduce a notation. If

$$\varphi^{\#}(z) = \lim_{w\to z} \frac{\rho(\varphi(w), \varphi(z))}{\rho(w, z)} = \frac{|\varphi'(z)|(1 - |z|^2)}{1 - |\varphi(z)|^2}$$

is the pseudo-hyperbolic derivative of φ, we set:

$$[\varphi] = \sup_{z\in\mathbb{D}} \varphi^{\#}(z) = \|\varphi^{\#}\|_\infty. \tag{3.1}$$

In our first theorem, we get that the approximation numbers cannot supersede a geometric speed.

Theorem 3.1 *For any Schur function φ, there exist positive constants $c > 0$ and $0 < r < 1$ such that, for $C_\varphi : \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$, we have:*

$$a_n(C_\varphi) \geq c\, r^n, \qquad n = 1, 2, \ldots \tag{3.2}$$

More precisely, one has $\beta(C_\varphi) \geq [\varphi]^2$ and hence, for each $\kappa < [\varphi]$, there exists a constant $c_\kappa > 0$ such that:

$$a_n(C_\phi) \geq c_\kappa \kappa^{2n}. \tag{3.3}$$

For the proof, we need the following lemma.

Lemma 3.2 *Let $T\colon H \to H$ be a compact operator. Suppose that $(\lambda_n)_{n\geq 1}$, the sequence of eigenvalues of T rearranged in non-increasing order, satisfies, for some $\delta > 0$ and $r \in (0,1)$:*

$$|\lambda_n| \geq \delta r^n, \quad n = 1, 2, \dots$$

Then there exists $\delta_1 > 0$ such that

$$a_n(T) \geq \delta_1 r^{2n}, \quad n = 1, 2, \dots$$

In particular $\beta(T) \geq r^2$.

Proof. By Weyl's inequality (Lemma 2.1), we have

$$\prod_{k=1}^{n} a_k(T) \geq \prod_{k=1}^{n} |\lambda_k| \geq \delta^n r^{n(n+1)/2}.$$

Since $a_k(T)$ is non-increasing and $a_k(T) \leq \|T\|$ for every k, changing n into $2n$, we get:

$$\|T\|^n a_n(T)^n \geq \prod_{k=1}^{2n} a_k(T) \geq \delta^{2n} r^{n(2n+1)} \geq \delta^{2n} r^{2n^2}$$

and therefore $a_n(T) \geq \frac{\delta^2}{\|T\|}\, r^{2n} = \delta_1 r^{2n}$, as claimed. $\square$

By applying this lemma to composition operators, we get the following result, which ends the proof of Theorem 3.1.

Proposition 3.3 *For every composition operator $C_\varphi\colon \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ of symbol $\varphi\colon \mathbb{D} \to \mathbb{D}$, we have $\beta(C_\varphi) \geq [\varphi]^2$.*

Proof. For every $a \in \mathbb{D}$, let Φ_a be the (involutive) automorphism of the unit disk defined by

$$\Phi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad z \in \mathbb{D}.$$

Observe that we have

$$\Phi_a(a) = 0, \quad \Phi_a(0) = a, \quad \Phi_a'(a) = \frac{1}{|a|^2 - 1}, \quad \Phi_a'(0) = |a|^2 - 1.$$

Define now $\psi = \Phi_{\varphi(a)} \circ \varphi \circ \Phi_a$. We have that 0 is a fixed point of ψ, whose derivative is, by the chain rule:

$$\psi'(0) = \Phi'_{\varphi(a)}(\phi(a))\varphi'(a)\Phi'_a(0) = \frac{\varphi'(a)(1 - |a|^2)}{1 - |\varphi(a)|^2} \stackrel{def}{=} \varphi^{\#}(a). \tag{3.4}$$

By Schwarz's lemma, we know that $|\psi'(0)| \leq 1$ and so $\frac{|\varphi'(a)|(1-|a|^2)}{1-|\varphi(a)|^2} \leq 1$ (Schwarz-Pick's inequality).

Let us first assume that the composition operator C_φ is compact. Then, so is C_ψ, since we have

$$C_\psi = C_{\Phi_a} \circ C_\varphi \circ C_{\Phi_{\varphi(a)}}. \tag{3.5}$$

If $\psi'(0) \neq 0$, the sequence of eigenvalues of C_ψ is $\big([\psi'(0)]^n\big)_{n\geq 0}$ ([41], page 96; the result given for the space H^2 holds for $\mathfrak{B}_\alpha \subset H^2$, and would also hold for *any* space of analytic functions in $\mathbb{D}$ on which C_ψ is compact). Lemma 3.2 then gives us:

$$\beta(C_\psi) \geq |\psi'(0)|^2 = [\varphi^{\#}(a)]^2 \geq 0.$$

This trivially still holds if $\psi'(0) = 0$.

Now, since C_{Φ_a} and $C_{\Phi_{\varphi(a)}}$ are invertible operators, (3.5) clearly implies that $\beta(C_\varphi) = \beta(C_\psi)$, and therefore, with the notation of (3.4):

$$\beta(C_\varphi) \geq [\varphi^{\#}(a)]^2, \quad \text{for all } a \in \mathbb{D}.$$

By passing to the supremum on $a \in \mathbb{D}$, we end the proof of Proposition 3.3, and that of Theorem 3.1 in the compact case. If C_φ is not compact, the proposition trivially holds. Indeed, in this case, we have $\beta(C_\varphi) = 1 \geq [\varphi]^2$. □

Remark. It is easy to see that the composition operator C_φ is always of infinite rank, contrary to the case of a Hankel operator, so that in some sense it refuses to be approached by finite-rank operators. Theorem 3.1 quantifies things: it is a well-known and easy fact (see for example [41], page 25 and see Theorem 5.1 to come) that, in the case $\|\varphi\|_\infty < 1$, we have $a_n(C_\varphi) \leq c\|\varphi\|_\infty^n$ (and hence $\beta(C_\varphi) \leq \|\varphi\|_\infty < 1$), showing that the approximation numbers can decrease at an exponential speed. Theorem 3.1 shows that this speed is the maximal possible one. The next theorem says that this maximal speed is *only* obtained when $\|\varphi\|_\infty < 1$.

Theorem 3.4 *For every $\alpha \geq -1$, there exists, for any $0 < r < 1$, $s = s(r) < 1$, satisfying $\lim_{r\to 1^-} s(r) = 1$, such that, for $C_\varphi \colon \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$, one has, with the notation coined in* (1.3)*:*

$$\|\varphi\|_\infty > r \quad \Longrightarrow \quad \beta(C_\varphi) \geq s^2. \tag{3.6}$$

In particular, the exponential speed of convergence to 0 *of the approximation numbers of a composition operator C_ϕ of symbol φ takes place if and only if $\|\varphi\|_\infty < 1$; in other words, we have:*

$$\|\varphi\|_\infty = 1 \quad \Longleftrightarrow \quad \beta(C_\varphi) = 1. \tag{3.7}$$

The proof will proceed through a series of lemmas. Throughout that proof, we assume, without loss of generality, that $\varphi(0) = 0$.

Lemma 3.5 *Let K be a compact subset of $\varphi(\mathbb{D})$ and μ be a probability supported by K. Then, there exists a constant $\delta > 0$ such that, if $R_\mu\colon \mathfrak{B}_\alpha \to L^2(\mu)$ denotes the restriction operator, we have:*

$$a_n(C_\varphi) \geq \delta\, a_n(R_\mu).$$

In particular:

$$\beta(C_\varphi) \geq \beta(R_\mu).$$

Proof. Since φ is an open map, there exists a compact set $L \subset \mathbb{D}$ and a Borel subset $A \subset L$ such that $\varphi(A) = K$ and $\varphi\colon A \to K$ is a bijection (see [36], Chapter I, Theorem 4.2). Then $\mu = \varphi(\nu)$, where $\nu = \varphi^{-1}(\mu)$ is a probability measure supported by L, and we have automatically $\|R_\nu\| < \infty$. Then, for every $f \in \mathfrak{B}_\alpha$:

$$\|f\|^2_{L^2(\mu)} = \int_K |f|^2\, d\mu = \int_L |f \circ \varphi|^2\, d\nu = \|C_\varphi f\|^2_{L^2(\nu)}.$$

This yields $\|R_\mu f\| = \|(R_\nu \circ C_\varphi) f\|$, so C_φ acts as an isometry from $L^2(\mu)$ into $L^2(\nu)$, and the lemma follows, since we have then:

$$a_n(R_\mu) = a_n(R_\nu \circ C_\varphi) \leq \|R_\nu\|\, a_n(C_\varphi)$$

for every $n \geq 1$. $\square$

Observe that this provides a new proof of Theorem 3.1. Indeed, if $K \subset \varphi(\mathbb{D})$ is a small ball of center 0 and radius r, we can take for μ the normalized area measure on K; then Parseval's formula easily shows that $\beta(R_\mu) \geq r$ in that case.

The strategy of the proof of Theorem 3.4 will consist of refining this observation. More precisely, we shall show that the situation can be reduced to the case $K = [0, r]$, and that an appropriate choice of μ can be made in that case, giving a sharp lower bound for $\beta(R_\mu)$. We begin with explaining that choice in the next two lemmas.

Lemma 3.6 *For every $r \in (0,1)$ there exists $s = s(r) < 1$ and $f = f_r \in H^\infty$ with the following properties:*

1) $\lim_{r\to 1^-} s(r) = 1$*;*

2) $\|f\|_\infty \leq 1$*;*

3) $f((0,r]) = s\,\partial\mathbb{D}$ *in a one-to-one way.*

Proof. Let $\rho = \frac{1-\sqrt{1-r^2}}{r}$. Then $r = \frac{2\rho}{1+\rho^2}$ and the automorphism $\varphi_\rho(z) = \frac{\rho - z}{1-\rho z}$ maps $[0,r]$ onto $[-\rho,\rho]$. We define $\varepsilon = \varepsilon(r)$ and $s = s(r)$ by the following relations:

$$\varepsilon(r) = \frac{\pi}{\log\frac{1+\rho}{1-\rho}}\,, \qquad \text{and} \quad s = \mathrm{e}^{-\varepsilon\pi/2}. \tag{3.8}$$

Let now

$$\chi(z)=\varepsilon\log\frac{1+\varphi_\rho(z)}{1-\varphi_\rho(z)} \tag{3.9}$$

and

$$f(z)=s\,\mathrm{e}^{i\chi(z)}\,. \tag{3.10}$$

Note that $f=\mathrm{e}^h$, where

$$h(z)=i\varepsilon\log\frac{1+\varphi_\rho(z)}{1-\varphi_\rho(z)}-\varepsilon\,\frac{\pi}{2}$$

is a conformal mapping from $\mathbb{D}$ onto a small vertical strip of the left-half plane. This function f fulfills all the requirements of the lemma. Indeed, we have $|f(z)|\leq 1$ for all $z\in\mathbb{D}$ and

$$h([0,r])=\Big[-i\varepsilon\log\frac{1+\rho}{1-\rho}\,,\,i\varepsilon\log\frac{1+\rho}{1-\rho}\Big]-\varepsilon\,\frac{\pi}{2}=[-i\pi,i\pi]-\varepsilon\,\frac{\pi}{2}\,,$$

so that $f((0,r])=\{w=s\mathrm{e}^{i\theta}\,;\ -\pi\leq\theta\leq\pi\}$, in a one-to-one way. □

Lemma 3.6 allows a good choice of the measure μ as follows.

Lemma 3.7 *Let f be as in Lemma 3.6. Then, there exists a probability measure $\mu=\mu_r$ supported by $[0,r]$ and a constant $\delta_r>0$ such that, for any integer $n\geq 1$ and any choice of scalars $c_0,c_1,\ldots,c_{n-1}$, we have:*

$$\Big\|\sum_{j=0}^{n-1}c_jR_\mu(f^j)\Big\|_{L^2(\mu)}\geq\frac{s^n}{\sqrt{n}}\Big\|\sum_{j=0}^{n-1}c_jf^j\Big\|_{H^2}\geq\frac{s^n}{\sqrt{n}}\Big\|\sum_{j=0}^{n-1}c_jf^j\Big\|_{\mathfrak{B}_\alpha}.$$

As a consequence, we can claim that, for $C_\varphi\colon\mathfrak{B}_\alpha\to\mathfrak{B}_\alpha$:

$$\varphi(\mathbb{D})\supset[0,r]\quad\Longrightarrow\quad\beta(C_\varphi)\geq s=s(r). \tag{3.11}$$

Proof. With our previous notations, we know that χ is a bijective map from $]0,r]$ onto the unit circle $\partial\mathbb{D}$. Let $\mu=\chi^{-1}(m)$ be the image of the Haar measure m of $\partial\mathbb{D}$ by χ^{-1}. We have by definition of μ:

$$\begin{aligned}\Big\|\sum_{j=0}^{n-1}c_jR_\mu(f^j)\Big\|^2_{L^2(\mu)}&=\int_0^r\Big|\sum_{j=0}^{n-1}c_jf^j(x)\Big|^2d\mu(x)=\int_0^r\Big|\sum_{j=0}^{n-1}c_js^j\mathrm{e}^{ij\chi(x))}\Big|^2d\mu(x)\\&=\int_{-\pi}^{\pi}\Big|\sum_{j=0}^{n-1}c_js^j\mathrm{e}^{ij\theta}\Big|^2\frac{d\theta}{2\pi}=\sum_{j=0}^{n-1}|c_j|^2s^{2j}\\&\geq s^{2n}\sum_{j=0}^{n-1}|c_j|^2.\end{aligned}$$

Now, $\|f^j\|_{H^2} \leq \|f^j\|_\infty \leq 1$, so that we have, using the Cauchy-Schwarz inequality:

$$\Big\| \sum_{j=0}^{n-1} c_j f^j \Big\|_{H^2} \leq \sum_{j=0}^{n-1} |c_j| \, \|f^j\|_{H^2} \leq \sum_{j=0}^{n-1} |c_j| \leq \sqrt{n} \Big(\sum_{j=0}^{n-1} |c_j|^2 \Big)^{1/2},$$

giving the first inequality, since $\| \, \|_{H^2} \geq \| \, \|_{\mathfrak{B}_\alpha}$. Finally, let $R\colon \mathfrak{B}_\alpha \to L^2(\mu)$ be an operator of rank $< n$. We can find a function $g = \sum_{j=0}^{n-1} c_j f^j$ such that $\|g\|_{\mathfrak{B}_\alpha} = 1$ and $R(g) = 0$. The first part of the proof gives:

$$\begin{aligned} \|R_\mu - R\| \geq \|R_\mu(g) - R(g)\| = \|R_\mu(g)\| &= \Big\| \sum_{j=0}^{n-1} c_j f^j \Big\|_{L^2(\mu)} \\ &\geq \frac{s^n}{\sqrt{n}} \Big\| \sum_{j=0}^{n-1} c_j f^j \Big\|_{\mathfrak{B}_\alpha} = \frac{s^n}{\sqrt{n}} \cdot \end{aligned}$$

Therefore $a_n(R_\mu) \geq s^n/\sqrt{n}$ and, in view of Lemma 3.5, the last conclusion of Lemma 3.7 follows. □

The next lemma explains how to reduce the situation to the case $K = [0, r]$ when we only know that $\|\varphi\|_\infty > r$. It was inspired to us by the proof of the Lindelöf theorem that convergence along a curve implies non-tangential convergence for functions in Hardy spaces ([39] page 300).

Lemma 3.8 *Suppose that 0 and r belong to $\varphi(\mathbb{D})$, with $0 < r < 1$. Let μ be a probability measure carried by $[0, r]$. Then, there exists a probability measure ν carried by a compact set $K \subset \varphi(\mathbb{D})$ such that, for any $f \in \mathcal{H}(\mathbb{D})$:*

$$\int_{[0,r]} |f(x)|^2 \, d\mu(x) \leq \frac{1}{2} \int_K \big(|f(z)|^2 + |f(\bar{z})|^2 \big) \, d\nu(z). \tag{3.12}$$

Proof. Since $\varphi(\mathbb{D})$ is open and connected and $0, r \in \varphi(\mathbb{D})$, there is a curve with image $K \subset \varphi(\mathbb{D})$ connecting 0 and r. Put $\tilde{K} = \{\bar{z}\,;\ z \in K\}$. Then, there exists a compact set L such that $[0, r] \subset L$ and whose boundary $\partial L \subset (K \cup \tilde{K})$. Now, the existence of ν carried by K will be provided by an appropriate application of the Pietsch factorization Theorem. To that effect, let X be the real subspace of $\mathcal{C}(L)$ formed by the real functions which are harmonic in the interior of L. By the maximum principle for harmonic functions, X can be viewed as a subspace of $\mathcal{C}(K \cup \tilde{K})$. Now, the inclusion map j of X into $L^2(\mu)$ has 2-summing norm less than one ([1] page 208, or [24], Chapitre 5, Proposition I.3). Therefore, the Pietsch factorization Theorem ([1] page 209, or [24], Chapitre 5, Théorème I.5) implies the existence of a probability σ on $K \cup \tilde{K}$ such that, for every $u \in X$:

$$\|u\|_{L^2(\mu)}^2 = \int_{[0,r]} u^2 \, d\mu \leq \int_{K \cup \tilde{K}} u^2 \, d\sigma. \tag{3.13}$$

For any harmonic function u on $\mathbb{D}$, we can apply (3.13) to $u(z)$ and $u(\bar{z})$ to get:

$$2\int_{[0,r]} u^2\,d\mu \leq \int_{K\cup\tilde{K}} \big[u^2(z)+u^2(\bar{z})\big]\,d\sigma(z) = \int_{K\cup\tilde{K}} \big[u^2(z)+u^2(\bar{z})\big]\,d\tilde{\sigma}(z),$$

where $\tilde{\sigma}$ is the symmetric measure of σ, defined by $\tilde{\sigma}(E)=\sigma(\bar{E})$. There is a probability ν on K such that $\nu+\tilde{\nu}=\sigma+\tilde{\sigma}$. For this probability ν, we thus have, for any real harmonic function u on $\mathbb{D}$:

$$\|u\|^2_{L^2(\mu)} \leq \int_K \big[u^2(z)+u^2(\bar{z})\big]\,d\nu(z). \tag{3.14}$$

Now, given $f\in\mathcal{H}(\mathbb{D})$, we use (3.14) with u the real and imaginary parts of f, and sum up to get (3.12). □

We can now finish the proof of Theorem 3.4 as follows.

Suppose that $\|\varphi\|_\infty > r$. Then, making a rotation if necessary, we may assume that $0, r\in\varphi(\mathbb{D})$ (recall that $\varphi(0)=0$). Let μ as in Lemma 3.7. Using Lemma 3.8, we find a probability measure ν, compactly supported by $\varphi(\mathbb{D})$, such that (3.12) holds. This inequality shows that:

$$\|R_\mu f\|^2 \leq \frac{1}{2}\big(\|R_\nu f\|^2+\|R_{\tilde{\nu}} f\|^2\big),$$

so that $R_\mu = A(R_\nu\oplus R_{\tilde{\nu}})$ with $\|A\|\leq 1/\sqrt{2}\leq 1$. Therefore, by the ideal and sub-additivity properties (1.4):

$$a_{2n}(R_\mu)\leq a_{2n}(R_\nu\oplus R_{\tilde{\nu}})\leq a_n(R_\nu)+a_n(R_{\tilde{\nu}}) = 2\,a_n(R_\nu),$$

implying $\beta(R_\nu)\geq\beta(R_\mu)^2$. Finally, Lemma 3.5 and Lemma 3.7 give:

$$\beta(C_\varphi)\geq\beta(R_\nu)\geq\beta(R_\mu)^2\geq s(r)^2,$$

and this ends the proof of Theorem 3.4. □

Remark. The proof of Theorem 3.4 is strongly influenced by the papers [8] and [45]. In the first one, it is proved that, if K is a continuum of a connected open set Ω and if the doubly connected region $\Omega\setminus K$ is conformally equivalent to the annulus $1<|z|<R$, then there exists a linearly independent sequence (f_n) in $H^\infty(\Omega)$ satisfying, for all scalars c_j:

$$\Big\|\sum_{j=1}^n c_j f_j\Big\|_{H^\infty(\Omega)} \leq R^n \Big\|\sum_{j=1}^n c_j f_j\Big\|_{\mathcal{C}(K)}.$$

As a consequence, the author proves that $\lim_{n\to\infty} d_n^{1/n}=1/R$, where the numbers d_n are the Kolmogorov numbers (see [6] page 49 for the definition) of the restriction map $H^\infty(\Omega)\to\mathcal{C}(K)$. This statement led us to Lemma 3.7. In the second paper, it is proved that, for the same operator, one has $\lim_{n\to\infty} d_n^{1/n} =$

$\mathrm{e}^{-1/C(K,\Omega)}$, where $C(K,\Omega)$ is the Green capacity of K relative to Ω. So that one has $1/R = \mathrm{e}^{-1/C(K,\Omega)}$. In the case we were interested in, namely $\Omega = \mathbb{D}$ and $K_r = [-r, r]$, it seemed to us, for topological and analytic reasons, that R should tend to 1 as $r \to 1$, in other terms that we should have $\lim_{r\to 1^-} C(K_r, \mathbb{D}) = \infty$. This is indeed the case ([40], Example II.1), but the proof is fairly involved, and the desire to get a reasonably simple and self-contained proof of Theorem 3.4 led us to the previous series of lemmas, once we were sure that the result was true.

4 Slow speed

In this section, we shall see that the speed of convergence to 0 of the approximation numbers of a compact composition operator can be as slow as one wants. This answers in the positive to a question which was first asked to us by C. Le Merdy ([26]) in the OT Conference 2008 of Timisoara.

Theorem 4.1 *Let $(\varepsilon_n)_{n\geq 1}$ be a non-increasing sequence of positive real numbers of limit zero. Then, there exists an injective Schur function φ such that $\varphi(0) = 0$ and $C_\varphi \colon \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ is compact, i.e. $a_n(C_\varphi) \to 0$, but:*

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0. \tag{4.1}$$

Equivalently, we have for some positive number $\delta > 0$, independent of n:

$$a_n(C_\varphi) \geq \delta\, \varepsilon_n \qquad \text{for all } n \geq 1.$$

As in the case of Hankel operators, an immediate consequence of Theorem 4.1 is the following:

Corollary 4.2 *There exists a composition operator $C_\varphi \colon H^2 \to H^2$ which is compact, but in no Schatten class.*

This corollary, which Theorem 4.1 reinforces and precises, was an answer to a question of Sarason, and has been first proved in [7]. Other proofs appeared in [2], [14], [19], [20], [47] (for a positive result on Schatten-ness, we refer to [28]).

The construction of the symbol φ in Theorem 4.1 follows that given in [7], but we have to proceed to some necessary adjustments. In order to exploit (2.8), we shall use, as in [7], the following two results due to Hayman ([12]) concerning the hyperbolic distance $d(z, w; U)$ of two points z, w of a simply connected domain U (see also [15]), whose proof uses in particular the comparison principle (2.7):

Proposition 4.3 *Suppose that U contains the rectangle*

$$R = \{z \in \mathbb{C}\,;\ a_1 - b < \mathfrak{Re}\, z < a_2 + b,\ |\mathfrak{Im}\, z| < b\},$$

where $a_1 < a_2$ and $b > 0$. Then, we have the upper estimate:

$$d(a_1, a_2; U) \leq \frac{\pi}{4b}(a_2 - a_1) + \frac{\pi}{2}\cdot \tag{4.2}$$

Proposition 4.4 *Suppose that U contains the rectangle*

$$R = \{z \in \mathbb{C}\,;\ a_1 - c < \mathfrak{Re}\, z < a_2 + c,\ |\mathfrak{Im}\, z| < c\},$$

where $a_1 < a_2$ and $c > 0$, but that the horizontal sides

$$\{z \in \mathbb{C}\,;\ a_1 - c \leq \mathfrak{Re}\, z \leq a_2 + c,\ |\mathfrak{Im}\, z| = c\}$$

of that rectangle are disjoint from U. Then, we have the lower estimate:

$$d(a_1, a_2; U) \geq \frac{\pi}{4c}(a_2 - a_1) - \frac{\pi}{2}\cdot \tag{4.3}$$

We now proceed to the construction of our Schur function φ.

We first define a continuous map $\psi\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$\psi(t) = \begin{cases} K(1 + |t|) & \text{if } |t| \leq 1 \\ |t|/A(|t|) & \text{if } |t| > 1, \end{cases}$$

where K is a positive constant adjusted below and $A\colon [0, \infty[\to [0, \infty[$ an increasing piecewise linear function on the intervals $(0, 1)$ and $(\mathrm{e}^{n-1}, \mathrm{e}^n)$ such that

$$A(0) \overset{def}{=} A_0 = 0, \quad A(\mathrm{e}^{n-1}) \overset{def}{=} A_n \quad \text{for } n \geq 1, \quad \text{and} \quad 2K = 1/A(1),$$

the increasing sequence (A_n) being positive and concave for $n \geq 1$, and tending to ∞. It then follows that the sequence of slopes $\frac{A_n - A_{n-1}}{\mathrm{e}^n - \mathrm{e}^{n-1}}$ is decreasing, since $A_{n+1} - A_n \leq A_n - A_{n-1} \leq \mathrm{e}\,(A_n - A_{n-1})$, that the function A is increasing and concave on $(0, \infty)$ and vanishing at 0, implying that $A(t)/t$ is decreasing on $(0, \infty)$, and that in particular ψ is increasing on $(1, \infty)$.

We then define a domain Ω of the complex plane by:

$$\Omega = \{w \in \mathbb{C}\,;\ |\mathfrak{Im}\, w| < \psi(|\mathfrak{Re}\, w|)\}. \tag{4.4}$$

Let $\sigma\colon \mathbb{D} \to \Omega$ be the unique Riemann map such that $\sigma(0) = 0$ and $\sigma'(0) > 0$. This map exists in view of the following simple fact.

Lemma 4.5 *The domain Ω defined by* (4.4) *is star-shaped with respect to the origin and $\sigma\colon (-1, 1) \to \mathbb{R}$ is an increasing bijection such that $\sigma(-1) = -\infty$ and $\sigma(1) = \infty$.*

Proof. The star-shaped character of Ω will follow from the implication:

$$|\mathfrak{Im}\, w| < \psi(|\mathfrak{Re}\, w|) \quad \text{and} \quad 0 < \lambda < 1 \quad \Longrightarrow \quad |\mathfrak{Im}\,(\lambda w)| < \psi(|\mathfrak{Re}\,(\lambda w)|).$$

We may assume that both $\mathfrak{Re}\, w, \mathfrak{Im}\, w$ are positive, and it is enough to prove:

$$\lambda\psi(x) \leq \psi(\lambda x), \qquad \forall \lambda \in [0, 1],\ \forall x > 0. \tag{4.5}$$

This is easy to check separating three cases:

1) $x \leq 1$; then $\lambda\psi(x) = \lambda K(1+x) \leq K(1+\lambda x) = \psi(\lambda x)$;

2) $\lambda x \leq 1 < x$; then, since $A(x) > A(1)$,

$$\lambda\psi(x) = \lambda \frac{x}{A(x)} < 2K\lambda x \leq K(1+\lambda x) = \psi(\lambda x);$$

3) $\lambda x > 1$; we then have, since ψ increases,

$$\lambda\psi(x) = \lambda \frac{x}{A(x)} \leq \frac{\lambda x}{A(\lambda x)} = \psi(\lambda x)$$

and this ends the proof of (4.5). Now, since σ is determined by the value of $\sigma(0)$ and the sign of $\sigma'(0)$, we have $\sigma(\overline{z}) = \overline{\sigma(z)}$ for all $z \in \mathbb{D}$, so that $\sigma[(-1,1)] \subset \mathbb{R}$. And since the derivative of an injective analytic function does not vanish and $\sigma'(0) > 0$, we get that σ is increasing on $(-1,1)$. Finally, if $w \in \mathbb{R}$ and $w = \sigma(z)$, we have $\overline{w} = w$, so that $\sigma(\overline{z}) = \sigma(z)$ and $\overline{z} = z$, which proves the surjectivity of $\sigma\colon (-1,1) \to \mathbb{R}$. □

We now choose A_n as follows, $\eta > 0$ denoting a positive numerical constant to be specified later.

$$A_n = \eta \log \frac{1}{\varepsilon_n}\,, \quad n \geq 1. \tag{4.6}$$

Observe that this is an increasing, concave sequence tending to ∞ since we assumed that (ε_n) is log-convex and decreasing to 0.

Finally, we define our Schur function φ and our sequence (r_n) under the form of the following lemma, in which the increasing character of ψ is important.

Lemma 4.6 *Let φ be defined by*

$$\varphi(z) = \sigma^{-1}(\mathrm{e}^{-1}\sigma(z)),$$

and let $r_n = \sigma^{-1}(\mathrm{e}^n)$. Then we have:

1. *φ is univalent and maps $\mathbb{D}$ to $\mathbb{D}$, (r_n) increases, and $\varphi(0) = 0$;*
2. *$\varphi(r_{n+1}) = r_n$;*
3. *$\frac{1-r_{n+1}}{1-r_n} \to 0$ and therefore (r_n) is an interpolation sequence;*
4. *$C_\varphi\colon \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ is compact.*

Proof. 1. Since Ω is star-shaped, $\mathrm{e}^{-1}\sigma(z) \in \Omega$ when $z \in \mathbb{D}$, so φ is well-defined and maps $\mathbb{D}$ to itself in a univalent way. Moreover, $\varphi(0) = \sigma^{-1}(0) = 0$, and (r_n) increases since σ^{-1} increases on $\mathbb{R}$.

2. We have $\varphi(r_{n+1}) = \sigma^{-1}\big(\frac{1}{\mathrm{e}}\sigma(r_{n+1})\big) = \sigma^{-1}\big(\frac{1}{\mathrm{e}}\,\mathrm{e}^{n+1}\big) = \sigma^{-1}(\mathrm{e}^n) = r_n$.

3. This assertion is more delicate and relies on Proposition 4.4 as follows.

Set $d_n = \psi(\mathrm{e}^n)$. We have clearly $\mathrm{e}^{n+1} + d_{n+2} < \mathrm{e}^{n+2}$ for large n (recall that $\psi(t) = o\,(t)$ as $t \to \infty$), so that $\psi(\mathrm{e}^{n+1} + d_{n+2}) < \psi(\mathrm{e}^{n+2}) = d_{n+2}$ since ψ is increasing. By the intermediate value theorem for the function $\psi(\mathrm{e}^{n+1} + x) - x$, we can therefore find a positive number $c_n < d_{n+2}$ such that $\psi(\mathrm{e}^{n+1} + c_n) = c_n$.

Now, consider the open sets:

$$R_n = \{z \in \mathbb{C}\,;\ \mathrm{e}^n - c_n < \Re\mathrm{e}\, z < \mathrm{e}^{n+1} + c_n \text{ and } |\Im\mathrm{m}\, z| < c_n\}, \quad U_n = R_n \cup \Omega.$$

Those sets U_n satisfy the assumptions of Proposition 4.4 in view of (4.4). Indeed, if z belongs to the horizontal sides of R_n, we have $z \notin U_n$ since

$$\mathrm{e}^n - c_n \leq \Re\mathrm{e}\, z \leq \mathrm{e}^{n+1} + c_n \quad \Longrightarrow \quad \psi(\Re\mathrm{e}\, z) \leq \psi(\mathrm{e}^{n+1} + c_n) = c_n = |\Im\mathrm{m}\, z|.$$

This proposition then gives, since $\Omega \subset U_n$ and $c_n < d_{n+2}$, and since the hyperbolic metric is conformally invariant,

$$\begin{aligned} d(r_n, r_{n+1}) = d(\mathrm{e}^n, \mathrm{e}^{n+1}; \Omega) \geq d(\mathrm{e}^n, \mathrm{e}^{n+1}; U_n) &\geq \frac{\pi}{4c_n}(\mathrm{e}^{n+1} - \mathrm{e}^n) - \frac{\pi}{2} \\ &\geq c\,\frac{\mathrm{e}^{n+2}}{\psi(\mathrm{e}^{n+2})} = cA(\mathrm{e}^{n+2}) \geq cA_n, \end{aligned}$$

where c is a positive constant. Now, we use Lemma 2.5 to obtain:

$$\frac{1 - r_{n+1}}{1 - r_n} \leq 2\,\mathrm{e}^{-2d(r_n, r_{n+1})} \leq 2\,\mathrm{e}^{-2cA_n},$$

which proves that $\frac{1-r_{n+1}}{1-r_n} \to 0$, and implies that (r_n) is an interpolation sequence.

4. Since φ is univalent, the compactness of $C_\varphi \colon \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ amounts to proving that $\lim_{|z|\to 1} \frac{1-|\varphi(z)|}{1-|z|} = \infty$. For $\alpha > -1$, this follows from [30], Theorem 3.5 and for $\alpha = -1$ from [41], page 39. By the Julia-Carathéodory Theorem ([41], page 57), this in turn is equivalent to proving that for any u, v on the unit circle, the quotient $\frac{\varphi(z)-v}{z-u}$ has no finite limit as z tends to u radially. This latter fact requires some precise justification.

First, we notice that σ extends continuously to an injective map of the open upper half of the unit circle onto the upper part of the boundary of Ω (and similarly for lower parts). This follows from the Carathéodory extension theorem ([39], page 290), applied to the restriction of σ^{-1} to the Jordan region limited by $\partial\Omega$ and two vertical lines $\Re\mathrm{e}\, w = \pm R$ where $R > 0$ is arbitrarily large. Now, let $u \in \partial\mathbb{D}$ with $u \neq \pm 1$. Then, $\sigma(ru) \to w \in \partial\Omega$ as $r \to 1^-$, so that $\mathrm{e}^{-1}\sigma(ru) \to \mathrm{e}^{-1}w = w' \in \Omega$ and that $\varphi(ru) \to \sigma^{-1}(w') \in \mathbb{D}$. Therefore the image of φ touches the unit circle only at ± 1, and the assumption of the Julia-Carathéodory Theorem is fulfilled if $u \neq \pm 1$. By symmetry, it remains to test the point $u = 1$ for which we have:

$$\limsup_{r \xrightarrow{<} 1} \frac{1 - \varphi(r)}{1 - r} \geq \limsup_{n\to\infty} \frac{1 - \varphi(r_{n+1})}{1 - r_{n+1}} = \limsup_{n\to\infty} \frac{1 - r_n}{1 - r_{n+1}} = \infty$$

by the preceding point 3. Since $|v - \varphi(r)| \geq 1 - \varphi(r)$, this ends the proof of Lemma 4.6. $\square$

We now want a good lower bound for the weights w_n appearing in (2.5). To that effect, we apply Proposition 4.3 with

$$U = \Omega, \quad a_1 = \mathrm{e}^n,\ a_2 = \mathrm{e}^{n+1} \quad \text{and} \quad b_n = \psi(\mathrm{e}^{n-1}),$$

as well as

$$R'_n = \{z \in \mathbb{C}\,;\ \mathrm{e}^n - b_n < \Re\mathrm{e}\, z < \mathrm{e}^{n+1} + b_n \text{ and } |\Im\mathrm{m}\, z| < b_n\}.$$

We have $\mathrm{e}^n - b_n > \mathrm{e}^{n-1}$ for large n, since this amounts to

$$\mathrm{e}^n - \mathrm{e}^{n-1} > b_n = \frac{\mathrm{e}^{n-1}}{A(\mathrm{e}^{n-1})}\,, \quad \text{or} \quad \mathrm{e} - 1 > \frac{1}{A(\mathrm{e}^{n-1})},$$

which holds for large n since $A(t)$ tends to ∞ with t. We then observe that $R'_n \subset \Omega$. Indeed, $z \in R'_n \Longrightarrow \Re\mathrm{e}\, z > \mathrm{e}^n - b_n > \mathrm{e}^{n-1}$ and, since ψ is increasing, we have $\psi(\Re\mathrm{e}\, z) > \psi(\mathrm{e}^{n-1}) = b_n > |\Im\mathrm{m}\, z|$. Therefore, we can apply (4.2) and get, for all $n \geq 1$:

$$d(\mathrm{e}^n, \mathrm{e}^{n+1}; \Omega) \leq \frac{\pi}{4\psi(\mathrm{e}^{n-1})}(\mathrm{e}^{n+1} - e^n) + \frac{\pi}{2} \leq C_0 A(\mathrm{e}^{n-1}) = C_0 A_n,$$

where C_0 is a *numerical* constant. By conformal invariance, we have as well $d(r_n, r_{n+1}) \leq C_0 A_n$. It then follows from (2.8) that:

$$\frac{1 - r_{n+1}}{1 - r_n} \geq \exp\big(-2d(r_n, r_{n+1})\big) \geq \exp(-2C_0 A_n). \tag{4.7}$$

Now, we take $h(z) = z - r_1$ in Lemma 2.4 and use the ideal property (1.4) of the approximation numbers. We get, denoting by C the interpolation constant of the sequence (r_n), and using the fact that $\|M_h\| = \|h\|_\infty \leq 2$:

$$a_n(\mathbf{B}) \leq \|J^{-1}\|\, a_n(C_\varphi)\, \|M_h\|\, \|J\| \leq 2C^2 a_n(C_\varphi). \tag{4.8}$$

Next, we choose $\eta = 1/C_0$ in (4.6) and we set $d = (r_2 - r_1)/\sqrt{2}$. Using Lemma 2.3 and relations (1.1), (2.5) and (4.7), we see that the weights w_n associated with $\mathbf{B}$ verify:

$$\begin{aligned} |w_n| &= h(r_{n+1})\frac{\|K_{r_n}\|}{\|K_{r_{n+1}}\|} = h(r_{n+1})\sqrt{\frac{1 - r_{n+1}^2}{1 - r_n^2}} \geq \frac{r_2 - r_1}{\sqrt{2}}\sqrt{\frac{1 - r_{n+1}}{1 - r_n}} \\ &\geq d\exp(-C_0 A_n) \geq d\varepsilon_n \qquad \text{for all } n \geq 1. \end{aligned} \tag{4.9}$$

Finally, using Lemma 2.4, (4.8) and (4.9):

$$a_n(C_\varphi) \geq \frac{1}{2C^2} a_n(\mathbf{B}) \geq \frac{1}{2C^2}\, d\,\varepsilon_n \overset{def}{=} \delta\varepsilon_n \quad \text{for all } n \geq 1.$$

We thus get the desired conclusion (4.1) of Theorem 4.1. $\square$

5 An upper bound

We do not obtain a fairly good upper bound, and we shall content ourselves with the following result, whose proof is quite simple and, for the case $\alpha = -1$, partly contained in [35], but under a very cryptic form which is not easy to decipher.

Theorem 5.1 *Let φ be a Schur function and $\alpha \geq -1$. Then, we have for the approximation numbers of $C_\varphi \colon \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ the upper bound:*

$$a_n(C_\varphi) \leq C \inf_{0<h<1} \left[n^{\frac{\alpha+1}{2}} (1-h)^n + \sqrt{\frac{\rho_{\varphi,\alpha+2}(h)}{h^{2+\alpha}}} \right], \qquad n = 1, 2, \ldots \tag{5.1}$$

where C is a constant. In particular, if $\frac{\rho_{\varphi,\alpha+2}(h)}{h^{2+\alpha}} \leq \mathrm{e}^{-h/A(h)}$, where the function $A \colon [0,1] \to [0,1]$ is increasing, with $A(0) = 0$ and with inverse function A^{-1}, we have:

$$a_n(C_\varphi) \leq C n^{\frac{\alpha+1}{2}} \mathrm{e}^{-nA^{-1}(1/2n)}, \qquad n = 1, 2, \ldots. \tag{5.2}$$

The proof of (5.1) uses a contraction principle which was first proved for $\alpha = -1$ ([18]) and $\alpha = 0$ ([23]), but is also valid for any $\alpha \geq -1$, as follows from the forthcoming work [25].

To prove Theorem 5.1, it will be convenient to prove first the following simple lemma.

Lemma 5.2 *Let n be a positive integer, $g \in \mathfrak{B}_\alpha$ and $f(z) = z^n g(z)$. Then, we have:*

$$\|g\|_\alpha \leq C n^{\frac{\alpha+1}{2}} \|f\|_\alpha. \tag{5.3}$$

Proof. Let $w_n = \frac{n!\Gamma(2+\alpha)}{\Gamma(n+2+\alpha)}$ and $f(z) = \sum_{n=0}^\infty a_n z^n$. We first observe that

$$\frac{w_k}{w_{k+n}} \leq C n^{\alpha+1}, \qquad \forall k \geq 0, \quad \forall n \geq 1. \tag{5.4}$$

Indeed, we have:

$$\begin{aligned} \frac{w_k}{w_{k+n}} &= \frac{k!}{(k+n)!} \frac{\Gamma(k+\alpha+2+n)}{\Gamma(k+\alpha+2)} = \prod_{j=1}^n \frac{(k+j+\alpha+1)}{(k+j)} \leq \prod_{j=1}^n \frac{j+\alpha+1}{j} \\ &= \prod_{j=1}^n \left(1 + \frac{\alpha+1}{j}\right) \leq \exp\left[(\alpha+1)\sum_{j=1}^n \frac{1}{j}\right] \leq C n^{\alpha+1}, \end{aligned}$$

which proves (5.4).

Now, if $f(z) = \sum_{k=n}^\infty a_k z^k$, we have $g(z) = \sum_{k=0}^\infty a_{k+n} z^k$ so that, using (5.4):

$$\|g\|_\alpha^2 = \sum_{k=0}^\infty |a_{k+n}|^2 w_k = \sum_{l=n}^\infty |a_l|^2 w_{l-n} \leq C n^{\alpha+1} \sum_{l=n}^\infty |a_l|^2 w_l = C n^{\alpha+1} \|f\|_\alpha^2,$$

proving (5.3). $\square$

We shall now majorize $a_{n+1}(C_\varphi)$, but provided that we change the constant C, this makes no difference with majorizing $a_n(C_\varphi)$. The choice of the approximating operator R of rank $\leq n$ for C_φ is quite primitive, but in counterpart we shall estimate $\|C_\varphi - R\|$ rather sharply. We denote by P_n the projection operator defined by $P_n f = \sum_{k=0}^{n-1} \hat{f}(k) z^k$ and we take $R = C_\varphi \circ P_n$, i.e. if we have $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k \in \mathfrak{B}_\alpha$, then $R(f) = \sum_{k=0}^{n-1} \hat{f}(k) \varphi^k$, so that $(C_\varphi - R)f = C_\varphi(r)$, with, making use of (5.3):

$$\text{(5.5)} \quad r(z) = \sum_{k=n}^{\infty} \widehat{f}(k) z^k = z^n s(z), \quad \text{with} \quad \|s\|_\alpha^2 \leq C n^{\alpha+1} \|r\|_\alpha^2, \ \|r\|_\alpha \leq \|f\|_\alpha.$$

Assume that $\|f\|_\alpha \leq 1$, fix $0 < h < 1$ and denote by μ_h the restriction of the measure $A_{\varphi,\alpha+2}$ to the annulus $1 - h < |z| \leq 1$. Then, we have:

$$\begin{aligned}
\|(C_\varphi - R)f\|_\alpha^2 = \|C_\varphi(r)\|_\alpha^2 &= \int_{\mathbb{D}} |r(z)|^2 \, dA_{\varphi,\alpha+2}(z) \\
&\leq (1-h)^{2n} \int_{|z| \leq 1-h} |s(z)|^2 \, dA_{\varphi,\alpha+2}(z) \\
&\qquad\qquad + \int_{1-h<|z|\leq 1} |r(z)|^2 \, dA_{\varphi,\alpha+2}(z) \\
&\leq (1-h)^{2n} \int_{\mathbb{D}} |s(z)|^2 \, dA_{\varphi,\alpha+2}(z) + \int_{\mathbb{D}} |r(z)|^2 \, d\mu_h(z) \\
&= (1-h)^{2n} \|C_\varphi(s)\|_\alpha^2 + \int_{\mathbb{D}} |r(z)|^2 \, d\mu_h(z) \\
&\leq C \bigg[(1-h)^{2n} \|s\|_\alpha^2 + \int_{\mathbb{D}} |r(z)|^2 \, d\mu_h(z) \bigg] \\
&\leq C \bigg[n^{\alpha+1} (1-h)^{2n} + \sup_{0<t\leq h} \frac{\rho_{\varphi,\alpha+2}(t)}{t^{2+\alpha}} \bigg]
\end{aligned}$$

if we use (5.5), as well as (1.6) under the form

$$\int_{\mathbb{D}} |r(z)|^2 \, d\mu_h(z) \leq C \sup_{0<t\leq h} \frac{\rho_{\varphi,\alpha+2}(t)}{t^{2+\alpha}} \, \|r\|_\alpha^2,$$

and we know that $\|r\|_\alpha \leq \|f\|_\alpha \leq 1$.

To get rid of the supremum with respect to t, we make use of the following inequality, which holds for $h \leq 1 - |\varphi(0)|$ and $0 < \varepsilon \leq 1$:

$$\text{(5.6)} \qquad \rho_{\varphi,\alpha+2}(\varepsilon h) \leq C \varepsilon^{\alpha+2} \rho_{\varphi,\alpha+2}(h).$$

For $\alpha = 0$ or $\alpha = -1$, this follows respectively from [18], Theorem 4.19, p. 55, and from [23], Theorem 3.1. The general case is proved in [25]. Setting $t = \varepsilon h$ for $0 < t \leq h$, this also reads $\frac{\rho_{\varphi,\alpha+2}(t)}{t^{\alpha+2}} \leq C \frac{\rho_{\varphi,\alpha+2}(h)}{h^{\alpha+2}}$, and we can forget the supremum in t in the previous inequalities. Taking square roots, we get the relation (5.1).

When $\frac{\rho_{\varphi,\alpha+2}(h)}{h^{2+\alpha}} \leq \mathrm{e}^{-h/A(h)}$, let us take for h the nearly optimal value $h = A^{-1}(1/2n)$, so that $h/A(h) = 2nh$. We then have from (5.1), since $(1-h)^{2n} \leq \mathrm{e}^{-2nh}$:

$$a_{n+1}(C_\varphi)^2 \leq \|C_\varphi - R\|_\alpha^2 \leq Cn^{\alpha+1}[\mathrm{e}^{-2nh} + \mathrm{e}^{-h/A(h)}] \leq 2Cn^{\alpha+1}\mathrm{e}^{-2nA^{-1}(1/2n)},$$

proving (5.2), and ending the proof of Theorem 5.1. □

Let us now indicate three corollaries, which improve results of [19], [22], and [23] respectively.

Corollary 5.3 *Suppose that $\rho_{\varphi,\alpha+2}(h) \leq Ch^{(2+\alpha)\beta}$ for some $\beta > 1$. Then:*

$$a_n(C_\varphi) \leq Cn^{-\frac{(\beta-1)(\alpha+2)}{2}}(\log n)^{\frac{(\beta-1)(\alpha+2)}{2}}.$$

In particular, C_φ belongs to the Schatten class $S_p = S_p(\mathfrak{B}_\alpha)$ for each $p > \frac{2}{(\beta-1)(\alpha+2)}$.

Proof. Set $\gamma = (\beta-1)(\alpha+2)/2$, $a = (\alpha+1)/2$, and $c = a + \gamma$. If we apply (5.1) of Theorem 5.1 with the value $h = c\log n/n$ which satisfies $n^a\mathrm{e}^{-nh} = n^{-\gamma}$, as well as the inequality $(1-h)^n \leq \mathrm{e}^{-nh}$, we get:

$$a_n(C_\varphi) \leq C\bigg[n^{-\gamma} + \Big(\frac{\log n}{n}\Big)^\gamma\bigg] \leq C\Big(\frac{\log n}{n}\Big)^\gamma,$$

ending the proof. □

In [19], we had only the assertion on Schatten classes, for the single value $\alpha = -1$, and not the upper bound for the individual approximation numbers $a_n(C_\varphi)$.

Corollary 5.4 *Let (ε_n) a sequence of positive numbers which tends to 0. Then, there exists a Schur function φ with the following properties:*

1. *$\varphi\colon \mathbb{D} \to \mathbb{D}$ is surjective and 4-valent;*
2. *$a_n(C_\varphi) \leq C\mathrm{e}^{-n\varepsilon_n}$, $n = 1, 2, \ldots$*

In particular, we can get $a_n(C_\varphi) \leq C\mathrm{e}^{-\frac{n}{\log(n+1)}}$ and C_φ is in every Schatten class $S_p(\mathfrak{B}_\alpha)$, $p > 0$.

Notice that the sequence (ε_n) in the statement cannot be dispensed with. Indeed, if φ is surjective, we surely have $\|\varphi\|_\infty = 1$! And we know from Theorem 3.4 that $\beta(C_\varphi) = 1$ in that case.

We begin with a lemma of independent interest.

Lemma 5.5 *Let $\delta\colon (0,1] \to \mathbb{R}$ be a positive and non-decreasing function. Then there exists a Schur function φ with the following properties:*

1. *$\varphi\colon \mathbb{D} \to \mathbb{D}$ is surjective and 4-valent;*
2. *$\rho_{\varphi,\alpha+2}(h) \leq \delta(h)$, for $h > 0$ small enough.*

Proof. We begin with the case $\alpha = -1$. Set, for $a = 1/2$:

$$\Phi_a(z) = \frac{a-z}{1-az}, \quad B = \Phi_a^2,$$

and $C = \frac{1+a}{2(1-a)} = 3/2$. Note that $B\big(\frac{2a}{a+1}\big) = B(0)$. Let now

$$b_n = \frac{1}{4n\pi}, \qquad \varepsilon(h) = \frac{1}{2}\delta(h/C), \qquad \varepsilon_n = \varepsilon(b_{n+1}).$$

In the proof of Theorem 4.1 of [22], using an argument of harmonic measure and of barrier, we have found a 2-valent symbol φ_1 with $\varphi_1(\mathbb{D}) = \mathbb{D}^*$ such that, noting ρ_φ for $\rho_{\varphi,1}$:

$$b_{n+1} < h \leq b_n \quad \Longrightarrow \quad \rho_{\varphi_1}(h) \leq \varepsilon_n. \tag{5.7}$$

This gives $\rho_{\varphi_1}(h) \leq \varepsilon(b_{n+1}) \leq \varepsilon(h)$. Let now, as in [22], $\varphi = B \circ \varphi_1$. This Schur function is surjective (since $\varphi(\mathbb{D}) = B(\mathbb{D}^*) = B(\mathbb{D}) = \mathbb{D}$), and 4-valent. Moreover, if $I = (u,v)$ is an arc of $\mathbb{T}$ of length $h < 1/2$ and $J = (\frac{u}{2}, \frac{v}{2})$, we have $B^{-1}(I) \subset \Phi_a(J) \cup \Phi_a(-J) = I_1 \cup I_2$, where I_1, I_2 are two arcs of $\mathbb{T}$ of length at most $\|P_a\|_\infty (h/2) = Ch$, since Φ_a being an inner function, we have ([34]), P_a being the Poisson kernel at a:

$$m_{\Phi_a} = P_a m.$$

Hence, using (5.7), we obtain:

$$m_\varphi(I) = m_{\varphi_1}(B^{-1}(I)) \leq m_{\varphi_1}(I_1) + m_{\varphi_1}(I_2) \leq 2\rho_{\varphi_1}(Ch) \leq 2\varepsilon(Ch) = \delta(h),$$

and $\rho_\varphi(h) \leq \delta(h)$ for small h, by passing to the supremum on all I's.

For the general case $\alpha \geq -1$, we use the following extension of an inequality from [23] (which treats the case $\alpha = 0$, see Remark before Corollary 3.11):

Lemma 5.6 *For small h, namely $0 < h < (1 - |\varphi(0)|)/4$, we have, for every $\alpha > -1$:*

$$\rho_{\varphi,\alpha+2}(h) \leq C[\rho_\varphi(Ch)]^{\alpha+2}. \tag{5.8}$$

Proof. Let us define, as in [42], the generalized Nevanlinna counting function $N_{\varphi,\alpha+2}$ by the formula

$$N_{\varphi,\alpha+2}(w) = \sum_{\varphi(z)=w} [\log(1/|z|)]^{\alpha+2}, \qquad w \in \mathbb{D} \setminus \{\varphi(0)\}.$$

The case $\alpha = -1$ corresponds to the usual Nevanlinna counting function, which will be denoted by N_φ. The partial Nevanlinna counting function $N_\varphi(r, w)$ is defined, for $0 \leq r \leq 1$, by:

$$N_\varphi(r, w) = \sum_{\varphi(z)=w} \log^+(r/|z|),$$

so that $N_\varphi(1, w) = N_\varphi(w)$.

Since $\alpha + 2 \geq 1$, we have the obvious but useful inequality:

$$N_{\varphi,\alpha+2}(w) \leq [N_\varphi(w)]^{\alpha+2}. \tag{5.9}$$

We shall also make use of the following identity, due to J. Shapiro ([42], Proposition 6.6, where a weight $1/r$ is missing), and which can easily be checked after two integrations by parts:

$$N_{\varphi,\alpha+2}(w) = (\alpha + 2)(\alpha + 1) \int_0^1 N_\varphi(r, w)[\log(1/r)]^\alpha \, \frac{dr}{r} \cdot \tag{5.10}$$

As it was noticed in ([23], Theorem 3.10), this formula reads, for w close to the boundary, as follows, for $0 < h < (1 - |\varphi(0)|)/4$ and $|w| > 1 - h$:

$$N_{\varphi,\alpha+2}(w) = (\alpha + 2)(\alpha + 1) \int_{1/3}^1 N_\varphi(r, w)[\log(1/r)]^\alpha \, \frac{dr}{r} \cdot \tag{5.11}$$

Under the same conditions on h and w, this obviously implies:

$$N_{\varphi,\alpha+2}(w) \geq \frac{1}{C} \int_{1/3}^1 N_\varphi(r, w)(1 - r^2)^\alpha r \, dr = \frac{1}{C} \int_0^1 N_\varphi(r, w)(1 - r^2)^\alpha r \, dr.$$

Now, using the same arguments as in [23], Theorem 3.10 and in particular using (5.11) for $\varphi_r(z) = \varphi(rz)$, the identity $N_\varphi(r, w) = N_{\varphi_r}(w)$ and an integration in polar coordinates, we get:

$$\sup_{|w|\geq 1-h} N_{\varphi,\alpha+2}(w) \geq \frac{1}{C} \, \rho_{\varphi,\alpha+2}(h/C). \tag{5.12}$$

The end of the proof is easy: changing h into Ch and using successively (5.12) and (5.9), we get for small h, depending on φ:

$$\rho_{\varphi,\alpha+2}(h) \leq C \sup_{|w|\geq 1-Ch} N_{\varphi,\alpha+2}(w) \leq C \sup_{|w|\geq 1-Ch} [N_\varphi(w)]^{\alpha+2} \leq C \, [\rho_\varphi(Ch)]^{\alpha+2},$$

the last inequality coming from [21], Theorem 3.1. This ends the proof of (5.8).

Going back to the proof of Lemma 5.5, if we apply the already settled case $\alpha = -1$ to the function $\tilde{\delta}(h) = [\delta(h/C)/C]^{\frac{1}{\alpha+2}}$, we obtain a surjective and 4-valent Schur function φ such that:

$$\rho_{\varphi,\alpha+2}(h) \leq C \, [\rho_\varphi(Ch)]^{\alpha+2} \leq C \, [\tilde{\delta}(Ch)]^{\alpha+2} = \delta(h),$$

for h small enough. $\square$

Proof of Corollary 5.4. Set $a = (\alpha + 1)/2$. Provided that we replace (ε_n) by the decreasing sequence (ε'_n) with $\varepsilon'_n = \frac{1}{n} + \sup_{k\geq n} \varepsilon_k \geq \varepsilon_n$, we can assume that (ε_n) decreases. Let $A\colon [0,1] \to [0,1]$ be a function such that $A(0) = 0$, and which increases (as well as $A(t)/t$) so slowly that $A(\varepsilon_n + a(\log n/n)) \leq 1/2n$; therefore $A^{-1}(1/2n) \geq \varepsilon_n + a(\log n/n)$ and

$$n^a \mathrm{e}^{-nA^{-1}(1/2n)} \leq \mathrm{e}^{-n\varepsilon_n}.$$

We now apply Lemma 5.5 to the non-decreasing function $\delta(h) = h^{2+\alpha}\mathrm{e}^{-h/A(h)}$ to get the result, in view of (5.2) of Theorem 5.1. $\square$

Our last corollary involves Hardy-Orlicz spaces H^ψ and Bergman-Orlicz spaces $\mathfrak{B}^\psi$. For the definitions, we refer to [18].

Corollary 5.7 *There exists a Schur function φ and an Orlicz function ψ such that $C_\varphi\colon H^\psi \to H^\psi$ is compact whereas $C_\varphi\colon \mathfrak{B}^\psi \to \mathfrak{B}^\psi$ is not compact. Moreover, the approximation numbers $a_n(C_\varphi)$ of $C_\varphi\colon \mathfrak{B}_\alpha \to \mathfrak{B}_\alpha$ satisfy the upper estimate $a_n(C_\varphi) \leq a\,\mathrm{e}^{-b\sqrt{n}}$ where a, b are positive constants independent of n, and therefore C_φ belongs to $\bigcap_{p>0} S_p(\mathfrak{B}_\alpha)$.*

Proof. Let $\alpha \geq -1$ be fixed. The Schur function constructed in the proof of Theorem 4.2 of [23] satisfies the two first assertions, as well as $\rho_\varphi(h)/h \leq \mathrm{e}^{-d/h}$ for some positive constant $d > 0$. We now apply (5.8) to get for small h:

$$\frac{\rho_{\varphi,\alpha+2}(h)}{h^{\alpha+2}} \leq C\,\frac{[\rho_\varphi(Ch)]^{\alpha+2}}{h^{\alpha+2}} \leq C^{\alpha+3}\mathrm{e}^{-(\alpha+2)d/Ch} \leq a\,\mathrm{e}^{-b/h}$$

for positive constants a and b. We can thus apply (5.2) of Theorem 5.1, for some $\delta > 0$, with the increasing function $A(h) = h^2/\delta$ (hence $A^{-1}(x) = \sqrt{\delta x}$) to get the result, diminishing slightly b to absorb the power factor $n^{\frac{\alpha+1}{2}}$. $\square$

Remark. Let us alternatively consider the entropy numbers $e_n(C_\varphi)$ (see [4] or [17], page 69 for the definition) of composition operators. Those numbers are also a very good indicator of the "degree of compactness" of general operators $T\colon X \to Y$ where X, Y are Banach spaces and are smaller than the approximation numbers, in the following weak sense ([38], page 64).

$$\sup_{1\leq k\leq n} [k^\alpha e_k(T)] \leq C_\alpha \sup_{1\leq k\leq n} [k^\alpha a_k(T)], \qquad \forall \alpha > 0. \tag{5.13}$$

$$(a_n(T)) \in \ell_q \quad \Longrightarrow \quad (e_n(T)) \in \ell_q, \qquad \forall q > 0. \tag{5.14}$$

The converse of (5.14) does not hold in Banach spaces, but it does for operators between Hilbert spaces, by polar decomposition. More precisely, we have ([38], page 68) $a_n(T) \leq 4\,e_n(T)$ and, in particular, $(e_n(T)) \in \ell_q$ if and only if $(a_n(T)) \in \ell_q$.

We now have the following improved version of Theorem 3.1. Recall that $\varphi^\#(z) = \frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}$ and $[\varphi] = \|\varphi^\#\|_\infty$.

Theorem 5.8 *Let $T = C_\varphi$ be a compact composition operator on $\mathfrak{B}_\alpha$, and $\gamma(T) = \liminf_{n\to\infty} [e_n(T)]^{1/n}$. Then:*

$$\gamma(T) \geq [\varphi]^{1/2}. \tag{5.15}$$

Proof. We proceed as in the proof of Theorem 3.1. First, recall that the entropy numbers $e_n(T)$ also have the ideal property ([17], page 69), namely:

$$e_n(ATB) \leq \|A\|\, e_n(T)\, \|B\|.$$

Then, we use an improved Weyl-type inequality for entropy numbers, due to Carl and Triebel ([5]), in which $(\lambda_n(T))_{n\geq 1}$ denotes the sequence of eigenvalues of T rearranged in non-increasing order of moduli and $C = \sqrt{2}$:

$$\Big(\prod_{k=1}^{n} |\lambda_k(T)|\Big)^{1/n} \leq C e_n(T). \tag{5.16}$$

It should be noted that this inequality can itself be improved ([11]):

$$\Big(\prod_{k=1}^{n} a_k(T)\Big)^{1/n} \leq C e_n(T). \tag{5.17}$$

Yet, the tempting similar inequality $\Big(\prod_{k=1}^{n} |\lambda_k(T)|\Big)^{1/n} \leq C a_n(T)$ is wrong (even the inequality $|\lambda_n(T)| \leq C a_n(T)$ is wrong) as follows from an example of ([17], pages 133–134). Note that (5.17) implies the following:

$$a_n(T) \geq \delta r^n \quad \Longrightarrow \quad e_n(T) \geq \frac{\delta}{C} r^{1/2}\, r^{n/2}.$$

This might explain why a square root appears in (5.15), and tends to indicate that $[\varphi]$ should appear instead of $[\varphi]^2$ in Theorem 3.1.

Now, for every $a \in \mathbb{D}$, let again Φ_a be defined by $\Phi_a(z) = \frac{a-z}{1-\bar{a}z}$, for $z \in \mathbb{D}$. Set $b = \varphi(a)$ and define $\psi = \Phi_b \circ \varphi \circ \Phi_a$. We already know that 0 is a fixed point of ψ with derivative $\psi'(0) = \phi^{\#}(a)$ and that $C_\psi = C_{\Phi_a} \circ C_\phi \circ C_{\Phi_b}$. We may assume that $\psi'(0) = \phi^{\#}(a) \neq 0$. The sequence of eigenvalues of C_ψ is then, as we have seen, $((\psi'(0)^n)_{n\geq 0}$ ([41], p. 96). The equation (5.16) then gives us, setting $r = |\psi'(0)| = \phi^{\#}(a)$:

$$e_n(C_\psi) \geq \frac{1}{C}\Big(\prod_{k=0}^{n-1} r^k\Big)^{1/n} = \frac{1}{C}\, r^{(n-1)/2}.$$

This clearly gives us $\gamma(C_\psi) \geq \sqrt{r}$. Now, since C_{Φ_a} and C_{Φ_b} are invertible operators, the relation $C_\psi = C_{\Phi_a} \circ C_\phi \circ C_{\Phi_b}$ and the ideal property of the numbers $e_n(T)$ imply that $\gamma(C_\varphi) = \gamma(C_\psi)$, and therefore, with the notation of (3.4), $\gamma(C_\varphi) \geq \big(\phi^{\#}(a)\big)^{1/2}$, for all $a \in \mathbb{D}$. Passing to the supremum on $a \in \mathbb{D}$, we end the proof of Theorem 5.8. $\square$

6 The explicit example of lens maps

To ease notation, we shall suppose in this section that $\alpha = -1$, i.e. we are concerned with the Hardy space H^2. Fix $0 < \theta < 1$. Denote by $\mathbb{H} = \{z \in \mathbb{C}\,;\ \mathfrak{Re}\, z > 0\}$ the right half-plane, by $T\colon \mathbb{D} \to \mathbb{C} \setminus \{-1\}$ the involutive transformation defined by $T(z) = \frac{1-z}{1+z}$, which maps $\mathbb{D}$ to $\mathbb{H}$, and by τ_θ the transformation $z \in \mathbb{H} \mapsto z^\theta \in \mathbb{H}$. Recall that the associated lens map $\varphi_\theta\colon \mathbb{D} \to \mathbb{D}$ is:

$$\varphi_\theta = T \circ \tau_\theta \circ T.$$

It is known that the associated composition operator on H^2 is in all Schatten classes S_p ([43], Theorem 6.3). Alternatively, one could use Luecking's criterion ([27]). Therefore, its approximation numbers decrease rather quickly. Still more precisely, adapting techniques of Parfenov ([35], page 511), we might show the following (where $\beta_\theta, \gamma_\theta, \ldots$ are positive constants):

$$a_n(C_{\varphi_\theta}) \leq \gamma_\theta e^{-\beta_\theta \sqrt{n}}. \tag{6.1}$$

We shall not detail this adaptation of Parfenov's methods from Carleson embeddings to composition operators, but shall dwell on the converse inequality, which is not proved in [35]. First, the proof of the second assertion being postponed, we show that there is no converse to the inequality of Theorem 3.1.

Proposition 6.1 *The value of $[\varphi_\theta]$ for the lens map is*

$$[\varphi_\theta] = \theta. \tag{6.2}$$

In particular, $[\varphi_\theta]$ can be as small as we wish, although $\beta(C_{\varphi_\theta}) = 1$.

Recall that β is defined in (1.3) and $[\varphi]$ in (3.1).

Proof. First note the simple

Lemma 6.2 *Let $z \in \mathbb{D}$ and $v = T(z) \in \mathbb{H}$. Then:*

$$|T'(z)|(1 - |z|^2) = 2\mathfrak{Re}\,(T(z)) \quad \textit{and} \quad \frac{|T'(v)|}{1 - |T(v)|^2} = \frac{1}{2\mathfrak{Re}\, v}\cdot$$

The two equalities are the same because $|T'(v)| = \frac{1}{|T'(z)|}$ in view of $T = T^{-1}$. For the first one, we have:

$$|T'(z)|(1 - |z|^2) = \frac{2(1 - |z|^2)}{|1 + z|^2} = 2\mathfrak{Re}\,(T(z)).$$

Let now $z \in \mathbb{D}$ and $w = T(z) \in \mathbb{H}$. By the chain rule, we have:

$$\varphi_\theta'(z) = T'(\tau_\theta(w))\, \tau_\theta'(w)\, T'(z).$$

Taking moduli and using the lemma with z and $v = \tau_\theta(w)$, we obtain:

$$\frac{|\varphi_\theta'(z)|(1 - |z|^2)}{1 - |\varphi_\theta(z)|^2} = \frac{|T'(\tau_\theta(w))|}{1 - |T(\tau_\theta(w))|^2} |\tau_\theta'(w)|\, |T'(z)|(1 - |z|^2) = \frac{|\tau_\theta'(w)| \mathfrak{Re}\, w}{\mathfrak{Re}\,(\tau_\theta(w))}\cdot$$

Now, setting $w = r\mathrm{e}^{it}$ with $r > 0$ and $-\pi/2 < t < \pi/2$, this writes as well:

$$\varphi_\theta^{\#}(z) = \frac{\theta r^{\theta-1} r \cos t}{r^\theta \cos \theta t} = \frac{\theta \cos t}{\cos \theta t}.$$

Using the fact that w runs over $\mathbb{H}$ as z runs over $\mathbb{D}$ and that the cosine decreases on $(0, \pi/2)$, we obtain (6.2) by taking $t = 0$. □

We now prove the second assertion of Proposition 6.1 under the following form (the small roman and Greek letters $a_\theta, \ldots, \beta_\theta, \ldots$ will denote positive constants depending only on θ):

Proposition 6.3 *There exist constants $b_\theta, c_\theta, , \beta_\theta, \gamma_\theta$ with $b_\theta = \pi\sqrt{\frac{2(1-\theta)}{\theta}}$ such that:*

$$c_\theta \, \mathrm{e}^{-b_\theta \sqrt{n}} \le a_n(C_{\varphi_\theta}) \le \gamma_\theta \, \mathrm{e}^{-\beta_\theta \sqrt{n}}. \tag{6.3}$$

In particular, we have $\beta(C_{\varphi_\theta}) = 1$ and C_{φ_θ} is in all Schatten classes S_p, $p > 0$ but its approximation numbers do not decrease exponentially.

The upper bound is (6.1). For the lower bound, we shall need two simple lemmas.

Lemma 6.4 *Let $0 < \sigma < 1$ and $u = (u_j)$ be a sequence of points of $\mathbb{D}$ such that $\frac{1-|u_{j+1}|}{1-|u_j|} \le \sigma$. Then, the Carleson constant δ_u of the sequence u satisfies:*

$$\delta_u \ge \exp\Big(-\frac{a}{1-\sigma}\Big), \qquad \textit{with } a = \frac{\pi^2}{2}\cdot$$

Proof. We use the following fact ([13], pages 203–204):

$$\delta_u \ge \prod_{j=1}^{\infty} \Big(\frac{1-\sigma^j}{1+\sigma^j}\Big)^2. \tag{6.4}$$

This implies $\log \delta_u \ge 2\sum_{j=1}^\infty \log(\frac{1-\sigma^j}{1+\sigma^j})$. Now, expanding the logarithm in power series and permuting sums, we note that:

$$\begin{aligned} 2\sum_{j=1}^{\infty} \log\Big(\frac{1+\sigma^j}{1-\sigma^j}\Big) &= 4\sum_{k=0}^{\infty} \frac{\sigma^{2k+1}}{(2k+1)(1-\sigma^{2k+1})} \\ &\le 4\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2(1-\sigma)} = \frac{a}{1-\sigma}, \end{aligned}$$

where we used $1 - \sigma^{2k+1} \ge (2k+1)(1-\sigma)\sigma^{2k+1}$ and $\sum_{k=0}^\infty \frac{1}{(2k+1)^2} = \pi^2/8$. So that $\delta_u \ge \exp\big(-a/(1-\sigma)\big)$, which was to be proved. □

The second lemma is similar.

Lemma 6.5 *Let $0 < \sigma < 1$, $u_j = 1 - \sigma^j$, $v_j = \varphi_\theta(u_j)$ and $v = (v_j)$. Then, the Carleson constant δ_v of the sequence v satisfies:*

$$\delta_v \geq \exp\Big(- \frac{a_\theta}{1-\sigma}\Big), \qquad \textit{with } a_\theta = \frac{\pi^2}{2^\theta \theta}.$$

Proof. We first note that $1 - \varphi_\theta(r) = \frac{2(1-r)^\theta}{(1+r)^\theta + (1-r)^\theta}$, and so

$$\frac{1 - v_{j+1}}{1 - v_j} = \sigma^\theta \frac{\sigma^{j\theta} + (2 - \sigma^j)^\theta}{\sigma^{(j+1)\theta} + (2 - \sigma^{j+1})^\theta} = \sigma_j,$$

with $\sigma_j \leq \sigma' = 1 - \frac{\theta}{2} 2^\theta (1-\sigma)$. To see this, observe that:

$$1 - \sigma_j = \frac{(2 - \sigma^{j+1})^\theta - (2\sigma - \sigma^{j+1})^\theta}{\sigma^{(j+1)\theta} + (2 - \sigma^{j+1})^\theta} \overset{def}{=} \frac{N}{D} \geq \theta 2^{\theta - 1}(1-\sigma) = 1 - \sigma'.$$

Indeed, the function $f(x) = x^\theta + (2-x)^\theta$ increases on $[0,1]$, so $D \leq f(1) = 2$. On the other hand, the mean-value theorem gives $N = 2(1-\sigma)\theta\, c^{\theta-1} \geq \theta(1-\sigma)2^\theta$ for some $c \in (0,2)$. Lemma 6.4 then gives the result for the sequence v. □

Proof of Proposition 6.3. Fix an integer $n \geq 1$, and take (u_j), (v_j) as in Lemma 6.5. We have $\varphi_\theta(0) = 0$, $|\varphi_\theta(z)| \leq |z|$ and so for $0 < r < 1$:

$$\frac{1 - r^2}{1 - \varphi_\theta(r)^2} \geq \frac{1 - r}{1 - \varphi_\theta(r)} = \frac{(1-r)[(1-r)^\theta + (1+r)^\theta]}{2(1-r)^\theta} \geq \frac{(1-r)^{1-\theta}}{2},$$

implying

$$\frac{1 - u_j^2}{1 - v_j^2} \geq \frac{1}{2}\, \sigma^{n(1-\theta)}, \quad \text{for } 1 \leq j \leq n.$$

Let now R be an operator of rank $< n$. There exists a function $f = \sum_{j=1}^n \lambda_j K_{u_j} \in H^2 \cap \ker R$ with $\|f\| = 1$. We thus have, denoting by C_u and C_v the interpolation constants of the sequences u and v, and using Lemma 2.2 twice:

$$\begin{aligned}
\|C_{\varphi_\theta}^* - R\|^2 &\geq \|C_{\varphi_\theta}^*(f) - R(f)\|^2 = \|C_{\varphi_\theta}^*(f)\|^2 = \Big\| \sum_{j=1}^n \lambda_j K_{v_j} \Big\|^2 \\
&\geq C_v^{-2} \sum_{j=1}^n |\lambda_j|^2 \|K_{v_j}\|^2 = C_v^{-2} \sum_{j=1}^n \frac{|\lambda_j|^2}{1 - v_j^2} \\
&\geq \frac{1}{2}\, C_v^{-2} \sigma^{n(1-\theta)} \sum_{j=1}^n \frac{|\lambda_j|^2}{1 - u_j^2} \geq \frac{1}{2}\, C_u^{-2} C_v^{-2} \sigma^{n(1-\theta)} \|f\|^2 \\
&= \frac{1}{2}\, C_u^{-2} C_v^{-2} \sigma^{n(1-\theta)}.
\end{aligned}$$

Therefore, $a_n(C_{\varphi_\theta}) = a_n(C_{\varphi_\theta}^*) \geq \frac{1}{2}\, C_u^{-1} C_v^{-1} \sigma^{n(1-\theta)/2}$. But it follows from (2.2), Lemma 6.4 and Lemma 6.5 that C_u, C_v satisfy, provided that we now take the value $a_\theta = \frac{\pi^2}{\theta} > \frac{\pi^2}{2} + \frac{\pi^2}{2^\theta \theta}$, since $\theta + 2^{1-\theta} < 2$, to absorb the logarithmic factor of (2.2):

$$C_u C_v \leq c_\theta^{-1} \exp\big(a_\theta/(1-\sigma)\big).$$

The preceding now gives us (c_θ changing from line to line):

$$a_n(C_{\varphi_\theta}) \geq c_\theta \exp\Big(-\frac{a_\theta}{1-\sigma}\Big) \exp\Big(\frac{n(1-\theta)}{2}\log\sigma\Big).$$

Finally, adjust $\sigma = 1 - \lambda n^{-1/2}$ so that $\frac{a_\theta}{\lambda} = \frac{1-\theta}{2}\lambda$, i.e. $\lambda = \sqrt{\frac{2a_\theta}{1-\theta}}$ and use $\log(1-x) \geq -x - x^2$ for $0 \leq x \leq 1/2$; this gives (6.3) with the value

$$b_\theta = \frac{2a_\theta}{\lambda} = \sqrt{2a_\theta(1-\theta)} = \pi\sqrt{\frac{2(1-\theta)}{\theta}},$$

and that ends the proof of Proposition 6.3. □

Remarks.

1) The procedure used here to get lower estimates for the approximation numbers for lens maps might be easily adapted to a general symbol, to provide a new proof of Theorem 3.1. But the value of $\beta(C_\varphi)$ which we obtain in the general case is worse than the one obtained in Section 3, therefore we did not think it useful to include this second proof.

2) It is easy to see that, for the lens map φ_θ, one has $\rho_{\varphi_\theta}(h) \approx h^{1/\theta}$. Then Corollary 5.3 gives $a_n(C_{\varphi_\theta}) \leq C\, n^{-\frac{1-\theta}{2\theta}} (\log n)^{\frac{1-\theta}{2\theta}}$ and so $C_{\varphi_\theta} \in S_p$ for all $p > 2\theta/(1-\theta)$. On the other hand, we know ([43]) that $C_{\varphi_\theta} \in \bigcap_{p>0} S_p$, so that $a_n(C_{\varphi_\theta})$ must be rapidly decreasing: $a_n(C_{\varphi_\theta}) \leq C_q n^{-q}$ for all $q > 0$. This shows that Theorem 5.1 is very imprecise in general, becoming more accurate when ρ_φ is very small, as this is the case in Corollary 5.7.

We hope to return to upper bounds for approximation numbers in another work.

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