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# Some revisited results about composition operators on Hardy spaces

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**Abstract.** *We generalize, on one hand, some results known for composition operators on Hardy spaces to the case of Hardy-Orlicz spaces  $H^\Psi$ : construction of a “slow” Blaschke product giving a non-compact composition operator on  $H^\Psi$ ; construction of a surjective symbol whose composition operator is compact on  $H^\Psi$  and, moreover, is in all the Schatten classes  $S_p(H^2)$ ,  $p > 0$ . On the other hand, we revisit the classical case of composition operators on  $H^2$ , giving first a new, and simpler, characterization of closed range composition operators, and then showing directly the equivalence of the two characterizations of membership in the Schatten classes of Luecking and Luecking and Zhu.*

**Mathematics Subject Classification.** Primary: 47B33 – Secondary: 47B10

**Key-words.** Blaschke product – Carleson function – Carleson measure – composition operator – Hardy-Orlicz space – Nevanlinna counting function – Schatten classes

## 1 Introduction

The study of composition operators on Hardy spaces is now a classical subject (see [18], [3] for example). In [8] (see also [7]), we considered a more general setting and studied composition operators on Hardy-Orlicz spaces; we gave there a characterization of their compactness in terms of the Carleson function of their symbol (and in terms of the Nevanlinna counting function in [11]). This work was continued in [10]: we compared the compactness on Hardy spaces versus the compactness on Hardy-Orlicz spaces. For instance, we showed that there is, for every  $1 \leq p < \infty$ , an Orlicz function  $\Psi$  such that  $H^{p+\varepsilon} \subseteq H^\Psi \subseteq H^p$  for every  $\varepsilon > 0$ , and a composition operator  $C_\varphi$  such that  $C_\varphi$  is compact on  $H^p$  and  $H^{p+\varepsilon}$ , but which is not compact on  $H^\Psi$ .

We carry on this study in the present work. In a first part (Section 3 and Section 4), we shall improve, and extend to the Hardy-Orlicz case, results known for Hardy spaces; in a second part (Section 5 and Section 6), we shall give new

lights on some results concerning Hardy spaces. More precisely, the content of this paper is as following.

B. McCluer and J. Shapiro ([14], Theorem 3.10; see also [18], § 3.2) proved that, when their symbol  $\varphi$  is finitely-valent, compactness of composition operators  $C_\varphi$  on the Hardy space  $H^2$  can be characterized by the behaviour of the modulus of  $\varphi$  near the frontier of  $\mathbb{D}$ : compactness is equivalent to  $1 - |z| = o(1 - |\varphi(z)|)$  as  $|z| \rightarrow 1$ , but that is not equivalent in general ([14], Example 3.8; see also [18], § 10.2). In [11], Theorem 5.3, we gave such a characterization for composition operators, with finitely-valent symbol, on Hardy-Orlicz spaces. In Section 3, we construct a “slow” Blaschke product (generalizing [18], § 10.2 and [8], Proposition 5.5) showing that this condition is not sufficient in general.

In Section 4, we construct a compact composition operator  $C_\varphi: H^\Psi \rightarrow H^\Psi$  with surjective symbol  $\varphi$  and such that  $C_\varphi: H^2 \rightarrow H^2$  is in all the Schatten classes  $S_p(H^2)$ ,  $p > 0$ . This generalizes and improves a result of B. McCluer and J. Shapiro ([14], Example 3.12; see also the survey [16], § 2).

In Section 5, we give a characterization of composition operators  $C_\varphi: H^p \rightarrow H^p$ ,  $1 \leq p < \infty$ , with a closed range, simpler than the former ones (see [1] and [20]).

Finally, based on the main result of [11], we show directly, in Section 6, the equivalence of Luecking’s and Luecking-Zhu’s criteria ([12], [13]) for the membership of  $C_\varphi: H^2 \rightarrow H^2$  in the Schatten classes.

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## 2 Notation

The open unit disk is denoted by  $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$  and its boundary, the unit circle, by  $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ . The normalized Lebesgue measure  $dt/2\pi$  on  $\mathbb{T}$  is denoted by  $m$ . The normalized area measure  $dx dy/\pi$  is denoted by  $A$ .

The Hardy space  $H^1$  is the space of analytic functions  $f: \mathbb{D} \rightarrow \mathbb{C}$  such that  $\sup_{r < 1} \int_0^{2\pi} |f(re^{i\theta})| d\theta < \infty$ . Every  $f \in H^1$  has almost everywhere boundary values on  $\mathbb{T}$ , which are denoted by  $f^*$ .

An Orlicz function is a convex nondecreasing function  $\Psi: [0, \infty) \rightarrow [0, \infty)$  such that  $\Psi(0) = 0$  and  $\Psi(\infty) = \infty$ . If  $\mu$  is a positive measure on some measurable space  $S$ , the Orlicz space  $L^\Psi(\mu)$  is the set of all (classes of) measurable functions  $f: S \rightarrow \mathbb{C}$  such that  $\int_S \Psi(|f|/C) d\mu < \infty$  for some  $C > 0$ ; the norm  $\|f\|_\Psi$  is defined as the infimum of the positive numbers  $C$  for which  $\int_S \Psi(|f|/C) d\mu \leq 1$ .

The Hardy-Orlicz space  $H^\Psi$  is the linear subspace of  $f \in H^1$  such that  $f^* \in L^\Psi(m)$  (see [8]).

Every analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  defines a bounded composition operator  $C_\varphi: f \in H^\Psi \mapsto f \circ \varphi \in H^\Psi$  (see [8]).

For every  $\xi \in \mathbb{T}$  and  $0 < h < 1$ , the Carleson window is the set  $W(\xi, h) = \{z \in \mathbb{D}; |z| \geq 1-h \text{ and } |\arg(z\bar{\xi})| \leq h\}$ . The Carleson function  $\rho_\varphi$  of the analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  is defined, for  $0 < h < 1$ , by:

$$\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m(\{e^{i\theta} \in \mathbb{T}; \varphi^*(e^{i\theta}) \in W(\xi, h)\}).$$

Alternatively,  $\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m_\varphi[W(\xi, h)]$ , where  $m_\varphi$  is the pull-back measure of  $m$  by  $\varphi$ . We shall also use, instead of  $W(\xi, h)$ , the set  $S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\}$ , which has an equivalent size.

The Nevanlinna counting function  $N_\varphi$  is defined, for  $w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\}$ , by

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|},$$

each term  $\log \frac{1}{|z|}$  being repeated according to the multiplicity of  $z$ , and  $N_\varphi(w) = 0$  for the other  $w \in \mathbb{D}$ .

### 3 Slow Blaschke products

B. McCluer and J. Shapiro ([14], Theorem 3.10; see also [18], § 3.2) proved that, when  $\varphi$  is finitely-valent (meaning that, for some  $s \geq 1$ , the equation  $\varphi(z) = w$  has at most  $s$  solutions), the composition operators  $C_\varphi: H^p \rightarrow H^p$  is compact,  $1 \leq p < \infty$ , if and only if  $\varphi$  has an angular derivative at no point of  $\mathbb{T}$ ; that means that:

$$(3.1) \quad \lim_{|z| \rightarrow 1} \frac{1 - |z|}{1 - |\varphi(z)|} = 0.$$

In [11], Theorem 5.3, we generalized this result to Hardy-Orlicz spaces and proved that if  $\varphi$  is finitely-valent, the composition operator  $C_\varphi: H^\Psi \rightarrow H^\Psi$  is compact if and only if:

$$(3.2) \quad \lim_{|z| \rightarrow 1} \frac{\Psi^{-1}\left[\frac{1}{1 - |\varphi(z)|}\right]}{\Psi^{-1}\left[\frac{1}{1 - |z|}\right]} = 0.$$

Without the assumption that  $\varphi$  is finitely-valent, condition (3.2) is no longer sufficient to ensure the compactness of  $C_\varphi: H^\Psi \rightarrow H^\Psi$ . Indeed, we are going to construct a Blaschke product satisfying (3.2), but whose associated composition operator is of course not compact on  $H^\Psi$ , as this is the case for every inner function. A Blaschke product satisfying (3.1) is constructed in [18], § 10.2; that construction uses Frostman's Theorem. Our construction, which is more general, is entirely elementary.

**Theorem 3.1** Let  $\delta: (0, 1) \rightarrow (0, 1/2]$  be any function such that  $\lim_{t \rightarrow 0} \delta(t) = 0$ . Then, there exists a Blaschke product  $B$  such that:

$$(3.3) \quad 1 - |B(z)| \geq \delta(1 - |z|), \quad \text{for all } z \in \mathbb{D}.$$

**Corollary 3.2** For every Orlicz function  $\Psi$  there exists a Blaschke product  $B$  which satisfies:

$$\lim_{|z| \rightarrow 1} \frac{\Psi^{-1} \left[ \frac{1}{1 - |B(z)|} \right]}{\Psi^{-1} \left[ \frac{1}{1 - |z|} \right]} = 0.$$

though the composition operator  $C_B: H^\Psi \rightarrow H^\Psi$  is not compact.

**Proof.**  $C_B$  is not compact since every compact composition operator should satisfy  $|\varphi^*| < 1$  a.e. (see [8], Lemma 4.8). It suffices then to chose  $\delta(t) = 1/\Psi(\sqrt{\Psi^{-1}(1/t)})$ , which satisfies the hypothesis of Theorem 3.1. Moreover:

$$\frac{\Psi^{-1}(1/\delta(t))}{\Psi^{-1}(1/t)} = \frac{1}{\sqrt{\Psi^{-1}(1/t)}} \xrightarrow{t \rightarrow 0} 0,$$

and condition (3.3) gives the result.  $\square$

**Proof of Theorem 3.1.** We shall essentially construct our Blaschke product  $B$  as an infinite product of finite Blaschke products

$$\prod_n B_n,$$

where each finite Blaschke product  $B_n$  has  $p_n$  zeros equidistributed in the circumference of radius  $r_n$ . That is, we will have, writing  $\theta_k = 2\pi k/p_n$  and  $z_k = r_n e^{i\theta_k}$ , for  $k = 1, 2, \dots, p_n$ :

$$(3.4) \quad B_n(z) = \prod_{k=1}^{p_n} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z} = \prod_{k=1}^{p_n} \frac{r_n - e^{-i\theta_k} z}{1 - r_n e^{-i\theta_k} z}.$$

We shall need the following estimate for the finite Blaschke product in (3.4).

**Lemma 3.3** Let  $p \in \mathbb{N}$ , and  $0 < r < 1$ . Consider the finite Blaschke product

$$(3.5) \quad G(z) = \prod_{k=1}^p \frac{r - e^{-i\theta_k} z}{1 - r e^{-i\theta_k} z},$$

where  $\theta_k = \frac{2k\pi}{p}$ , for  $k = 1, 2, \dots, p$ .

(a) Then, for every  $z \in \mathbb{D}$  with  $|z| = r$ ,

$$(3.6) \quad |G(z)| \leq \frac{2r^p}{1 + r^{2p}} = 1 - \frac{(1 - r^p)^2}{1 + r^{2p}}.$$

(b) If besides we have  $ph \leq 1/2$ , where  $h = 1 - r$ , we also have, for every  $z \in \mathbb{D}$  with  $|z| = r$ ,

$$(3.7) \quad |G(z)| \leq 1 - \frac{(ph)^2}{2e}.$$

Let us continue the proof of the theorem. Define  $\chi: (0, 1) \rightarrow (0, 1]$  by:

$$(3.8) \quad \chi(x) = \sup_{t \leq x} [\max\{2\delta(t), \sqrt{t}\}].$$

Then  $\chi$  is non-decreasing,  $\lim_{x \rightarrow 0} \chi(x) = 0$  and  $\lim_{x \rightarrow 1} \chi(x) = 1$ . We can find a decreasing sequence  $(h_n)_{n \geq 0}$  of point  $h_n \in (0, 1)$ , such that  $\chi(h_n) \leq 2^{-n}$ . This sequence converges to 0; in fact,  $\sqrt{h_n} \leq \chi(h_n) \leq 2^{-n}$ , by (3.8), and hence:

$$(3.9) \quad h_n \leq 2^{-2n}.$$

We now define, for every  $n \in \mathbb{N}$ , a positive integer  $p_n$ , by:

$$(3.10) \quad p_n = \min\{p \in \mathbb{N}; \frac{p^2 h_n^2}{2e} > 2^{-n}\}.$$

We have  $p_n > 1$  because  $h_n^2/2e < h_n^2 \leq 2^{-4n}$ . So, for every  $n$ , we have  $4(p_n - 1)^2 \geq p_n^2$ , and then:

$$(3.11) \quad 4 \cdot 2^{-n} \geq \frac{4(p_n - 1)^2 h_n^2}{2e} \geq \frac{p_n^2 h_n^2}{2e}.$$

This yields, for  $n \geq 7$ , that  $(p_n h_n)^2 \leq 8e 2^{-n} \leq 1/4$ . Therefore  $p_n h_n \leq 1/2$ , and we can use the estimate in part (b) of Lemma 3.3.

Now, for  $n \geq 7$ , let  $B_n$  be the finite Blaschke product defined by (3.4), where  $r_n = 1 - h_n$ . Using (b) in Lemma 3.3, the Maximum Modulus Principle and the definition of  $p_n$  in (3.10), we have:

$$(3.12) \quad |B_n(z)| \leq 1 - \frac{p_n^2 h_n^2}{2e} < 1 - 2^{-n}, \quad \text{for } |z| \leq r_n.$$

Consider then the Blaschke product  $D$  defined by:

$$(3.13) \quad D(z) = \prod_{n=7}^{\infty} B_n(z).$$

This product is convergent since, by (3.11), we have:

$$\sum p_n(1 - r_n) = \sum p_n h_n \leq \sum \sqrt{8e 2^{-n}} < +\infty.$$

Finally, take  $N \in \mathbb{N}$  big enough to have  $r_6^N < 1/2$ , and define:

$$(3.14) \quad B(z) = z^N D(z).$$

Thus  $B$  is a Blaschke product, and, if  $|z| \leq r_6$ , we have, since  $\delta(t) \leq 1/2$ :

$$(3.15) \quad |B(z)| \leq |z^N| \leq r_6^N < 1/2 \leq 1 - \delta(1 - |z|).$$

If  $1 > |z| > r_6$ , there exists  $k \geq 7$  such that  $r_k \geq |z| > r_{k-1}$ . Therefore, thanks to (3.12),

$$(3.16) \quad |B(z)| \leq |D(z)| \leq |B_k(z)| \leq 1 - 2^{-k}.$$

On the other hand  $r_k \geq |z| > r_{k-1}$  implies  $h_k \leq 1 - |z| < h_{k-1}$ , and so:

$$(3.17) \quad \delta(1 - |z|) \leq \frac{1}{2}\chi(1 - |z|) \leq \frac{1}{2}\chi(h_{k-1}) \leq 2^{-k}.$$

Combining (3.16) and (3.17) we get  $|B(z)| \leq 1 - \delta(1 - |z|)$ , when  $1 > |z| > r_6$ . From this and (3.15), Theorem 3.1 follows.  $\square$

**Proof of Lemma 3.3.** It is obvious that, for all  $a, z \in \mathbb{C}$ ,

$$\prod_{k=1}^p (z - ae^{i\theta_k}) = z^p - a^p.$$

Using this we have:

$$(3.18) \quad G(z) = \prod_{k=1}^p \frac{r - e^{-i\theta_k} z}{1 - re^{-i\theta_k} z} = \prod_{k=1}^p \frac{z - re^{i\theta_k}}{rz - e^{i\theta_k}} = \frac{z^p - r^p}{(rz)^p - 1}.$$

Now, if  $|z| = r$ , we can write  $z^p = r^p u$ , for some  $u$  with  $|u| = 1$ . Then  $|G(z)| = |T(u)|$ , where  $T$  is the Moebius transformation

$$T(u) = \frac{r^p(u - 1)}{r^{2p}u - 1}.$$

This transformation  $T$  maps the unit circle  $\partial\mathbb{D}$  onto a circumference  $C$ . As  $T$  maps the extended real line  $\mathbb{R}_\infty$  to itself, and  $\partial\mathbb{D}$  is orthogonal to  $\mathbb{R}_\infty$  at the intersection points 1 and  $-1$ ,  $C$  is the circumference orthogonal to  $\mathbb{R}_\infty$  crossing through the points  $T(1) = 0$  and  $T(-1) = \alpha$ . It is easy to see that  $|w| \leq |\alpha|$ , for every  $w \in C$ ; consequently:

$$|G(z)| \leq \sup_{u \in \partial\mathbb{D}} |T(u)| = |T(-1)| = \frac{2r^p}{1 + r^{2p}}.$$

This finishes the proof of the statement (a).

To prove part (b), observe that,  $1 + r^{2p} \leq 2$ , and so, for  $|z| = r$ ,

$$(3.19) \quad |G(z)| \leq 1 - \frac{(1 - r^p)^2}{1 + r^{2p}} \leq 1 - \frac{(1 - r^p)^2}{2}.$$

Remember that  $r = 1 - h$ , so  $r \leq e^{-h}$ , and  $r^p \leq e^{-ph}$ . Thus  $1 - r^p \geq 1 - e^{-ph}$ . Now, if  $x \in [0, 1/2]$ , we have, by the Mean Value theorem:

$$1 - e^{-x} \geq \frac{x}{\sqrt{e}}.$$

Since  $ph \leq 1/2$ , we can apply this last estimate to (3.19) to get, as promised,

$$|G(z)| \leq 1 - \frac{(1 - e^{-ph})^2}{2} \leq 1 - \frac{p^2 h^2}{2e},$$

and ending the proof of Lemma 3.3.  $\square$

**Remark.** The key point in the proof of Theorem 3.1 is the inequality (3.6) in Lemma 3.3. This inequality may be viewed as a consequence of the strong triangle inequality (applied to  $a = z^p$ ,  $b = r^p$  and  $c = 0$ ):

$$(3.20) \quad d(a, b) \leq \frac{d(a, c) + d(c, b)}{1 + d(a, c)d(c, b)}$$

for the pseudo-hyperbolic distance  $d(u, v) = \frac{|u-v|}{|1-\bar{u}v|}$  on  $\mathbb{D}$ . Let us recall a proof for the convenience of the reader: by conformal invariance, we may assume that  $c = 0$ ; then:

$$1 - [d(a, b)]^2 = \frac{(1 - |a|^2)(1 - |b|^2)}{|1 - \bar{a}b|^2} \geq \frac{(1 - |a|^2)(1 - |b|^2)}{(1 + |a||b|)^2} = 1 - [d(|a|, -|b|)]^2,$$

so that:

$$d(a, b) \leq d(|a|, -|b|) = \frac{|a| + |b|}{1 + |a||b|},$$

proving (3.20), since  $d(a, 0) = |a|$  and  $d(0, b) = |b|$ .

## 4 A compact composition operator with a surjective symbol

A well-known result of J. H. Schwartz ([17], Theorem 2.8) asserts that the composition operator  $C_\varphi: H^\infty \rightarrow H^\infty$  is compact if and only if  $\|\varphi\|_\infty < 1$ . In particular, the compactness of  $C_\varphi: H^\infty \rightarrow H^\infty$  prevents the surjectivity of  $\varphi$ . It may be therefore to be expected that, the bigger  $\Psi$ , the more difficult it will be to obtain both the compactness of  $C_\varphi: H^\Psi \rightarrow H^\Psi$  and the surjectivity of  $\varphi$ . Nevertheless, this is possible, as says the following theorem, and the case  $H^\infty$  appears really as a singular case (corresponding to an ‘‘Orlicz function’’ which is discontinuous and can take the value infinity).

**Theorem 4.1** *For every Orlicz function  $\Psi$ , there exists a symbol  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  which is 4-valent and surjective and such that  $C_\varphi: H^\Psi \rightarrow H^\Psi$  is compact. Moreover,  $\varphi$  can be taken so as  $C_\varphi: H^2 \rightarrow H^2$  is in all the Schatten classes  $S_p(H^2)$ ,  $p > 0$ .*

In the case of  $H^2$  ( $\Psi(x) = x^2$ ), B. McCluer and J. Shapiro ([14], Example 3.12) gave an example based on the Riemann mapping theorem and on the fact that, for a finitely valent symbol  $\varphi$ , we have the equivalence:

$$(4.1) \quad C_\varphi: H^2 \rightarrow H^2 \text{ compact} \iff \lim_{|z| \nearrow 1} \frac{1 - |\varphi(z)|}{1 - |z|} = \infty.$$



A specific example is as follows. Take

$$(4.2) \quad R = \left\{ z = x + iy \in \mathbb{C}; x > 0 \text{ and } \frac{1}{x} < y < \frac{1}{x} + 4\pi \right\},$$

let  $g: \mathbb{D} \rightarrow R$  be a Riemann map and set  $\varphi = e^{-g}$ . Then,  $\varphi$  is 2-valent,  $\varphi(\mathbb{D}) = \mathbb{D}^*$  (where  $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ ), and the validity of (4.1) is tested through the use of the Julia-Carathéodory theorem (see [16] for details). To get a fully surjective mapping  $\varphi_1$ , just compose  $\varphi$  with the square of a Blaschke product:

$$\varphi_1(z) = B \circ \varphi, \quad \text{with } B(z) = \left( \frac{z - \alpha}{1 - \bar{\alpha}z} \right)^2, \quad \alpha \in D^* = \mathbb{D} \setminus \{0\}$$

(note that  $B(0) = B(2\alpha/1 + |\alpha|^2)$ ). Since  $C_{\varphi_1} = C_\varphi \circ C_B$ , we see that  $C_{\varphi_1}$  is compact as well and we are done.

Here, we can no longer rely on the Julia-Carathéodory theorem. But we shall use the following necessary and sufficient condition, in terms of the maximal Carleson function  $\rho_\varphi$ , which is valid for any symbol, finitely-valent or not (see [8], Theorem 4.18 – or [7], Théorème 4.2, where a different, but equivalent, formulation is given):

$$(4.3) \quad C_\varphi: H^\Psi \rightarrow H^\Psi \text{ compact} \iff \lim_{h \searrow 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\rho_\varphi(h))} = 0.$$

For the sequel, we shall set:

$$(4.4) \quad \Delta(h) = \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\rho_\varphi(h))}.$$

Our strategy will be to elaborate on the previous example to produce a (nearly) surjective  $\varphi$  such that  $\rho_\varphi(h)$  is very small (depending on  $\Psi$ ) for small  $h$ . The tool will be the notion of harmonic measure for certain open sets of the extended plane  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , called *hyperbolic* (see [2], Definition 19.9.3); for example, every conformal image of  $\mathbb{D}$  is hyperbolic (see [2], Proposition 19.9.2 (d) and Theorem 19.9.7). If  $G$  is a hyperbolic domain and  $a \in G$ , the *harmonic measure* of  $G$  at  $a$  is the probability measure  $\omega_G(a, \cdot)$  supported by  $\partial G$  (here, and throughout the rest of this section, boundaries and closures will be taken in  $\hat{\mathbb{C}}$ ) such that:

$$u(a) = \int_{\partial G} u(z) d\omega_G(a, z)$$

for each bounded and continuous function  $u$  on  $\overline{G}$ , which is harmonic in  $G$  (see [2], Definition 21.1.3). The harmonic measure at  $a$  of a Borel set  $A \subseteq \partial G$  will be denoted by  $\omega_G(a, A)$ . Clearly,

$$\omega_{\mathbb{D}}(0, \cdot) = m,$$

the Haar measure (*i.e.* normalized Lebesgue measure) of  $\partial\mathbb{D}$ .

R. Nevanlinna (see [2], Proposition 21.1.6) showed that harmonic measures share a *conformal invariance property*. Namely, assume that  $G$  is a simply connected domain, in which the Dirichlet problem can be solved (a *Dirichlet domain*), and  $\tau: \mathbb{D} \rightarrow \overline{G}$  is a continuous function which maps conformally  $\mathbb{D}$  onto  $G$ ; then  $\tau$  maps  $\partial\mathbb{D}$  onto  $\partial G$ , and, if  $\tau(0) = a$ :

$$(4.5) \quad \omega_G(a, A) = m(\tau^{-1}(A))$$

for every Borel set  $A \subseteq \partial G$ . This explains why harmonic measures enter the matter when we consider composition operators  $C_\varphi$ : such an operator induces a map  $H^\Psi \rightarrow L^\Psi(m_\varphi)$ , where  $m_\varphi = \varphi^*(m)$  appears as an image measure of  $m$ , as it happens for the harmonic measure of  $G$  at  $a$  in (4.5).

A useful alternative way of defining the harmonic measure, due to S. Kakutani, and completed by J. Doob (see [19], page 454, and [6], Appendix F, page 477) is the following: Let  $(B_t)_{t>0}$  be the 2-dimensional Brownian motion starting at  $a \in G$  (*i.e.*  $B_0 = a$ ), and  $\tau$  be the stopping time defined by:

$$(4.6) \quad \tau = \inf\{t > 0; B_t \notin G\};$$

we have:

$$(4.7) \quad \omega_G(a, A) = \mathbb{P}_a(B_\tau \in A),$$

*i.e.* the harmonic measure of  $A$  at  $a$  is the probability that the Brownian motion starting at  $a$  exits from  $G$  through the Borel set  $A \subseteq \partial G$ . The following lemma will be basic for the construction of our example. We shall provide two proofs, the second one being more illuminating.

**Lemma 4.2 (Hole principle)** *Let  $G_0$  and  $G_1$  be two hyperbolic open sets and  $H \subseteq \partial G_0$  a Borel set such that*

$$G_0 \subseteq G_1 \quad \text{and} \quad \partial G_0 \subseteq \partial G_1 \cup H.$$

*Then, for every  $a \in G_0$ , we have the following inequality:*

$$(4.8) \quad \omega_{G_1}(a, \partial G_1 \setminus \partial G_0) \leq \omega_{G_0}(a, H).$$

**Proof 1.** From [2], Corollary 21.1.14, with  $\Delta = \partial G_0 \cap \partial G_1$ , one has  $\omega_{G_0}(a, \Delta) \leq \omega_{G_1}(a, \Delta)$ . But  $\partial G_1 \setminus \Delta = \partial G_1 \setminus \partial G_0$ , and hence, since harmonic measures are probability measures,

$$\omega_{G_1}(a, \partial G_1 \setminus \partial G_0) = \omega_{G_1}(a, \partial G_1 \setminus \Delta) = 1 - \omega_{G_1}(a, \Delta) \leq 1 - \omega_{G_0}(a, \Delta);$$

we get the result since  $\partial G_0 = H \cup \Delta$ , which implies  $1 \leq \omega_{G_0}(a, H) + \omega_{G_0}(a, \Delta)$ .  
□

**Proof 2.** Let us define

$$(4.9) \quad \tau_0 = \inf\{t > 0; B_t \notin G_0\}, \quad \tau_1 = \inf\{t > 0; B_t \notin G_1\}$$

and

$$(4.10) \quad E = \{B_{\tau_1} \in \partial G_1 \setminus \partial G_0\}, \quad F = \{B_{\tau_0} \in H\}.$$

Inequality (4.8) amounts to proving that  $\mathbb{P}_a(E) \leq \mathbb{P}_a(F)$ , which will follow from the inclusion  $E \subseteq F$ . Suppose that the event  $E$  holds. Since  $G_0 \subseteq G_1$ , one has  $\tau_0 \leq \tau_1$ . The Brownian path  $(B_s)_{0 \leq s \leq \tau_1}$  being continuous with  $B_0 = a \in G_0$ , one has  $B_{\tau_0} \in \partial G_0 \subseteq \partial G_1 \cup H$ . If we had  $B_{\tau_0} \in \partial G_1$ , we should have  $B_{\tau_0} \notin G_1$ , since  $G_1$  is open, and hence  $\tau_0 = \tau_1$ , since we know that  $\tau_0 \leq \tau_1$ . But then  $B_{\tau_1} = B_{\tau_0} \in \partial G_0$ , contrary to the definition of  $E$ . Therefore,  $B_{\tau_0} \in H$  and  $F$  holds.  $\square$

We also shall need the following result (see [2], Proposition 21.1.17).

**Proposition 4.3 (Continuity principle)** *If  $G$  is a hyperbolic open set and  $a \in G$ , then the harmonic measure  $\omega_G(a, \cdot)$  is atomless.*

**Proof of Theorem 4.1.** It will be enough to construct a 2-valent mapping  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  such that  $\varphi(\mathbb{D}) = \mathbb{D}^*$  and  $C_\varphi: H^\Psi \rightarrow H^\Psi$  is compact. We can then modify  $\varphi$  by the same trick as the one used by B. McCluer and J. Shapiro. Note that every point in  $\mathbb{D}^*$  is the image by  $e^{-z}$  of two distinct points of  $R$ , except those which are the image of points of the hyperbola  $y = (1/x) + 2\pi$ , which have only one pre-image.

For a positive integer  $n$ , set:

$$(4.11) \quad b_n = \frac{1}{4n\pi},$$

and let  $\varepsilon_n > 0$  such that:

$$(4.12) \quad \frac{\Psi^{-1}(2/b_{n+1})}{\Psi^{-1}(1/\varepsilon_n)} \leq \frac{1}{n}.$$

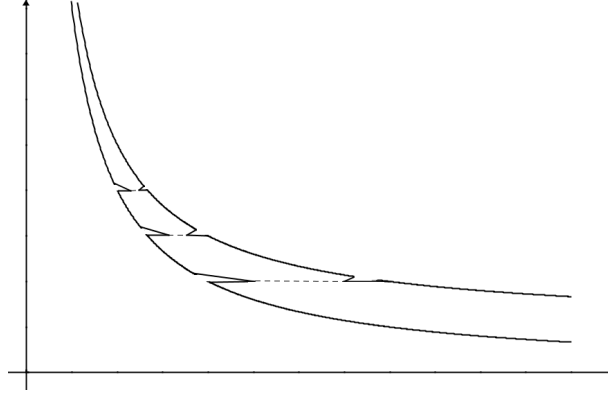
We now modify the domain  $R$ , including ‘‘barriers’’ in it (not in the sense of potential theory, nor of Perron!) in the following way.

Let, for every  $n \geq 1$ ,  $M_n$  be the intersection point of the horizontal line  $y = 4\pi n$  and of the hyperbola  $y = (1/x) + 2\pi$ , that is  $M_n = \frac{1}{4\pi n - 2\pi} + 4\pi n i$ .

Define inductively closed sets  $P_n^+$  and  $P_n^-$ , which are like small points of swords (two segments and a piece of hyperbola), in the following way:

- The lower part of  $P_n^+$  and  $P_n^-$  are horizontal segments of altitude  $4n\pi$ .
- Those two horizontal segments are separated by a small open horizontal segment  $H_n$  whose middle is  $M_n$ .
- The upper part of  $P_n^+$  is a slant segment whose upper extremity  $c_n^+$  lies on the hyperbola  $y = 1/x$ .
- The upper part of  $P_n^-$  is a slant segment whose upper extremity  $c_n^-$  lies on the hyperbola  $y = (1/x) + 4\pi$ .

- The curvilinear part of  $P_n^+$  is supported by the hyperbola  $y = 1/x$ .
- The curvilinear part of  $P_n^-$  is supported by the hyperbola  $y = (1/x) + 4\pi$ .
- One has  $4(n+1)\pi - \Im c_n^\pm > 2\pi$ .



The size of the small horizontal holes will be determined inductively in the following way. Fix once and for all  $a \in \mathbb{R}$  such that  $\Im a < 4\pi$ . Suppose that  $H_1, H_2, \dots, H_{n-1}$  have already been determined. Set:

$$(4.13) \quad \Omega_n = \left\{ z \in \mathbb{R} \setminus \bigcup_{j < n} (P_j^+ \cup P_j^-); \Im z < 4n\pi \right\}.$$

We can adjust  $H_n$  so small that:

$$(4.14) \quad \omega_{\Omega_n}(a, H_n) \leq \varepsilon_n.$$

Indeed,  $\Omega_n$  is bounded above by the horizontal segment  $[b_n + 4in\pi, b_{n-1} + 4in\pi]$ , where the point  $M_n$  lies. If  $H_n = [M_n - \delta, M_n + \delta]$ , we see that  $H_n$  decreases to the singleton  $\{M_n\}$  as  $\delta$  decreases to zero. Therefore, by Proposition 4.3, we can adjust  $\delta$  so as to realize (4.14).

We now define our modified open set  $\Omega$  by the formula

$$(4.15) \quad \Omega = \mathbb{R} \setminus \bigcup_{n \geq 1} (P_n^+ \cup P_n^-) = \bigcup_{n \geq 1} \Omega_n.$$

It is useful to observe that:

$$(4.16) \quad \inf_{w \in \partial\Omega_n} \Re w = b_n.$$

This is obvious by the way we defined the upper part of  $\partial\Omega_n$ .

Now, we can easily finish the proof. Fix  $h \leq b_1/2$  and let  $n$  be the integer such that:

$$(4.17) \quad b_{n+1} < 2h \leq b_n.$$

Let  $g: \mathbb{D} \rightarrow \Omega$  be a conformal mapping such that  $g(0) = a$ . Since  $\partial_\infty \Omega$  is connected, Caratheodory's Theorem (see [15]) ensures that  $g$  can be continuously extended from  $\overline{\mathbb{D}}$  onto  $\overline{\Omega}$ . More explicitly, using the Moebius transformation  $T(z) = 1/z$ , we see that there exists an automorphism of the extended complex plane such that  $\overline{\Omega}$  is sended onto a compact subset of  $\mathbb{C}$ ; so, we can apply to  $\Omega$  many results stated for bounded domains. For instance, the boundary of  $\Omega$  is a continuous path in the extended plane; so, by [2], Theorem 14.5.5,  $g$  can be extended to a continuous function (for the extended plane topology)  $g: \overline{\mathbb{D}} \rightarrow \overline{\Omega}$ . In particular,  $g$  has boundary values  $g^*$ .

We define  $\varphi = e^{-g}$ .

As in the proof of B. McCluer and J. Shapiro ([14]), we have that  $\varphi$  is 2-valent (see the remark made at the beginning of this proof), and we still have  $\varphi(\mathbb{D}) = \mathbb{D}^*$ , since, in the process for constructing  $\Omega$  from  $R$ , for every point of  $\mathbb{D}^*$ , at least one of the preimages by  $e^{-z}$  in  $R$  has not been removed. Observe that, in particular, we did not remove any point in the hyperbola  $y = (1/x) + 2\pi$ , thanks to the choice of  $M_n$ .

Moreover,  $\Omega$  is a Dirichlet domain (because each component of  $\partial\Omega$  has more than one point: see the comment after Definition 19.7.1 in [2]), so we can use the conformal invariance. Then by (4.5), (4.14), (4.16) and by the hole principle, we see that, if  $A = \{\Re e g^*(e^{it}) < 2h\}$ :

$$\begin{aligned}
(4.18) \quad \rho_\varphi(h) &\leq m_\varphi(\{|z| > 1 - h\}) = m(\{e^{-\Re e g^*(e^{it})} > 1 - h\}) \\
&= m(\{\Re e g^*(e^{it}) < \log(1/1 - h)\}) \\
&\leq m(\{\Re e g^*(e^{it}) < 2h\}) = \omega_{\mathbb{D}}(0, A) \\
&= \omega_{g(\mathbb{D})}(g(0), g(A)) = \omega_\Omega(a, \{\Re e w < 2h\}) \\
&\leq \omega_\Omega(a, \{\Re e w \leq b_n\}) \\
&\leq \omega_\Omega(a, \partial\Omega \setminus \partial\Omega_n) \leq \omega_{\Omega_n}(a, H_n) \leq \varepsilon_n.
\end{aligned}$$

It remains to observe that:

$$\Delta(h) = \frac{\Psi^{-1}(1/h)}{\Psi^{-1}(1/\rho_\varphi(h))} \leq \frac{\Psi^{-1}(2/b_{n+1})}{\Psi^{-1}(1/\varepsilon_n)} \leq \frac{1}{n} \leq Ch,$$

in view of (4.12) and of the choice of  $n$ ,  $C$  being a numerical constant. We should point out the fact that we applied the hole principle to the domains  $G_0 = \Omega_n$  and  $G_1 = \Omega$  and that this was licit because the assumptions of the hole principle (in particular the inclusion  $\partial\Omega_n \subseteq \partial\Omega \cup H_n$ ) are satisfied. We have therefore proved that:

$$\lim_{h \searrow 0} \Delta(h) = 0,$$

and this ends, as we already explained, the first part of the proof of Theorem 4.1.

To prove the last part, let us remark that in (4.12) we may take  $\varepsilon_n$  arbitrarily small. If one takes  $\varepsilon_n \leq e^{-n}$ , one has, for some constant  $c > 0$ ,  $\rho_\varphi(h) \leq e^{-c/h}$ , by using (4.17) and (4.18). In particular,  $\rho_\varphi(h) \leq Ch^\alpha$  for every  $\alpha > 1$ . By Luecking's criterion, that implies that  $C_\varphi \in S_p(\overline{H^2})$  for every  $p > 0$  (see [9], Corollary 3.2).  $\square$

**Remark.** Let us note that our result is stronger than McCluer-Shapiro's, since our  $C_\varphi$  is in all the Schatten classes  $S_p(H^2)$ ,  $p > 0$ . Though our construction follows McCluer-Shapiro's, it is the introduction of the "barriers"  $P_n^+$  and  $P_n^-$  which allows to get this improvement.

## 5 Composition operators with closed range

In [1], J. Cima, J. Thomson and W. Wogen gave a characterization of composition operators  $C_\varphi: H^p \rightarrow H^p$  with closed range. This characterization involves the Radon-Nikodym derivative of the restriction to  $\partial\mathbb{D}$  of  $m_\varphi$ . They found it not satisfactory, and asked a characterization with the range of  $\varphi$  itself. N. Zorboska ([20]) gave such a characterization, but her statement is somewhat complicated. We shall give here more explicit characterizations, either in terms of the Nevanlinna counting function  $N_\varphi$ , or in terms of the Carleson measure  $m_\varphi$ .

**Theorem 5.1** *Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  be a non-constant analytic self map. Then the composition operator  $C_\varphi: H^p \rightarrow H^p$ ,  $1 \leq p < \infty$ , has a closed range if and only if there is a constant  $c > 0$  such that, for  $0 < h < 1$ ,*

$$(5.1) \quad \frac{1}{A(S(\xi, h))} \int_{S(\xi, h)} N_\varphi(z) dA(z) \geq ch, \quad \forall \xi \in \partial\mathbb{D}.$$

Theorem 5.1 will follow immediately from the next theorem, applied to  $\mu = m_\varphi$ , and from [11], Theorem 4.2.

**Theorem 5.2** *Let  $\mu$  be a finite positive measure on  $\overline{\mathbb{D}}$ . Assume that the canonical map  $J: H^p \rightarrow L^p(\mu)$  is continuous,  $1 \leq p < \infty$ . Then  $J$  is one-to-one and has a closed range if and only if there is a constant  $c > 0$  such that, for  $0 < h < 1$ ,*

$$(5.2) \quad \mu[W(\xi, h)] \geq ch, \quad \forall \xi \in \partial\mathbb{D}.$$

**Proof.** 1) Assume that  $J$  has a closed range. By making a rotation on the variable  $z$ , we only have to find a constant  $c > 0$  such that

$$(5.3) \quad \mu(S_h) \geq ch,$$

for  $h > 0$  small enough, where  $S_h = S(1, h)$ .

Since  $J$  is one-to-one, there is a constant  $C > 0$  such that:

$$(5.4) \quad \|f\|_{L^p(\mu)}^p \geq C^p \|f\|_p^p, \quad \forall f \in H^p.$$

We are going to test (5.4) on

$$(5.5) \quad f_N(z) = \left( \frac{1+z}{2} \right)^N.$$

It is classical that there is a constant  $c_p > 0$  such that:

$$(5.6) \quad \|f_N\|_p^p = \int_{-\pi}^{\pi} \left| \cos \frac{t}{2} \right|^{pN} dt \geq \frac{c_p}{\sqrt{N}}.$$

Now, since  $|z+1|^2 + |z-1|^2 = 2(|z|^2 + 1) \leq 4$  for every  $z \in \overline{\mathbb{D}}$ , one has:

$$|f_N(z)| \leq \left( 1 - \frac{|z-1|^2}{4} \right)^{N/2} \leq e^{-\frac{N}{8}|z-1|^2}.$$

Hence, using  $|f_N(z)| \leq 1$  when  $|z-1| \leq h$ , one has:

$$\begin{aligned} \|f_N\|_{L^p(\mu)}^p &\leq \mu(S_h) + \int_{|z-1|>h} e^{-p\frac{N}{8}|z-1|^2} d\mu \\ &= \mu(S_h) + \int_0^{e^{-pNh^2/8}} \mu(\{e^{-p\frac{N}{8}|z-1|^2} > u\}) du, \end{aligned}$$

that is, making the change of variable  $u = e^{-p\frac{N}{8}x^2}$ ,

$$\|f_N\|_{L^p(\mu)}^p \leq \mu(S_h) + \int_h^\infty \mu(\{|z-1| \leq x\}) \frac{pN}{4} x e^{-p\frac{N}{8}x^2} dx.$$

Now, the continuity of  $J$  means, by Carleson's Theorem see [4], Theorem 9.3), that there is a constant  $K > 0$  such that:

$$(5.7) \quad \sup_{|\xi|=1} \mu(S(\xi, x)) \leq Kx, \quad 0 \leq x < 1.$$

We get hence:

$$\begin{aligned} \|f_N\|_{L^p(\mu)}^p &\leq \mu(S_h) + \int_h^\infty Kx \frac{pN}{4} x e^{-p\frac{N}{8}x^2} dx \\ &= \mu(S_h) + \frac{K\sqrt{8}}{\sqrt{p}} \frac{1}{\sqrt{N}} \int_{h\sqrt{\frac{pN}{8}}}^\infty y^2 e^{-y^2} dy. \end{aligned}$$

We take now for  $N$  the smaller integer  $> 1/h^2$ , multiplied by some constant integer  $a_p$ , large enough to have:

$$\frac{K\sqrt{8}}{\sqrt{p}} \int_{\sqrt{\frac{p a_p}{8}}}^\infty y^2 e^{-y^2} dy \leq \frac{c_p C^p}{2}.$$

We get then, from (5.4) and (5.6):

$$\mu(S_h) \geq \frac{C^p c_p}{2} \frac{1}{\sqrt{N}},$$

which gives (5.3).

2) Conversely, assume that (5.2) holds. Since the disk algebra  $A(\mathbb{D})$  is dense in  $H^p$ , it suffices to show that there exists a constant  $C > 0$  such that  $\|f\|_{L^p(\mu)} \geq C \|f\|_p$  for every  $f \in A(\mathbb{D})$ .

Let  $f \in A(\mathbb{D})$  such that  $\|f\|_p = 1$ . Choose an integer  $N$  such that:

$$\frac{1}{N} \sum_{n=1}^N |f(e^{2\pi in/N})|^p \geq \frac{1}{2} \int_{\partial\mathbb{D}} |f(\xi)|^p dm(\xi) = \frac{1}{2},$$

and such that, due to the uniform continuity of  $f$ ,

$$z, z' \in \overline{\mathbb{D}} \quad \text{and} \quad |z - z'| \leq \frac{2\pi}{N} \quad \implies \quad |f(z) - f(z')| \leq \frac{1}{2^{(p+1)/p}}.$$

Then, setting  $W_n = W(e^{2\pi in/N}, \pi/N)$ ,  $1 \leq n \leq N$ , one has:

$$\|f\|_{L^p(\mu)}^p = \int_{\mathbb{D}} |f|^p d\mu \geq \sum_{n=1}^N \int_{W_n} |f|^p d\mu.$$

If we choose  $z_n \in W_n$  such that  $|f(z_n)| = \min_{z \in W_n} |f(z)|$ , we get, using (5.2):

$$\|f\|_{L^p(\mu)}^p \geq \sum_{n=1}^N |f(z_n)|^p \mu(W_n) \geq \frac{c\pi}{N} \sum_{n=1}^N |f(z_n)|^p.$$

Since  $A^p \leq 2^{p-1}[(A-B)^p + B^p]$ , by Hölder's inequality, one has:

$$|f(z_n)|^p \geq \frac{1}{2^{p-1}} |f(e^{2\pi in/N})|^p - |f(z_n) - f(e^{2\pi in/N})|^p$$

and hence:

$$\|f\|_{L^p(\mu)}^p \geq \frac{c\pi}{N} \sum_{n=1}^N \left[ \frac{1}{2^{p-1}} |f(e^{2\pi in/N})|^p - |f(z_n) - f(e^{2\pi in/N})|^p \right].$$

Now, since  $z_n \in W_n$ , one has:

$$|z_n - e^{2\pi in/N}| \leq \left| z_n - \frac{z_n}{|z_n|} \right| + \left| \frac{z_n}{|z_n|} - e^{2\pi in/N} \right| \leq \frac{\pi}{N} + \frac{\pi}{N} = \frac{2\pi}{N};$$

therefore  $|f(z_n) - f(e^{2\pi in/N})| \leq 1/2^{p+1}$  and we get:

$$\begin{aligned} \|f\|_{L^p(\mu)}^p &\geq c\pi \left[ \frac{1}{N} \sum_{n=1}^N \frac{1}{2^{p-1}} |f(e^{2\pi in/N})|^p - \frac{1}{2^{p+1}} \right] \\ &\geq c\pi \left( \frac{1}{2^{p-1}} \frac{1}{2} - \frac{1}{2^{p+1}} \right) = \frac{c\pi}{2^{p+1}}. \end{aligned}$$

That ends the proof of Theorem 5.2.  $\square$



**Remark.** To make the link with Cima-Thomson-Wogen's criterion, we shall see that condition 5.2 implies that the restriction of  $\mu$  to the boundary  $\mathbb{T} = \partial\mathbb{D}$  of the disk dominates the Lebesgue measure  $m$ . In fact, let  $I$  be an arc of  $\mathbb{T}$ . If  $m(I) = h$ , we can write:

$$I = \bigcap_{n \geq 1} \bigcup_{j=1}^n W(\xi_{n,j}, h/2n),$$

with disjoint windows  $W(\xi_{n,1}, h/2n), \dots, W(\xi_{n,n}, h/2n)$ ; hence:

$$\mu(I) = \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu[W(\xi_{n,j}, h/2n)] \geq c \sum_{j=1}^n \frac{h}{2n} = \frac{c}{2} h.$$

## 6 Composition operators in Schatten classes

In [12], D. Luecking characterized composition operators  $C_\varphi: H^2 \rightarrow H^2$  which are in the Schatten classes, by using, essentially, the  $m_\varphi$ -measure of Carleson windows. Five years later, D. Luecking and K. Zhu ([13]) characterized them by using the Nevanlinna counting function of  $\varphi$ . We shall see in this section how the result of [11] makes these two characterizations directly equivalent.

It will be convenient here to work with *modified* Carleson windows, namely:

$$W_{n,j} = \left\{ z \in \overline{\mathbb{D}}; 1 - 2^{-n} \leq |z| \leq 1 \text{ and } \frac{(2j-1)\pi}{2^n} \leq \arg z < \frac{(2j+1)\pi}{2^n} \right\}$$

( $j = 0, 1, \dots, 2^n - 1$ ,  $n = 1, 2, \dots$ ). We shall say that  $W_{n,j}$  is the Carleson window centered at  $e^{2\pi i j/2^n}$  with size  $2^{-n}$ .

**Theorem 6.1** *For  $p > 0$  the two following conditions are equivalent:*

- a)  $\frac{N_\varphi(z)}{\log(1/|z|)} \in L^{p/2}(\lambda)$ , where  $d\lambda(z) = (1 - |z|)^{-2} dA(z)$  and  $A$  is the normalized area measure on  $\mathbb{D}$ ;
- b)  $\sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} [2^n m_\varphi(W_{n,j})]^{p/2} < \infty$ .

Condition *b*) in the last theorem yields that  $\lim_{n \rightarrow \infty} \max_j 2^n m_\varphi(W_{n,j}) = 0$ , and it is not difficult to see that this implies that  $m_\varphi(\partial\mathbb{D}) = 0$ , or equivalently, that  $|\varphi^*| < 1$  almost everywhere on  $\partial\mathbb{D}$ . In this situation we know ([9], Proposition 3.3) that *b*) in Theorem 6.1 is equivalent to Luecking's condition in [12]. In fact the characterization of belonging to a Schatten class in [12] includes the requirement  $m_\varphi(\partial\mathbb{D}) = 0$ .

**Proof.** We may, and do, assume that  $\varphi(0) = 0$ .

1) Assume first that condition *b*) is satisfied. Let:

$$R_{n,j} = \left\{ z \in \mathbb{D}; 1 - 2^{-n} \leq |z| < 1 - 2^{-n-1} \text{ and } \frac{(2j-1)\pi}{2^n} \leq \arg z < \frac{(2j+1)\pi}{2^n} \right\}$$

be the (disjoint) Luecking windows ( $0 \leq j \leq 2^n - 1$ ,  $n \geq 0$ ). One has  $R_{n,j} \subseteq W_{n,j}$ .

By [11], Theorem 3.1, there are a constant  $C > 0$  and an integer  $K$  such that  $N_\varphi(z) \leq C m_\varphi(\widetilde{W}_{n,j})$ , for every  $z \in R_{n,j}$ , where  $\widetilde{W}_{n,j}$  is the window centered at  $e^{2\pi i j/2^n}$ , as  $W_{n,j}$ , but with size  $2^{K-n}$ . The windows  $W_{n-K,j}$ ,  $j = 0, 1, \dots, 2^{n-K} - 1$ , have the same size as the windows  $\widetilde{W}_{n,j}$ , but may have a different center; nevertheless, each  $\widetilde{W}_{n,j}$  can be covered with two windows  $W_{n-K,l}$ : for  $n > K$ ,  $\widetilde{W}_{n,j} \subseteq W_{n-K,l} \cup W_{n-K,l+1}$ , for some  $l = 1, 2, \dots, 2^{n-K}$  (where  $l+1$  is understood as 0 if  $l = 2^{n-K} - 1$ ), we get (we shall use  $\lesssim$  to mean  $\leq$  up to a constant):

$$\begin{aligned} \int_{\mathbb{D}} \frac{(N_\varphi(z))^{p/2}}{(1-|z|)^{\frac{p}{2}+2}} dA(z) &\leq \sum_{n,j} \int_{R_{n,j}} (2^n)^{\frac{p}{2}+2} (N_\varphi(z))^{p/2} dA(z) \\ &\lesssim \sum_{n,j} \int_{R_{n,j}} (2^n)^{\frac{p}{2}+2} (m_\varphi(\widetilde{W}_{n,j}))^{p/2} dA(z) \\ &\lesssim \sum_{n,j} (2^n)^{p/2} (m_\varphi(\widetilde{W}_{n,j}))^{p/2} \\ &\lesssim \sum_{\nu,l} (2^\nu)^{p/2} (m_\varphi(W_{\nu,l}))^{p/2} < \infty, \end{aligned}$$

and  $a$ ) holds.

2) Conversely, assume that  $a$ ) is satisfied. We shall use the following inequality, whose proof will be postponed (for  $p \geq 2$ , (6.1) follows directly from [11], Theorem 4.2, and Hölder's inequality):

$$(6.1) \quad [m_\varphi(W_{n,j})]^{p/2} \lesssim \frac{1}{A(\widetilde{W}_{n,j})} \int_{\widetilde{W}_{n,j}} [N_\varphi(z)]^{p/2} dA(z),$$

where  $\widetilde{W}_{n,j}$  is a window with the same center as  $W_{n,j}$  but with a bigger proportional size; say of size  $2^{-n+L}$ . We get:

$$\begin{aligned} \sum_{n,j} [2^n m_\varphi(W_{n,j})]^{p/2} &\lesssim \sum_{n,j} 2^{np/2} 2^{2n} \int_{\widetilde{W}_{n,j}} [N_\varphi(z)]^{p/2} dA(z) \\ &= \int_{\mathbb{D}} \left( \sum_n 2^{n(2+\frac{p}{2})} \left[ \sum_j \mathbf{1}_{\widetilde{W}_{n,j}}(z) \right] \right) [N_\varphi(z)]^{p/2} dA(z). \end{aligned}$$

Let  $k = 0, 1, \dots$  such that  $1 - 2^{-k+1} < |z| \leq 1 - 2^{-k}$ . One has  $z \in \widetilde{W}_{n,j}$  only if  $n \leq k + L$ , and then, for each such  $n$ ,  $z$  is at most in  $2^L$  windows  $\widetilde{W}_{n,j}$ . It follows that:

$$\sum_n 2^{n(2+\frac{p}{2})} \sum_j \mathbf{1}_{\widetilde{W}_{n,j}}(z) \leq 2^{(k+L+1)(2+\frac{p}{2})} \times 2^L.$$

But  $|z| \geq 1 - 2^{-k+1}$  implies  $2^{(k+L+1)(2+\frac{p}{2})} \leq C_p/(1-|z|)^{2+\frac{p}{2}}$ ; hence:

$$\sum_{n,j} [2^n m_\varphi(W_{n,j})]^{p/2} \lesssim \int_{\mathbb{D}} \frac{[N_\varphi(z)]^{p/2}}{(1-|z|)^{\frac{p}{2}+2}} dA(z) < \infty,$$

and b) holds.

It remains to show (6.1).

By [11], Theorem 4.1, we can find a window  $W$  with the same center as  $W_{n,j}$ , but with greater size  $ch$  ( $h = 2^{-n}$  is the size of the window  $W_{n,j}$ ), such that:

$$m_\varphi(W_{n,j}) \lesssim \sup_{w \in W} N_\varphi(w).$$

There is hence some  $w_0 \in W$  such that:

$$m_\varphi(W_{n,j}) \lesssim N_\varphi(w_0).$$

Take  $R = |w_0| + ch$  (one has  $R \geq 1$  since  $w_0 \in W$  and  $W$  has size  $ch$ ) and set  $\varphi_0(z) = \varphi(z)/R$ . One has  $N_{\varphi_0}(z) = N_\varphi(Rz)$  for  $|z| < 1/R$  and  $N_{\varphi_0}(z) = 0$  if  $|z| \geq 1/R$ .

Let now  $u$  be the upper subharmonic regularization of  $N_{\varphi_0}$  ([13], Lemma 1, and its proof page 1140):  $u$  is a subharmonic function on  $\mathbb{D} \setminus \{0\}$  such that  $u \geq N_{\varphi_0}$  and  $u = N_{\varphi_0}$  almost everywhere, with respect to  $dA$ .

A result of C. Fefferman and E. M. Stein ([5], Lemma 2), generously attributed by them to Hardy and Littlewood, asserts that for any  $q > 0$ , there exists a constant  $C = C(q)$  such that

$$(6.2) \quad [u(a)]^q \leq \frac{C}{A(D(a,r))} \int_{D(a,r)} [u(z)]^q dA(z)$$

for every nonnegative subharmonic function  $u$  on a domain  $G$  and every disk  $D(a,r) \subseteq G$  (see also [13], Lemma 3).

If  $\Delta$  is the disk centered at  $w_0/R$  and of radius  $1 - |w_0|/R$  (which is contained in  $\mathbb{D} \setminus \{0\}$  since  $R > |w_0|$ ), one has, by (6.2):

$$\begin{aligned} [N_\varphi(w_0)]^{p/2} &= [N_{\varphi_0}(w_0/R)]^{p/2} \leq [u(w_0/R)]^{p/2} \\ &\leq \frac{C}{A(\Delta)} \int_{\Delta} [u(z)]^{p/2} dA(z) \\ &= \frac{C}{A(\Delta)} \int_{\Delta} [N_{\varphi_0}(z)]^{p/2} dA(z) \\ &= \frac{C}{A(\Delta)} \int_{\Delta \cap D(0,1/R)} [N_\varphi(Rz)]^{p/2} dA(z) \\ &= \frac{C}{A(\tilde{\Delta})} \int_{\tilde{\Delta} \cap \mathbb{D}} [N_\varphi(w)]^{p/2} dA(w), \end{aligned}$$

where  $\tilde{\Delta} = D(w_0, R - |w_0|) = D(w_0, ch)$ .

Since the center  $w_0$  of  $\tilde{\Delta}$  is in  $\mathbb{D}$ ,  $\tilde{\Delta} \cap \mathbb{D}$  contains more than a quarter of  $\tilde{\Delta}$  (at least for  $ch \leq 1$ ), and hence  $A(\tilde{\Delta} \cap \mathbb{D}) \geq A(\tilde{\Delta})/4 = c^2 h^2 / 4\pi$ . Now, let  $\tilde{W}_{n,j}$  be the window with the same center as  $W_{n,j}$  and of size  $2ch$ . Since  $2ch \geq ch + (1 - |w_0|)$ ,  $\tilde{W}_{n,j}$  contains  $\tilde{\Delta} \cap \mathbb{D}$  and  $A(\tilde{W}_{n,j}) \approx h^2 \approx A(\tilde{\Delta})$  ( $\approx$  meaning that the ratio is between two absolute constants). We therefore get:

$$[N_\varphi(w_0)]^{p/2} \lesssim \frac{1}{A(\tilde{W}_{n,j})} \int_{\tilde{W}_{n,j}} [N_\varphi(w)]^{p/2} dA(w),$$

proving (6.1). □

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