# The Gysin sequence for $S^3$-actions on manifolds.

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# The Gysin sequence for $\mathbb{S}^3$-actions on manifolds*

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**Abstract**

We construct a Gysin sequence associated to any smooth $\mathbb{S}^3$-action on a smooth manifold.

Given a semi-free smooth action $\Phi\colon \mathbb{S}^3 \times M \to M$, we have the Gysin sequence[1]:

$$\cdots \longrightarrow H^i(M) \xrightarrow{②} H^{i-3}\left(M/\mathbb{S}^3, M^{\mathbb{S}^3}\right) \xrightarrow{③} H^{i+1}\left(M/\mathbb{S}^3\right) \xrightarrow{①} H^{i+1}(M) \longrightarrow \cdots,$$

where the morphism ① is induced by the natural projection $\pi\colon M \to M/\mathbb{S}^3$, the morphism ② is induced by the integration along the fibers of $\pi$ and the morphism ③ is the multiplication by the *Euler class* $[e] \in \boldsymbol{H}^4_{\bar{4}}\left(M/\mathbb{S}^3\right)$ (cf. [4]).

The main goal of this work is to extend this result to any smooth action of $\mathbb{S}^3$. We obtain the following Gysin sequence (cf. Theorem 4.4 and paragraph 4.5)

$$\cdots \longrightarrow H^i(M) \xrightarrow{②} H^{i-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{i-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \xrightarrow{③} H^{i+1}\left(M/\mathbb{S}^3\right) \xrightarrow{①} H^{i+1}(M) \longrightarrow \cdots$$

where $\Sigma$ is the subset of points of $M$ whose isotropy group is not finite, the $\mathbb{Z}_2$-action is induced by the product by $j \in \mathbb{S}^3$ and $(-)^{-\mathbb{Z}_2}$ denotes the subspace of antisymmetric elements (cf. (11)).

The organization of the work is as follows.

| | |
|---|---|
| **1.** *Thom-Mather's structure*. | Description of the singular manifolds arising from the action. |
| **2.** *Verona's differential forms*. | Computation of the real cohomology of singular manifolds by using differential forms. |
| **3.** *Decomposition of a differential form*. | Writing of a differential form in terms of characteristic forms and horizontal forms. |
| **4.** *The Gysin sequence*. | Main result of this work. |

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[1]In this work, $H^*(X)$ stands for the singular cohomology of the space $X$ with real coefficients.

In the sequel $M$ is a connected, second countable, Haussdorff, without boundary and smooth (of class $C^\infty$) manifold. We fix a smooth action $\Phi\colon \mathbb{S}^3 \times M \to M$.

**1. Thom-Mather structure.** There are three possibilities for the dimension of the isotropy subgroup[2] $\mathbb{S}^3_x$ of a point $x \in M$, namely: 0,1 and 3. So, we have the *dimension type filtration*

$$F = \left\{x \in M \,\middle|\, \dim \mathbb{S}^3_x = 3\right\} \subset \Sigma = \left\{x \in M \,\middle|\, \dim \mathbb{S}^3_x \geq 1\right\} \subset M = \left\{x \in M \,\middle|\, \dim \mathbb{S}^3_x \geq 0\right\}.$$

In this section, we describe the geometry of the triple $(M, \Sigma, F)$. The subset $\Sigma$ is not necessarily a manifold, but subsets $F = M^{\mathbb{S}^3}$, $\Sigma\backslash F = \left\{x \in M \,\middle|\, \dim \mathbb{S}^3_x = 1\right\}$ and $M\backslash\Sigma = \left\{x \in M \,\middle|\, \dim \mathbb{S}^3_x = 0\right\}$ are proper invariant submanifolds[3] of $M$. So we can find two invariant tubular neighborhoods in $M$: $\tau_0\colon T_0 \to F$ and $\tau_1\colon T_1 \to \Sigma\backslash F$. Over each connected component the structural group is the orthogonal group. Associated to these tubular neighborhoods we have the following maps ($k = 0, 1$):

⇝ The *radius map* $\nu_k\colon T_k \to [0, \infty[$, defined fiberwiselly by $u \mapsto ||u||$. It is an invariant smooth map.

⇝ The *dilatation map* $\partial_k\colon [0, \infty[\times T_k \to T_k$, defined fiberwiselly by $(t, u) \mapsto t \cdot u$. It is a smooth equivariant map.

The family of tubular neighborhoods $\mathfrak{T}_M = \{T_0, T_1\}$ is a *Thom-Mather system* when:

$$\text{(TM)} \left\{ \begin{array}{l} \tau_0 = \tau_0 \circ \tau_1 \\ \nu_0 = \nu_0 \circ \tau_1 \end{array} \right\} \text{ on } T_0 \cap T_1 = \tau_1^{-1}(T_0 \cap (\Sigma\backslash F)).$$

**Lemma 1.1** *Thom-Mather systems exist.*

*Proof.* We fix an invariant tubular neighborhood $\tau_0\colon T_0 \to F$. It exists since $F$ is an invariant closed submanifold of $M$. Since the isotropy subgroup of any point of $F$ is the whole $\mathbb{S}^3$, we can find[4] an atlas $\mathcal{A} = \left\{\varphi\colon U \times \mathbb{R}^n \to \tau_0^{-1}(U)\right\}$ of $\tau_0$, having $O(n)$ as structural group, and an orthogonal action $\Psi\colon \mathbb{S}^3 \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\text{(1)} \qquad \varphi(x, \Psi(g, v)) = \Phi(g, \varphi(x, v)) \quad \forall x \in U, \forall v \in \mathbb{R}^n \text{ and } \forall g \in \mathbb{S}^3.$$

We write $\tau_0'\colon S_0 \to F$ the restriction of $\tau_0$, where $S_0$ is the submanifold $\nu_0^{-1}(1)$. It is a fiber bundle. The restriction $\tau_0''\colon (S_0 \cap (\Sigma\backslash F)) \to F$ is also a fiber bundle whose induced atlas is $\mathcal{A}'' = \left\{\varphi\colon U \times \mathbb{S}^{n-1}_\Sigma \to \tau_0''^{-1}(U)\right\}$, where $\mathbb{S}^{n-1}_\Sigma = \left\{w \in \mathbb{S}^{n-1} \,\middle|\, \dim \mathbb{S}^3_w = 1\right\}$.

The map $\mathfrak{L}_0\colon T_0\backslash F \to S_0 \times ]0, \infty[$, defined by $\mathfrak{L}_0(x) = \left(\partial_0\left(\nu_0(x)^{-1}, x\right), \nu_0(x)\right)$, is an equivariant diffeomorphism. Under $\mathfrak{L}_0$:

⇝ the map $\tau_0$ becomes $(y, t) \mapsto \tau_0'(y)$,

⇝ the map $\nu_0$ becomes $(y, t) \mapsto t$, and

⇝ the manifold $T_0 \cap (\Sigma\backslash F)$ becomes $(S_0 \cap (\Sigma\backslash F)) \times ]0, \infty[$.

Since the structural group of $\tau_0'$ is a compact Lie group, condition (1) allows us to construct an invariant Riemannian metric $\mu_0$ on $S_0$ such that the fibers of $\tau_0'$ are totally geodesic submanifolds and $(T(S_0 \cap (\Sigma\backslash F)))^\perp \subset \ker\left(\tau_0'\right)_*$. Then, if we consider the associated tubular neighborhood $\tau_1'\colon T_1' \to S_0 \cap (\Sigma\backslash F)$ we have $\tau_0' \circ \tau_1' = \tau_0'$.

We can construct now an invariant Riemannian metric $\mu$ on $M\backslash F$ such that under $\mathfrak{L}_0$:

⇝ the metric $\mu$ becomes $\mu_0 + dr^2$ on $S_0 \times ]0, \infty[$.

[2] We refer the reader to [2] for the notions related with compact Lie group actions, such as isotropy, invariant tubular neighborhoods,. . .

[3] In fact, these manifolds may have connected components whith different dimensions.

[4] For each connected component of $F$.

We consider the associated tubular neighborhood $\tau_1\colon T_1 \to \Sigma\backslash F$. Verification of the property (TM) must be done on $T_0 \cap T_1$, where using $\mathfrak{L}_0$, we get:

$\rightsquigarrow T_0 \cap T_1$ becomes $T_1' \times ]0,\infty[$.

$\rightsquigarrow \tau_1$ becomes $(y,t) \mapsto (\tau_1'(y), t)$.

A straightforward calculation gives (TM) and ends the proof. ♣

We fix a such system $\mathfrak{T}_M$. For each $k \in \{0,1\}$, we shall write $D_k \subset M$ the open subset $\nu_k^{-1}([0,1[)$ and call it the *soul* of the tubular neighborhood $\tau_k$. We shall write $\Delta_0 = D_0 \cap \Sigma$.

**2. Verona's differential forms.** As it is shown in [5], the singular cohomology of $M$ (resp. $\Sigma$) can be computed by using differential forms on $M\backslash\Sigma$ (resp. $\Sigma\backslash F$). These are the controlled differential forms, the tool we use in this work. The complex of *controlled forms* (or *Verona's forms*) of $M$ and $\Sigma$ is defined by

$$\begin{aligned}
\Omega^*_v(M) &= \left\{\omega \in \Omega^*(M\backslash\Sigma) \;\middle|\; \exists \omega_1 \in \Omega^*(\Sigma\backslash F) \text{ and } \omega_0 \in \Omega^*(F) \text{ with } \left\{\begin{array}{l} (a)\ \tau_1^*\omega_1 = \omega \ \text{ on } D_1\backslash\Sigma \\ (b)\ \tau_0^*\omega_0 = \omega \ \text{ on } D_0\backslash\Sigma \\ (c)\ \tau_0^*\omega_0 = \omega_1 \ \text{ on } \Delta_0\backslash F \end{array}\right\}\right\}.\\
\Omega^*_v(\Sigma) &= \left\{\gamma \in \Omega^*(\Sigma\backslash F) \;\middle|\; \exists \gamma_0 \in \Omega^*(F) \text{ with } \tau_0^*\gamma_0 = \gamma \ \text{ on } \Delta_0\backslash F\right\}.
\end{aligned}$$

Following [5] we know that the cohomology of the complex $\Omega^*_v(M)$ (resp. $\Omega^*_v(\Sigma)$) is the singular cohomology $H^*(M)$ (resp. $H^*(\Sigma)$). We also have the complexes of *relative controlled forms*: $\Omega^*_v(M,\Sigma) = \{\omega \in \Omega^*_v(M) \mid \omega_1 \equiv 0\}$ and $\Omega^*_v(\Sigma, F) = \{\gamma \in \Omega^*_v(\Sigma) \mid \gamma_0 \equiv 0\}$.

Since $M$ is a manifold, controlled forms are in fact differential forms.

**Lemma 2.1** *Any controlled form of $M$ is the restriction of a differential form of $M$.*

*Proof.* First, we construct a section $\sigma$ of the restriction $\rho\colon \Omega^*_v(M) \to \Omega^*_v(\Sigma)$ defined by $\rho(\omega) = \omega_1$. Let us consider a smooth function $f\colon ]0,\infty[\to [0,1]$ verifying $f \equiv 0$ on $[3,\infty[$ and $f \equiv 1$ on $]0,2]$. Notice that the compositions $f\circ\nu_0\colon M \to [0,1]$ and $f\circ\nu_1\colon M\backslash F \to [0,1]$ are smooth invariant maps. So, for each $\gamma \in \Omega^*_v(\Sigma)$ we have

$$\sigma(\gamma) = (f\circ\nu_0)\cdot\tau_0^*\gamma_0 + (1-(f\circ\nu_0))\cdot(f\circ\nu_1)\tau_1^*\gamma \in \Omega^*(M). \tag{2}$$

This differential form is a controlled form since

(a) Since $(f\circ\nu_1) \equiv 1$ on $D_1$, $(f\circ\nu_0) \equiv 0$ on $M\backslash T_0$ and (TM) then we have

$$\sigma(\gamma) = (f\circ\nu_0)\cdot\tau_1^*\tau_0^*\gamma_0 + (1-(f\circ\nu_0))\cdot\tau_1^*\gamma = \tau_1^*\left((f\circ\nu_0)\cdot\tau_0^*\gamma_0 + (1-(f\circ\nu_0))\cdot\gamma\right)$$

on $D_1\backslash\Sigma$. This gives $(\sigma(\gamma))_1 = (f\circ\nu_0)\cdot\tau_0^*\gamma_0 + (1-(f\circ\nu_0))\cdot\gamma$. Since $\tau_0^*\gamma_0 = \gamma$ on $\Delta_0\backslash F$ then $(\sigma(\gamma))_1 = (f\circ\nu_0)\cdot\gamma + (1-(f\circ\nu_0))\cdot\gamma = \gamma$.

(b) Since $(f\circ\nu_0) \equiv 1$ on $D_0$ then we have $\sigma(\gamma) = \tau_0^*\gamma_0$ on $D_0\backslash\Sigma$. This gives $(\sigma(\gamma))_0 = \gamma_0$.

(c) We have $(\sigma(\gamma))_1 = \gamma = \tau_0^*\gamma_0 = \tau_0^*(\sigma(\gamma))_0$ on $\Delta_0\backslash F$.

This map $\sigma$ is a section of $\rho$ since $\rho(\sigma(\gamma)) = (\sigma(\gamma))_1 = \gamma$.

In particular, $\rho(\omega - \sigma(\rho(\omega))) = 0$ for each $\omega \in \Omega^*_v(M)$. As $\sigma(\rho(\omega)) \in \Omega^*(M)$ (cf. (2)) and coincides with $\omega$ in the open set $(D_0 \cup D_1)\backslash\Sigma$ we conclude that $\omega$ can be extended to $M$. ♣

**2.2. Remark.** Notice that the original germ-like definition of Verona's differential forms (cf. [5]) is slightly different from ours. Our Verona forms are more rigid in a fixed neighborhood of the stratum, which will prove useful enough, as we use tools like excision and retraction. Of course, we prove in the Appendix that the cohomologies of our Verona complexes are the ordinary ones, like happens in [5].

**2.3. Invariant forms.** Denote by $\mathfrak{X}_\Phi(M)$ the subbundle of $TM$ formed by the vector fields of $M$ tangent to the orbits of $\Phi$. A controlled form $\omega$ of $M$ is an *invariant form* when

$$L_X\omega = 0$$

for each $X \in \mathfrak{X}_\Phi(M)$ of the action $\Phi$. The complex of invariant forms is denoted by $\underline{\Omega}^*_{\mathcal{V}}(M)$. The inclusion $\underline{\Omega}^*_{\mathcal{V}}(M) \hookrightarrow \Omega^*_{\mathcal{V}}(M)$ induces an isomorphism in cohomology. This a standard argument based on the fact that $\mathbb{S}^3$ is a connected compact Lie group (cf. [3, Theorem I, Ch. IV, vol. II]). So,

$$H^*\left(\underline{\Omega}^\cdot_{\mathcal{V}}(M)\right) = H^*\left(\Omega^\cdot_{\mathcal{V}}(M)\right) = H^*(M). \tag{3}$$

**2.4. Basic forms.** A controlled form $\omega$ of $M$ is a *basic form* when

$$i_X\omega = i_X d\omega = 0$$

for each $X \in \mathfrak{X}_\Phi(M)$. The complex of the basic forms is denoted by $\Omega^*_{\mathcal{V}}\left(M/\mathbb{S}^3\right)$. In this work, we shall use the following relative versions of this complex: $\Omega^*_{\mathcal{V}}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) = \Omega^*_{\mathcal{V}}\left(M/\mathbb{S}^3\right) \cap \Omega^*_{\mathcal{V}}(M,\Sigma)$, as well as $\Omega^*_{\mathcal{V}}\left(\Sigma/\mathbb{S}^3, F\right) = \Omega^*_{\mathcal{V}}\left(\Sigma/\mathbb{S}^3\right) \cap \Omega^*_{\mathcal{V}}(\Sigma, F)$.

**Lemma 2.5**

$$H^*\left(\Omega^\cdot_{\mathcal{V}}\left(M/\mathbb{S}^3\right)\right) = H^*\left(M/\mathbb{S}^3\right) \qquad \text{and} \qquad H^*\left(\Omega^\cdot_{\mathcal{V}}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right)\right) = H^*\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right).$$

*Proof.* See Appendix. ♣

**3. Decomposition of a differential form.** Put $N = M\backslash\Sigma$ which is an invariant open subset of $M$. We denote by $\Phi_N \colon \mathbb{S}^3 \times N \to N$ the restriction of $\Phi$. It is an *almost free action*.[5]

We denote by $\mathfrak{su}(2)$ the Lie algebra of $\mathbb{S}^3$. We fix $\{u_1, u_2, u_3\}$ a basis of $\mathfrak{su}(2)$ with $[u_1, u_2] = u_3$, $[u_2, u_3] = u_1$ and $[u_3, u_1] = u_2$. We denote by $X_u \in \mathfrak{X}_\Psi(N)$ the *fundamental vector field* associated to $u \in \mathfrak{su}(2)$. The map $X \colon \mathfrak{su}(2) \to \mathfrak{X}_{\Phi_N}(N)$, defined by $u \mapsto X_u$, is a Lie algebra morphism. For the sake of simplicity, we shall put $X_{u_i} = X_i$ for $i = 1, 2, 3$.

We endow $N$ with a $\mathbb{S}^3$-invariant Riemannian metric $\mu_0$, which exists because of the compactness of $\mathbb{S}^3$. We also fix a bi-invariant Riemannian metric $\nu$ on the Lie group $\mathbb{S}^3$. Consider now the $\mu_0$-orthogonal $\mathbb{S}^3$-invariant decomposition $TN = \ker(\pi_N)_* \oplus \xi$, where $\pi_N \colon N \to N/\mathbb{S}^3$ is the canonical projection (a submersion). Since the action $\Phi_N$ is almost free then, for each point $x \in N$, the family $\{X_1(x), X_2(x), X_3(x)\}$ is a basis of $\ker(\pi_N)_*$. We define the $\mathbb{S}^3$-Riemannian metric $\mu$ on $N$ by putting

$$\mu(w_1, w_2) = \begin{cases} \mu_0(w_1, w_2) & \text{if } w_1, w_2 \in \xi \\ 0 & \text{if } w_1 \in \xi, w_2 \in \ker(\pi_N)_* \\ \nu(u, v) & \text{if } w_1 = X_u(x), w_2 = X_v(x) \end{cases}$$

We denote by $\chi_u = i_{X_u}\mu \in \Omega^1(N)$ the *characteristic form* associated to $u \in \mathfrak{su}(2)$. For the sake of simplicity, we shall put $\chi_{u_i} = \chi_i$ for $i = 1, 2, 3$. Since

$$\chi_j(X_i) = \mu(X_i, X_j) = \nu(u_i, u_j) = \delta_{ij} \tag{4}$$

[5] All the isotropy subgroups are finite groups.

then the characteristic forms $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{X}_3$ do not vanish at any point of $N$.

Applying the well-known equality of operators

$$L_A i_B = i_B L_A + i_{[A,B]}, \quad \forall A, B \in \mathfrak{X}(M) \tag{5}$$

to the metric $\mu$, we obtain $L_{X_u}\mathcal{X}_v = \mathcal{X}_{[u,v]} \quad \forall u, v \in \mathfrak{su}(2)$. In particular:

$$\begin{array}{ll} L_{X_1}\mathcal{X}_1 = L_{X_2}\mathcal{X}_2 = L_{X_3}\mathcal{X}_3 = 0, & L_{X_1}\mathcal{X}_2 = -L_{X_2}\mathcal{X}_1 = \mathcal{X}_3 \\ L_{X_1}\mathcal{X}_3 = -L_{X_3}\mathcal{X}_1 = -\mathcal{X}_2 & L_{X_2}\mathcal{X}_3 = -L_{X_3}\mathcal{X}_2 = \mathcal{X}_1. \end{array} \tag{6}$$

A differential form $\omega \in \Omega^*(N)$ is *horizontal* when $i_{X_u}\omega = 0$ for each $u \in \mathfrak{su}(2)$. All the differential forms of $N$ can be expressed in terms of horizontal and characteristic forms. In fact, each differential form $\omega \in \Omega^*(N)$ possesses a unique writing,

$$\omega = {}_0\omega + \sum_{p=1}^{3} \mathcal{X}_p \wedge {}_p\omega + \sum_{1\le p<q\le 3} \mathcal{X}_p \wedge \mathcal{X}_q \wedge {}_{pq}\omega + \mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge {}_{123}\omega,$$

where the coefficients ${}_\bullet\omega$ are horizontal forms of $\Omega^*(N)$ (cf. (4)). This is the *canonical decomposition* of $\omega$. For example, we have:

$$\left\{ \begin{array}{rcl} d\beta & = & {}_0(d\beta) + \mathcal{X}_1 \wedge L_{X_1}\beta + \mathcal{X}_2 \wedge L_{X_2}\beta + \mathcal{X}_3 \wedge L_{X_3}\beta \\ d\mathcal{X}_1 & = & e_1 - \mathcal{X}_2 \wedge \mathcal{X}_3 \\ d\mathcal{X}_2 & = & e_2 + \mathcal{X}_1 \wedge \mathcal{X}_3 \\ d\mathcal{X}_3 & = & e_3 - \mathcal{X}_1 \wedge \mathcal{X}_2 \end{array} \right. \tag{7}$$

where $\beta \in \Omega^*(N)$ is a horizontal form.

Consider $U \subset N$ an equivariant open subset. If $\omega \in \Omega^*(N, U)$ then the coefficients of its canonical decomposition also belong to $\Omega^*(N, U)$.

**4. The Gysin sequence.** The starting point for the construction of the Gysin sequence associated to $\Phi$ is the inclusion $I\colon \Omega^*_V\left(M/\mathbb{S}^3\right) \hookrightarrow \underline{\Omega}^*_V(M)$. It is a well defined differential operator since every basic form is invariant. Associated to this operator we have the long exact sequence

$$\cdots \longrightarrow H^i(M) \xrightarrow{\textcircled{2}} H^{i-3}(\text{Coker } I) \xrightarrow{\textcircled{3}} H^{i+1}\left(M/\mathbb{S}^3\right) \xrightarrow{\textcircled{1}} H^{i+1}(M) \longrightarrow \cdots,$$

where we have performed the substitutions of (3) and Lemma 2.5. The operator $\textcircled{1}$ is just $\pi^*$ (cf. 5.6). This is the *Gysin sequence*.

It remains to compute the cohomology of the quotient[6] Coker $I = \dfrac{\underline{\Omega}^*_V(M)}{\Omega^*_V\left(M/\mathbb{S}^3\right)}$. For that purpose we consider the integration operator:

$$\fint\colon \text{ Coker } I \longrightarrow \Omega^{*-3}_V\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right),$$

given by:

$$\fint(<\omega>) = (-1)^{\deg\omega}\, i_{X_3} i_{X_2} i_{X_1}\omega.$$

It is a well defined differential operator since

[6] An element of this quotient is denoted by $< - >$.

- the tubular neighborhoods of the Thom-Mather's structure $\mathfrak{T}$ are invariant,
- the operator $i_{X_3} i_{X_2} i_{X_1}$ vanishes on $\Sigma$, and
- $i_X i_{X_3} i_{X_2} i_{X_1} \omega = i_X d i_{X_3} i_{X_2} i_{X_1} \omega = 0$ for each $X \in \mathfrak{X}_\Phi(M)$ (cf. (5)).

A form $\gamma \in \Omega_V^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right)$ vanishes in a neighborhood of $\Sigma/\mathbb{S}^3$. So, the product $\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \gamma$ belongs to $\underline{\Omega}_V^*(M)$ (cf. (6)). Since $i_{X_3} i_{X_2} i_{X_1} (\mathcal{X}_1 \wedge \mathcal{X}_2 \wedge \mathcal{X}_3 \wedge \gamma) = \gamma$ then we have the short exact sequence

(8) $$0 \longrightarrow \operatorname{Ker}^* \textstyle\oint \hookrightarrow \operatorname{Coker} I \xrightarrow{\oint} \Omega_V^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \longrightarrow 0$$

As we shall see later, the connecting morphim of (8) vanishes. Thus, we now have to focus on the computation of $H^*\left(\operatorname{Ker}^* \oint\right)$. For the sake of simplicity we put $A^*(M) = \operatorname{Ker}^* \oint$; in fact we have

$$A^*(M) = \frac{\left\{\omega \in \underline{\Omega}_V^*(M) \mid i_{X_3} i_{X_2} i_{X_1} \omega = 0\right\}}{\Omega_V^*\left(M/\mathbb{S}^3\right)}$$

Using this last expression, we define analogously $A^*(\Sigma)$ and $A^*(\Sigma, F)$. The following Lemmas are devoted to the computation of the cohomology of $A^*(M)$.

**Lemma 4.1**

$$H^*\left(A^\cdot(M)\right) = H^*\left(A^\cdot(\Sigma)\right)$$

*Proof.* Consider the inclusion $L \colon A^*(M, \Sigma) \longrightarrow A^*(M)$ and the restriction $R \colon A^*(M) \to A^*(\Sigma)$, which are differential morphisms. This gives the short sequence

(9) $$0 \longrightarrow A^*(M, \Sigma) \xrightarrow{L} A^*(M) \xrightarrow{R} A^*(\Sigma) \longrightarrow 0$$

In order to get this Lemma we prove the following:

(i) The sequence (9) is exact.

(ii) $H^*(A^\cdot(M, \Sigma)) = 0$.

(i)

Since $R \circ L = 0$ it suffices to prove that $R$ is an onto map and that Ker $R \subset$ Im $L$.

• *The operator $R$ is an onto map.* Consider $\gamma \in \underline{\Omega}_V^*(\Sigma)$. We know that $\sigma(\gamma) \in \Omega_V^*(M)$ (cf. Lemma 2.1). The result comes from:

⇝ $\sigma(\gamma) \in \underline{\Omega}_V^*(M)$. Since $\tau_0$, $\tau_1$ are equivariant maps and $f \circ \nu_0$, $f \circ \nu_1$ are invariant maps.

⇝ $i_{X_3} i_{X_2} i_{X_1} \sigma(\gamma) = 0$. Since $\tau_0$, $\tau_1$ are equivariant maps and rank $\{X_1(x), X_2(x), X_3(X)\} \leq 2$ for any $x \in \Sigma$.

⇝ $R(<\sigma(\gamma)>) =< (\sigma(\gamma))_1 >=< \gamma >$.

• Ker $R \subset$ Im $L$. Consider $\omega \in \underline{\Omega}_V^*(M)$ with $i_{X_3} i_{X_2} i_{X_1} \omega = 0$ and $i_{X_j} \omega_1 = 0$ for $j \in \{1, 2, 3\}$. Since $\tau_0$ and $\tau_1$ are equivariant maps and $X_j = 0$ on $F$ then $i_{X_j} \sigma(\omega_1) = 0$ for $j \in \{1, 2, 3\}$. This gives $< \sigma(\omega_1) >= 0$. Finally, we have $< \omega >=< \omega - \sigma(\omega_1) >= L(< \omega - \sigma(\omega_1) >)$ since $(\omega - \sigma(\omega_1))_1 = \omega_1 - (\sigma(\omega_1))_1 = \omega_1 - \omega_1 = 0$.

(ii)

By definition of Verona's differential forms we have

$$A^*(M,\Sigma) = A^*(M,D_1) \overset{excision}{=\!=} A^*(M\backslash\Sigma, D_1\backslash\Sigma).$$

Put $N = M\backslash\Sigma$ and $U = D_1\backslash\Sigma$ as in Section 3. A straightforward calculation gives:

$$H^*\left(A^\cdot(N,U)\right) = \frac{\left\{\omega \in \underline{\Omega}^*(N,U) \,\middle|\, i_{X_3} i_{X_2} i_{X_1}\omega = 0 \text{ and } i_{X_j} d\omega = 0 \text{ for } j \in \{1,2,3\}\right\}}{\Omega^*\left(N/\mathbb{S}^3, U/\mathbb{S}^3\right) + \left\{d\beta \,\middle|\, \beta \in \underline{\Omega}^{*-1}(N,U) \text{ and } i_{X_3} i_{X_2} i_{X_1}\beta = 0\right\}}$$

Let $\omega$ be a differential form of $\underline{\Omega}^*(N,U)$ verifying $i_{X_3} i_{X_2} i_{X_1}\omega = 0$ and $i_{X_j} d\omega = 0$ for $j \in \{1,2,3\}$. Then

$$\omega = \begin{array}{c} d\,\underbrace{(\mathcal{X}_1 \wedge i_{X_3} i_{X_2}\omega - \mathcal{X}_2 \wedge i_{X_3} i_{X_1}\omega + \mathcal{X}_3 \wedge i_{X_2} i_{X_1}\omega)}_{\beta} \\ + \\ \underbrace{-e_1 \wedge i_{X_3} i_{X_2}\omega + e_2 \wedge i_{X_3} i_{X_1}\omega - e_3 \wedge i_{X_2} i_{X_1}\omega + {}_0\omega}_{\alpha} \end{array} \tag{10}$$

(cf. (7) and (5)) with $\beta \in \underline{\Omega}^{*-1}(N,U)$, verifying $i_{X_3} i_{X_2} i_{X_1}\beta = 0$, and $\alpha \in \Omega^*\left(N/\mathbb{S}^3, U/\mathbb{S}^3\right)$. This implies $H^*(A^\cdot(N,U)) = 0$ and then $H^*(A^\cdot(M,\Sigma)) = 0$. ♣

**Lemma 4.2**

$$H^*\left(A^\cdot(\Sigma)\right) = H^*\left(A^\cdot(\Sigma,F)\right).$$

*Proof.* Consider the inclusion $L\colon A^*(\Sigma,F) \hookrightarrow A^*(\Sigma)$ which is a differential morphism. It suffices to prove that $L$ is an onto map.

Let us consider a smooth function $f\colon ]0,\infty[\to [0,1]$ verifying $f \equiv 0$ on $[3,\infty[$ and $f \equiv 1$ on $]0,2]$. Notice that the composition $f{\circ}\nu_0\colon M \to [0,1]$ is an smooth invariant map. So, for each $\gamma \in \Omega^*(F)$ we have $\sigma(\gamma) = (f{\circ}\nu_0)\tau_0^*\gamma \in \Omega^*(M)$. This differential form verifies

⇝ $\sigma(\gamma) \in \Omega^*_V(\Sigma)$. Since $(f{\circ}\nu_0) \equiv 1$ on $\Delta_0$ then $\sigma(\gamma) = \tau_0^*\gamma$ on $\Delta_0\backslash F$. This gives $(\sigma_0(\gamma))_0 = \gamma$.

⇝ $\sigma(\gamma) \in \underline{\Omega}^*_V(\Sigma)$. Since $\tau_0$ is an equivariant map and $f{\circ}\nu_0$ is an invariant map.

⇝ $i_{X_j}\sigma(\gamma) = 0$ for $j \in \{1,2,3\}$ since $\tau_0$ is an equivariant map and $X_j = 0$ on $F$.

Then $<\sigma(\gamma)>= 0$ on $A^*(\Sigma)$.

Let $<\omega>$ be a class of $A^*(\Sigma)$. We can write: $<\omega>=<\omega - \sigma((\omega)_0)>= L(<\omega - \sigma((\omega)_0)>)$ since $(\omega - \sigma((\omega)_0))_0 = \omega_0 - (\sigma(\omega_0))_0 = \omega_0 - \omega_0 = 0$. This proves that $L$ is an onto map. ♣

Given an action of $\mathbb{Z}_2$ on a vector space $E$, generated by the morphism $h\colon E \to E$, we shall write

$$E^{-\mathbb{Z}_2} = \{e \in E \,|\, h(e) = -e\}, \tag{11}$$

the subspace of *antisymmetric elements*.

**Lemma 4.3**

$$H^*\left(A^\cdot(\Sigma,F)\right) = \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$$

*where the $\mathbb{Z}_2$-action is induced by the product by $j \in \mathbb{S}^3$.*

*Proof.* By definition of Verona's differential forms we have

$$A^*(\Sigma, F) = A^*(\Sigma, \Delta_0) \stackrel{excision}{=\!=} A^*(\Sigma\backslash F, \Delta_0\backslash F) = \frac{\underline{\Omega}^*(\Sigma\backslash F, \Delta_0\backslash F)}{\Omega^*\left((\Sigma\backslash F)/\mathbb{S}^3, (\Delta_0\backslash F)/\mathbb{S}^3\right)}.$$

The isotropy subgroup of a point of $\Sigma\backslash F$ is conjugated to $\mathbb{S}^1$ or $N(\mathbb{S}^1)$ (cf. [2, Th. 8.5, pag. 153]). We consider

$$\Gamma = \{x \in \Sigma\backslash F \mid \mathbb{S}^3_x \text{ is conjugated to } N(\mathbb{S}^1)\},$$

which is an invariant submanifold[7] of $\Sigma\backslash F$. Proceeding as in the proof of Lemma 1.1 we can construct an invariant tubular neighborhood $\tau\colon T \to \Gamma$ of $\Gamma$ on $\Sigma\backslash F$ such that:

$$\text{(TM')} \left\{ \begin{array}{l} \tau_0 = \tau_0\circ\tau \\ \nu_0 = \nu_0\circ\tau \end{array} \right\} \text{ on } T \cap \Delta_0 = \tau^{-1}(\Gamma \cap \Delta_0).$$

The inclusion $L\colon A^*(\Sigma\backslash F, (\Delta_0\backslash F) \cup \Gamma) \to A^*(\Sigma\backslash F, \Delta_0\backslash F)$ and the restriction $R\colon A^*(\Sigma\backslash F, \Delta_0\backslash F) \to A^*(\Gamma, \Gamma \cap \Delta_0)$ are differential morphisms. In fact, $R(<\omega>) =< \iota^*\omega >$ where $\iota\colon \Gamma \to \Sigma\backslash F$ is the natural inclusion. This gives the short sequence

$$0 \longrightarrow A^*(\Sigma\backslash F, (\Delta_0\backslash F) \cup \Gamma) \xrightarrow{L} A^*(\Sigma\backslash F, \Delta_0\backslash F) \xrightarrow{R} A^*(\Gamma, \Gamma \cap \Delta_0) \longrightarrow 0$$

In order to get this Lemma we prove the following facts:

(i) The sequence is exact.

(ii) $H^*(A^{\cdot}(\Gamma, \Gamma \cap \Delta_0)) = 0$.

(iii) $H^*(A^{\cdot}(\Sigma\backslash F, (\Delta_0\backslash F) \cup \Gamma)) = \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$.

(i)

•*The operator R is an onto map.* Let us consider a smooth function $f\colon ]0, \infty[\to [0, 1]$ verifying $f \equiv 0$ on $[3, \infty[$ and $f \equiv 1$ on $]0, 2]$. Notice that the composition $f\circ\nu\colon \Sigma\backslash F \to [0, 1]$ is an smooth invariant map, where $\nu$ is the radius map associated to the invariant tubular neighborhood $\tau\colon T \to \Gamma$. So, for each $\gamma \in \underline{\Omega}^*(\Gamma, \Gamma \cap \Delta_0)$ the differential form $\sigma(\gamma) = (f\circ\nu)\tau^*\gamma$ belongs to $\underline{\Omega}^*(\Sigma\backslash F, \Delta_0\backslash F)$ (since (TM') and $f \equiv 0$ outside $T$). Since $f \equiv 1$ and $\tau = \mathrm{Id}$ on $\Gamma$ then $R(<\sigma(\gamma)>) =< \gamma >$. So, the operator $R$ is an onto map.

• Ker $R \subset$ Im $L$. Let $\omega \in \underline{\Omega}^*(\Sigma\backslash F, \Delta_0\backslash F)$ with $\iota^*\omega \in \Omega^*\left(\Gamma/\mathbb{S}^3, (\Gamma \cap \Delta_0)/\mathbb{S}^3\right)$. Since $f\circ\nu$ is invariant and $\tau$ is equivariant then $\sigma(\iota^*\omega) \in \Omega^*\left((\Sigma\backslash F)/\mathbb{S}^3, (\Delta_0\backslash F)/\mathbb{S}^3\right)$. Then $<\omega>=<\omega - \sigma(\iota^*\omega)> = L(<\omega - \sigma(\iota^*\omega)>)$ since $\iota^*(\omega - \sigma(\iota^*\omega)) = \iota^*\omega - \iota^*\sigma(\iota^*\omega) = \iota^*\omega - \iota^*\omega = 0$.

(ii)

The isotropy subgroup of any point of $\Gamma$ is conjugated to the normalizer $N(\mathbb{S}^1)$. So, we have the homeomorphisms

$$\Gamma = \mathbb{S}^3 \times_{N(N(\mathbb{S}^1))} \Gamma^{N(\mathbb{S}^1)} = \mathbb{S}^3 \times_{N(\mathbb{S}^1)} \Gamma^{N(\mathbb{S}^1)} = \mathbb{RP}^2 \times \Gamma^{N(\mathbb{S}^1)} = \mathbb{RP}^2 \times \Gamma/\mathbb{S}^3.$$

[7]This manifold may have connected components with different dimensions.

Analogously we have $\Gamma \cap \Delta_0 = \mathbb{RP}^2 \times (\Gamma \cap \Delta_0)/\mathbb{S}^3$. The Künneth formula gives

$$\begin{aligned} H^*\left(A^\cdot(\Gamma, \Gamma \cap \Delta_0)\right) &= H^*\left(\frac{\underline{\Omega}^\cdot(\Gamma, \Gamma \cap \Delta_0)}{\Omega^\cdot\left(\Gamma/\mathbb{S}^3, (\Gamma \cap \Delta_0)/\mathbb{S}^3\right)}\right) = H^*\left(\frac{\underline{\Omega}^\cdot\left(\mathbb{RP}^2\right) \otimes \Omega^\cdot\left(\Gamma/\mathbb{S}^3, (\Gamma \cap \Delta_0)/\mathbb{S}^3\right)}{\Omega^\cdot\left(\Gamma/\mathbb{S}^3, (\Gamma \cap \Delta_0)/\mathbb{S}^3\right)}\right) \\ &= H^{>0}\left(\mathbb{RP}^2\right) \otimes H^*\left(\Gamma/\mathbb{S}^3, (\Gamma \cap \Delta_0)/\mathbb{S}^3\right) = 0. \end{aligned}$$

(iii)

Write $\Delta = \nu^{-1}([0,1[)$ the soul of the tubular neighborhood $\tau\colon T \to \Gamma$. Since this retraction preserves $\Delta_0 \backslash F$ (cf. (TM')) then

$$\begin{aligned} H^*\left(A^\cdot(\Sigma\backslash F, (\Delta_0\backslash F) \cup \Gamma)\right) &= H^*\left(A^\cdot(\Sigma\backslash F, (\Delta_0\backslash F) \cup \Delta)\right) \\ &\overset{excision}{=\!=} H^*\left(A^\cdot(\Sigma\backslash(F \cup \Gamma), (\Delta_0\backslash(F \cup \Gamma) \cup (\Delta\backslash\Gamma))\right). \end{aligned}$$

The isotropy subgroup of any point of $\Sigma' = \Sigma\backslash(F \cup \Gamma)$ is conjugated to $\mathbb{S}^1$. So,

$$\Sigma' = \mathbb{S}^3 \times_{N(\mathbb{S}^1)} \Sigma'^{\mathbb{S}^1} = \left(\mathbb{S}^3/\mathbb{S}^1\right) \times_{N(\mathbb{S}^1)/\mathbb{S}^1} \Sigma'^{\mathbb{S}^1} = \mathbb{S}^2 \times_{\mathbb{Z}_2} \Sigma'^{\mathbb{S}^1}.$$

Notice that $\Sigma'^{\mathbb{S}^1}/\mathbb{Z}_2 = \Sigma'/\mathbb{S}^3$. Put $U$ the open subset $(\Delta_0\backslash(F \cup \Gamma) \cup (\Delta\backslash\Gamma)$ of $\Sigma'$. Analogously we have $U = \mathbb{S}^2 \times_{\mathbb{Z}_2} U^{\mathbb{S}^1}$ and $U^{\mathbb{S}^1}/\mathbb{Z}_2 = U/\mathbb{S}^3$.

The $\mathbb{Z}_2$-action on $\mathbb{S}^2$ is generated by $(x_0, x_1, x_2) \mapsto (-x_0, -x_1, -x_2)$[8]. Then, the $\mathbb{Z}_2$-action on $H^0\left(\mathbb{S}^2\right)$ (resp. $H^2\left(\mathbb{S}^2\right)$) is the identity $\mathrm{Id}$ (resp. $-\mathrm{Id}$). The $\mathbb{Z}_2$-action on $\Sigma'^{\mathbb{S}^1}$ is induced by the product by $j \in \mathbb{S}^3$. The Künneth formula gives

$$\begin{aligned} H^*\left(\underline{\Omega}^*(\Sigma', U)\right) &= H^*\left(\underline{\Omega}\left(\mathbb{S}^2 \times_{\mathbb{Z}_2} \Sigma'^{\mathbb{S}^1}, \mathbb{S}^2 \times_{\mathbb{Z}_2} U^{\mathbb{S}^1}\right)\right) = H^*\left(\underline{\Omega}\left(\mathbb{S}^2 \times \Sigma'^{\mathbb{S}^1}, \mathbb{S}^2 \times U^{\mathbb{S}^1}\right)^{\mathbb{Z}_2}\right) \\ &= H^*\left(\underline{\Omega}\left(\mathbb{S}^2 \times \Sigma'^{\mathbb{S}^1}, \mathbb{S}^2 \times U^{\mathbb{S}^1}\right)\right)^{\mathbb{Z}_2} = \left(H^*\left(\mathbb{S}^2\right) \otimes H^*\left(\Sigma'^{\mathbb{S}^1}, U^{\mathbb{S}^1}\right)\right)^{\mathbb{Z}_2} \\ &= \left(H^0\left(\mathbb{S}^2\right) \otimes H^*\left(\Sigma'^{\mathbb{S}^1}, U^{\mathbb{S}^1}\right)\right)^{\mathbb{Z}_2} \oplus \left(H^2\left(\mathbb{S}^2\right) \otimes H^{*-2}\left(\Sigma'^{\mathbb{S}^1}, U^{\mathbb{S}^1}\right)\right)^{\mathbb{Z}_2} \\ &= \left(H^*\left(\Sigma'^{\mathbb{S}^1}, U^{\mathbb{S}^1}\right)\right)^{\mathbb{Z}_2} \oplus \left(H^{*-2}\left(\Sigma'^{\mathbb{S}^1}, U^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = H^*\left(\Sigma'^{\mathbb{S}^1}/\mathbb{Z}_2, U^{\mathbb{S}^1}/\mathbb{Z}_2\right) \oplus \left(H^{*-2}\left(\Sigma'^{\mathbb{S}^1}, U^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \\ &= H^*\left(\Sigma'/\mathbb{S}^3, U/\mathbb{S}^3\right) \oplus \left(H^{*-2}\left(\Sigma'^{\mathbb{S}^1}, U^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}, \end{aligned}$$

and then

$$\begin{aligned} H^*\left(A^\cdot(\Sigma\backslash F, (\Delta_0\backslash F) \cup \Gamma)\right) &= H^*\left(\frac{\underline{\Omega}^\cdot(\Sigma', U)}{\Omega^\cdot\left(\Sigma'/\mathbb{S}^3, U/\mathbb{S}^3\right)}\right) = \left(H^{*-2}\left(\Sigma'^{\mathbb{S}^1}, U^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \\ &= \left(H^{*-2}\left((\Sigma\backslash(F \cup \Gamma))^{\mathbb{S}^1}, ((\Delta_0\backslash(F \cup \Gamma) \cup (\Delta\backslash\Gamma))^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \\ &\overset{excision}{=\!=} \left(H^{*-2}\left(\Sigma^{\mathbb{S}^1}, (\Delta_0 \cup \Delta)^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \\ &\overset{retraction}{=\!=} \left(H^{*-2}\left(\Sigma^{\mathbb{S}^1}, (F \cup \Gamma)^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}. \end{aligned}$$

[8]This map is induced by $j\colon \mathbb{S}^3 \to \mathbb{S}^3$ defined by $j(u) = u \cdot j$ (see [1, Example 17.23]).

Consider the long exact sequence associated to the $\mathbb{Z}_2$-invariant pair $\left(\Sigma^{\mathbb{S}^1}, (F\cup\Gamma)^{\mathbb{S}^1}\right)$:

$$\cdots \to \left(H^{i-1}\left((F\cup\Gamma)^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \to \left(H^{i}\left(\Sigma^{\mathbb{S}^1}, (F\cup\Gamma)^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \to \left(H^{i}\left(\Sigma^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \to \left(H^{i}\left((F\cup\Gamma)^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \to \cdots.$$

Since the action of $\mathbb{Z}_2$ on $(F\cup\Gamma)^{\mathbb{S}^1}$ is the trivial one then $\left(H^{i}\left((F\cup\Gamma)^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = 0$. On the other hand, we have $\Sigma^{\mathbb{S}^1} = M^{\mathbb{S}^1}$. This gives $\left(H^{*-2}\left(\Sigma^{\mathbb{S}^1}, (F\cup\Gamma)^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} = \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$. ♣

**Theorem 4.4** *Associated to any smooth action $\Phi\colon \mathbb{S}^3\times M \longrightarrow M$ we have the Gysin sequence*

$$\cdots \longrightarrow H^{i}(M) \longrightarrow H^{i-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{i-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2} \longrightarrow H^{i+1}\left(M/\mathbb{S}^3\right) \longrightarrow H^{i+1}(M) \longrightarrow \cdots$$

*where $\Sigma$ is the subset of points of $M$ whose isotropy group is not finite, the $\mathbb{Z}_2$-action is induced by the product by $j\in\mathbb{S}^3$ and $(-)^{-\mathbb{Z}_2}$ denotes the subspace of antisymmetric elements.*

*Proof.* If we prove that the connecting morphism $\delta\colon H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \longrightarrow H^{*+1}\left(\mathrm{Ker}^{\cdot} \textstyle\fint\right)$ of the long exact sequence associated to (8) vanishes then we shall get that

$$H^{*}(\mathrm{Coker}\ I) \xlongequal{\text{Lemma 2.5}} H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus H^{*}\left(\mathrm{Ker}^{\cdot} \textstyle\fint\right) \xlongequal{\text{Lemma 4.3}} H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}.$$

Now the Gysin sequence will come from the short exact sequence

(12) $$0 \longrightarrow \Omega^{*}_{V}\left(M/\mathbb{S}^3\right) \xrightarrow{\ I\ } \underline{\Omega}^{*}_{V}(M) \longrightarrow \mathrm{Coker}\ I \longrightarrow 0,$$

since (3) and Lemma 2.5.

Notice that the connecting morphism $\delta$ is defined by $\delta([\zeta]) = \pm[< d(\mathcal{X}_1\wedge\mathcal{X}_2\wedge\mathcal{X}_3)\wedge\zeta >]$. We have $\delta\equiv 0$ since $\zeta_1 = 0$ (cf. Lemma 4.1). ♣

**4.5. Morphisms.** We describe the morphisms of the above Gysin sequence.

$$①\colon H^{*}\left(M/\mathbb{S}^3\right) \longrightarrow H^{*}(M)$$

It is the pull-back $\pi^*$ of the canonical projection $\pi\colon M\to M/\mathbb{S}^3$ (cf. Remark 5.9).

$$②\colon H^{*}(M) \longrightarrow H^{*-3}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) \oplus \left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$$

We have already seen that the first component of this morphism is induced by the integration along the fibers of $\pi$, that is, it is given by $\fint_{\mathbb{S}^3}[\omega] = [i_{X_3} i_{X_2} i_{X_1}\omega]$.

For the second component we keep track of the isomorphisms given by Lemma 4.3 and we get that it is defined by:

$$[\omega] \mapsto \text{class}\left(\fint_{\mathbb{S}^2} (\omega_1 - \sigma(\iota^*\omega_1))\right),$$

where $\fint_{\mathbb{S}^2}$ is the integration along the fibers of the canonical projection $\mathrm{pr}_2\colon \mathbb{S}^2\times(\Sigma\backslash(F\cup\Gamma))^{\mathbb{S}^1} \longrightarrow (\Sigma\backslash(F\cup\Gamma))^{\mathbb{S}^1}$.

$$③\colon H^{*-3}\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right)\oplus\left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}\longrightarrow H^{*+1}\left(M/\mathbb{S}^3\right)$$

A straightforward calculation using sequences (8) and (12) gives that the connecting morphism ③ of the Gysin sequence sends:

- $[\zeta]\in H^{*-3}\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right)$ to -$\left[\left(e_1^2+e_2^2+e_3^2\right)\wedge\zeta\right]$;
- $[\xi]\in\left(H^{*-2}\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$ to $[d\sigma(\epsilon_2\wedge \mathrm{pr}_2^*\xi)]$ where $\epsilon_2$ is the canonical volume form of $\mathbb{S}^2$.

Since $e_1^2+e_2^2+e_3^2$ is not a Verona's form, then it does not define a class of $H^4\left(M/\mathbb{S}^3\right)$. Nevertheless, it does generate a class of the intersection cohomology $\boldsymbol{H}^4_{\bar{4}}\left(M/\mathbb{S}^3\right)$ (as in the semi-free case of [4]).

**4.6. Remarks.**

(a) We have

$$\left(H^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}=\frac{H^*\left(M^{\mathbb{S}^1}\right)}{H^*\left(M^{\mathbb{S}^1}/\mathbb{Z}_2\right)}.$$

Let us see that. The assignment $\omega\mapsto\left(\dfrac{\omega+j^*\omega}{2},\dfrac{\omega-j^*\omega}{2}\right)$ establishes the isomorphism

$$\Omega^*\left(M^{\mathbb{S}^1}\right)=\left(\Omega^*\left(M^{\mathbb{S}^1}\right)\right)^{\mathbb{Z}_2}\oplus\left(\Omega^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}=\Omega^*\left(M^{\mathbb{S}^1}/\mathbb{Z}_2\right)\oplus\left(\Omega^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$$

and then $H^*\left(M^{\mathbb{S}^1}\right)=H^*\left(M^{\mathbb{S}^1}/\mathbb{Z}_2\right)\oplus\left(H^*\left(M^{\mathbb{S}^1}\right)\right)^{-\mathbb{Z}_2}$. This gives the claim.

(b) Let us suppose that there is not a point of $M$ whose isotropy subgroup is one dimensional[9]. Then, we have a long exact sequence

$$\cdots\to H^i(M)\to H^{i-3}\left(M/\mathbb{S}^3,F\right)\to H^{i+1}\left(M/\mathbb{S}^3\right)\to H^{i+1}(M)\to\cdots,$$

since $j$ acts trivially on $M^{\mathbb{S}^1}=F$. This case contains the semi-free, almost-free and free actions.

(c) Let us suppose that there is not a point of $M$ whose isotropy subgroup is conjugated to $\mathbb{S}^1$[10]. Then, we have a long exact sequence

$$\cdots\to H^i(M)\to H^{i-3}\left(M/\mathbb{S}^3,\Sigma/\mathbb{S}^3\right)\to H^{i+1}\left(M/\mathbb{S}^3\right)\to H^{i+1}(M)\to\cdots.$$

since $j$ acts trivially on $M^{\mathbb{S}^1}=\left\{x\in M\;\middle|\;\mathbb{S}^3_x=\mathbb{S}^3\text{ or }N(\mathbb{S}^1)\right\}$.

**5. Appendix.** We give a proof of Lemma 2.5.

**5.1. Stratification.** We shall use in this Appendix the orbit type stratification $\mathcal{S}_M$ of the action $\Phi$ (cf. [2]). This partition is determined by the equivalence relation $x\sim y\Leftrightarrow\mathbb{S}^3_x$ conjugated to $\mathbb{S}^3_y$, and its elements are invariant submanifolds. For example, the connected components of $F$, $\Gamma$ and $\Sigma\backslash\Gamma$ are elements of $\mathcal{S}_M$ while those of $\Sigma$ and $M\backslash\Sigma$ may not be.

[9] That is, $\Sigma=F=M^{\mathbb{S}^3}$.
[10] That is, $\Sigma=F\cup\Gamma$.

This stratification is endowed with the order: $S_1 \preceq S_2 \Leftrightarrow S_1 \subset \overline{S_2}$. The *depth* of $\mathcal{S}_M$, written depth $\mathcal{S}_M$, is defined to be the largest $i$ for which there exists a chain of strata $S_0 \prec S_1 \prec \cdots \prec S_i$.

**5.2. Integration.** The isomorphism we construct for proving Lemma 2.5 is given by integration. Since the integration of controlled forms cannot be done directly on singular simplices we introduce an auxiliary complex.

⇝ $S_*\left(M/\mathbb{S}^3\right)$ denotes the complex generated by the singular simplices of $M/\mathbb{S}^3$. The associated cochain complex is $S^*\left(M/\mathbb{S}^3\right) = \text{Hom}\left(S_*\left(M/\mathbb{S}^3\right), \mathbb{R}\right)$.

⇝ A *liftable singular simplex* of $M/\mathbb{S}^3$ is a continuous map $\varphi\colon \Delta \to M/\mathbb{S}^3$ such that:

(a) there exists a smooth map $\widetilde{\varphi}\colon \Delta \to M$ with $\pi \circ \widetilde{\varphi} = \varphi$, and

(b) the pullback $\widetilde{\varphi}^{-1}\left(\overline{S}\right)$ is a face of $\Delta$, for each stratum $S \in \mathcal{S}_M$.

We shall say that $\widetilde{\varphi}$ is a *lifting* of $\varphi$.

⇝ $LS_*\left(M/\mathbb{S}^3\right)$ denotes the complex generated by the liftable singular simplices of $M/\mathbb{S}^3$. The associated cochain complex is $LS^*\left(M/\mathbb{S}^3\right) = \text{Hom}\left(LS_*\left(M/\mathbb{S}^3\right), \mathbb{R}\right)$.

Now, integration over these simplices makes sense.

**Lemma 5.3** *The map* $\Omega^*_V\left(M/\mathbb{S}^3\right) \xrightarrow{\int} LS^*\left(M/\mathbb{S}^3\right)$*, defined by* $\int_\varphi \omega = \int_\Delta \widetilde{\varphi}^* \omega$*, is a well defined differential operator.*

*Proof.* The only problem might come from the non uniqueness of the lifting $\widetilde{\varphi}$ (cf. (a)). Since the differential form $\omega$ is invariant, then it suffices to prove the following statement:

> "Let $\omega$ be a differential form of $\Omega^*_V\left(M/\mathbb{S}^3\right)$. Given two lifting maps $\psi_1$ and $\psi_2$ of a liftable singular simplex $\varphi\colon \Delta \to M/\mathbb{S}^3$, there exists a smooth map[11] $g\colon \overset{\circ}{\Delta} \to \mathbb{S}^3$ with $\psi_2 = g \circ \psi_1$."

Let $t$ be a point of $\overset{\circ}{\Delta}$ and let $S_1$ be the stratum containing $\psi_1(t)$. Condition (b) gives $\psi_1(\Delta) \subset \overline{S_1}$. Consider a stratum $S \preceq S_1$ meeting $\psi_1(\overset{\circ}{\Delta})$. Condition (b) gives $\psi_1(\Delta) \subset \overline{S}$. Then $S_1 \cap \overline{S} \neq \emptyset$, that is, $S_1 \preceq S$. We get $S = S_1$ and therefore $\psi_1(\Delta) \subset S_1$. By the same reason, we find a stratum $S_2$ containing $\psi_2(\Delta)$. Since $\pi \circ \psi_1 = \pi \circ \psi_2$ then $\pi(S_1) \cap \pi(S_2) \neq \emptyset$. This implies $S_1 \cap S_2 \neq \emptyset$ and therefore $S_1 = S_2$. The result comes now from the fact that the projection $\pi\colon S_1 \to S_1/\mathbb{S}^3$ is a homogeneous bundle. ♣

So, we have the following diagram of differential morphisms:

$$\Omega^*_V\left(M/\mathbb{S}^3\right) \xrightarrow{\int} LS^*\left(M/\mathbb{S}^3\right) \xleftarrow{\rho} S^*\left(M/\mathbb{S}^3\right),$$

where $\rho$ is the restriction. We define the statement

$$\mathfrak{P}(M) := \text{"The morphisms } \int \text{ and } \rho \text{ are quasi-isomorphisms"}$$

In a similar way we define $\mathfrak{P}(M, \Sigma)$. The goal of the Appendix is to prove $\mathfrak{P}(M)$ and $\mathfrak{P}(M, \Sigma)$, which will give Lemma 2.5. This is done in several steps.

[11] Here, $\overset{\circ}{\Delta}$ is the interior of $\Delta$.

**5.4. First Step: If all the orbits have the same dimension then $\mathfrak{P}(M)$ is true.** Notice first that we have $\Omega^*_v\left(M/\mathbb{S}^3\right) = \Omega^*\left(M/\mathbb{S}^3\right)$. We proof $\mathfrak{P}(M)$ by induction on the depth of $\mathcal{S}_M$.

• depth $\mathcal{S}_M = 0$. Since the action $\Phi\colon \mathbb{S}^3 \times M \to M$ is a homogenous bundle then the complex $LS_*\left(M/\mathbb{S}^3\right)$ is generated by the smooth simplices of the manifold $M/\mathbb{S}^3$. We know that the restriction $LS^*\left(M/\mathbb{S}^3\right) \xleftarrow{\rho} S^*\left(M/\mathbb{S}^3\right)$ is a quasi-isomorphism. Now the de Rham Theorem gives $\mathfrak{P}(M)$.

• Induction. Consider $S \subset M$ the union of the minimal strata of $\mathcal{S}_M$. It is an invariant submanifold[12] with depth $\mathcal{S}_S = 0$ which gives $\mathfrak{P}(S)$. Fix $\tau\colon T \to S$ an equivariant tubular neighborhood of $S$. Associated to the open covering $\{M\backslash S, T\}$ of $M\backslash\Sigma$ we have the Mayer-Vietoris commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & S^*\left((T\backslash S)/\mathbb{S}^3\right) & \to & S^*\left(T/\mathbb{S}^3\right)\oplus LS^*\left((M\backslash S)/\mathbb{S}^3\right) & \to & S^*\left(M/\mathbb{S}^3\right) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & LS^*\left((T\backslash S)/\mathbb{S}^3\right) & \to & LS^*\left(T/\mathbb{S}^3\right)\oplus LS^*\left((M\backslash S)/\mathbb{S}^3\right) & \to & LS^*\left(M/\mathbb{S}^3\right) & \to & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \Omega^*\left((T\backslash S)/\mathbb{S}^3\right) & \to & \Omega^*\left(T/\mathbb{S}^3\right)\oplus \Omega^*\left((M\backslash S)/\mathbb{S}^3\right) & \to & \Omega^*\left(M/\mathbb{S}^3\right) & \to & 0
\end{array}$$

We shall denote this kind of diagrams by:

$$0 \Rrightarrow \mathfrak{P}(T\backslash S) \Rrightarrow \mathfrak{P}(T)\oplus\mathfrak{P}(M\backslash S) \Rrightarrow \mathfrak{P}(M) \Rrightarrow 0.$$

The first and second lines are exact since linear subdivisions preserve (liftable) singular simplices. The third row is exact since the covering $\{M\backslash S, T\}$ possesses a subordinated partition of unity made up of smooth invariant maps (for example, $\{1-(f\circ\nu), f\circ\nu\}$ where $\nu$ is the radius map associated to $\tau$). The Five's Lemma gives: $\mathfrak{P}(T\backslash S), \mathfrak{P}(T), \mathfrak{P}(M\backslash S) \Longrightarrow \mathfrak{P}(M)$.

Since depth $\mathcal{S}_{T\backslash S} <$ depth $\mathcal{S}_M$ and depth $\mathcal{S}_{M\backslash S} <$ depth $\mathcal{S}_M$ then, by induction hypothesis, we have $\mathfrak{P}(T\backslash S)$ and $\mathfrak{P}(M\backslash S)$.

Consider the map $H\colon T\times[0,1]\to T$ defined by $H(x,t) = \partial(t,x)$ (the dilatation map associated to $\tau$). It is an equivariant homotopy between $\jmath\circ\tau$ and the identity map of $T$, where $\jmath\colon S \hookrightarrow T$ is the natural inclusion. So, restrictions $LS^*\left(T/\mathbb{S}^3\right) \to LS^*\left(S/\mathbb{S}^3\right)$, $S^*\left(T/\mathbb{S}^3\right) \to S^*\left(S/\mathbb{S}^3\right)$ and $\Omega^*\left(T/\mathbb{S}^3\right) \to \Omega^*\left(S/\mathbb{S}^3\right)$ are quasi-isomorphisms. Then we get $\mathfrak{P}(T)$ since we have $\mathfrak{P}(S)$ (depth $\mathcal{S}_S = 0$).

**5.5. Second Step: $\mathfrak{P}(M\backslash F) \Longrightarrow \mathfrak{P}(M)$.** Associated to the open covering $\{M\backslash F, T_0\}$ of $M$ we have the Mayer-Vietoris commutative diagram

$$0 \Rrightarrow \mathfrak{P}(T_0\backslash F) \Rrightarrow \mathfrak{P}(T_0)\oplus\mathfrak{P}(M\backslash F) \Rrightarrow \mathfrak{P}(M) \Rrightarrow 0.$$

The first and second rows are exact since linear subdivisions preserve (liftable) singular simplices. The third row is exact since the covering $\{M\backslash F, T_0\}$ possesses a subordinated partition of unity made up of invariant controlled functions (for example, $\{1-(f\circ\nu_0), f\circ\nu_0\}$ cf. 4.2). The Five's Lemma gives: $\mathfrak{P}(T_0\backslash F), \mathfrak{P}(T_0), \mathfrak{P}(M\backslash F) \Longrightarrow \mathfrak{P}(M)$.

Since $T_0\backslash F$ does not possess any fixed point then $\mathfrak{P}(M\backslash F) \Longrightarrow \mathfrak{P}(T_0\backslash F)$.

Consider the map $H\colon T_0\times[0,1]\to T_0$ defined by $h(x,t) = \partial_0(t,x)$. It is an equivariant homotopy between $\jmath\circ\tau_0$ and the identity map of $T_0$, where $\jmath\colon F \hookrightarrow T_0$ is the natural inclusion. So, restrictions $LS^*\left(T_0/\mathbb{S}^3\right) \to LS^*(F)$ and $S^*\left(T_0/\mathbb{S}^3\right) \to S^*(F)$ are quasi-isomorphisms. Moreover, since the homotopy

[12] This manifold may have connected components with different dimensions.

$H$ preserves the Thom-Mather structure, then the restriction $\Omega^*_{\nu}\left(T_0/\mathbb{S}^3\right) \to \Omega^*(F)$ is a quasi-isomorphism. Then we get $\mathfrak{P}(T_0)$ since we have $\mathfrak{P}(F)$ (cf. First Step).

**5.6. Third Step: $\mathfrak{P}(M\backslash F)$ is true.** Associated to the open covering $\{M\backslash\Sigma, T_1\}$ of $M\backslash F$ we have the Mayer-Vietoris commutative diagram

$$0 \Rrightarrow \mathfrak{P}(T_1\backslash\Sigma) \Rrightarrow \mathfrak{P}(T_1) \oplus \mathfrak{P}(M\backslash\Sigma) \Rrightarrow \mathfrak{P}(M\backslash F) \Rrightarrow 0.$$

The first and second rows are exact since linear subdivisions preserve (liftable) singular simplices. The third row is exact since the covering $\{M\backslash\Sigma, T_1\}$ possesses a subordinated partition of unity made up of invariant controlled functions (for example, $\{1 - (f\circ\nu_1), f\circ\nu_1\}$ cf. 2.1). The Five's Lemma gives: $\mathfrak{P}(T_1\backslash\Sigma), \mathfrak{P}(T_1)$, and $\mathfrak{P}(M\backslash\Sigma) \Longrightarrow \mathfrak{P}(M\backslash F)$.

From First Step we get $\mathfrak{P}(M\backslash\Sigma)$ and $\mathfrak{P}(T_1\backslash\Sigma)$.

Consider the map $H\colon T_1 \times [0,1] \to T_1$ defined by $H(x,t) = \partial_1(t,x)$. It is an equivariant homotopy between $\jmath\circ\tau_1$ and the identity map of $T_1$, where $\jmath\colon (\Sigma\backslash F) \hookrightarrow T_1$ is the natural inclusion. So, the restrictions $LS^*\left(T_1/\mathbb{S}^3\right) \to LS^*\left((\Sigma\backslash F)/\mathbb{S}^3\right)$ and $S^*\left(T_1/\mathbb{S}^3\right) \to S^*\left((\Sigma\backslash F)/\mathbb{S}^3\right)$ are quasi-isomorphisms. Moreover, since the homotopy $H$ preserves the tubular neighborhood $\tau_1$ then the restriction $\Omega^*_{\nu}\left(T_1/\mathbb{S}^3\right) \to \Omega^*\left((\Sigma\backslash F)/\mathbb{S}^3\right)$ is a quasi-isomorphism. Then we get $\mathfrak{P}(T_1)$ since we have $\mathfrak{P}(\Sigma\backslash F)$ (cf. First Step).

**5.7. Fourth Step: Notice that we have proved that $\mathfrak{P}(M)$ is true.**

**5.8. Fifth Step: $\mathfrak{P}(M,\Sigma)$ is true.** By definition of controlled forms we have $\Omega^*_{\nu}\left(M/\mathbb{S}^3, \Sigma/\mathbb{S}^3\right) = \Omega^*\left(M/\mathbb{S}^3, (D_0 \cup D_1)/\mathbb{S}^3\right)$. We have seen in Second Step and Third Step that $D_0$ retracts equivariantly to $F$ and that $D_1$ retracts equivariantely to $\Sigma\backslash F$, this last retraction preserving $D_0$. So, we have

$$\mathfrak{P}(M, D_0 \cup (\Sigma\backslash F)) \Longrightarrow \mathfrak{P}(M, D_0 \cup D_1) \Longleftrightarrow \mathfrak{P}(M,\Sigma).$$

Associated to the triple $(M, D_0, \Sigma\backslash F)$ we have the commutative diagram

$$0 \Rrightarrow \mathfrak{P}(M, D_0 \cup (\Sigma\backslash F)) \Rrightarrow \mathfrak{P}(M, D_0) \Rrightarrow \mathfrak{P}(\Sigma\backslash F, D_0 \cap (\Sigma\backslash F)) \Rrightarrow 0.$$

The first and second rows are clearly exact. The third row is exact since (2). The Five's Lemma gives: $\mathfrak{P}(M, D_0), \mathfrak{P}(\Sigma\backslash F, D_0 \cap (\Sigma\backslash F)) \Longrightarrow \mathfrak{P}(M, D_0 \cup (\Sigma\backslash F))$. The two following items end the proof, the Appendix and the article.

- $\mathfrak{P}(M, D_0)$ is true. Since $D_0$ retracts equivariantly to $F$ then $\mathfrak{P}(M,F) \Longleftrightarrow \mathfrak{P}(M, D_0)$. Associated to the pair $(M,F)$ we have the commutative diagram $0 \Rrightarrow \mathfrak{P}(M,F) \Rrightarrow \mathfrak{P}(M) \Rrightarrow \mathfrak{P}(F) \Rrightarrow 0$. The first and second rows are clearly exact. The third row is also exact (see proof of Lemma 4.2). The Five's Lemma gives: $\mathfrak{P}(M), \mathfrak{P}(F) \Longrightarrow \mathfrak{P}(M,F)$. But we have $\mathfrak{P}(M)$ (cf. Forth Step) and $\mathfrak{P}(F)$ (First Step).

- $\mathfrak{P}(\Sigma\backslash F, D_0 \cap (\Sigma\backslash F))$ is true. Associated to the open covering $\{\Sigma\backslash \nu_0^{-1}([0,1.5]), \nu_0^{-1}(]0,3.5[) \cap \Sigma\}$ of $\Sigma\backslash F$ we have the Mayer-Vietoris commutative diagram:

$$0 \Rrightarrow \mathfrak{P}(\Sigma\backslash F, D_0\cap(\Sigma\backslash F)) \Rrightarrow \mathfrak{P}(\Sigma\backslash\nu_0^{-1}([0,1.5]))\oplus\mathfrak{P}(\nu_0^{-1}(]0,3.5[)\cap\Sigma), D_0\cap(\Sigma\backslash F)) \Rrightarrow \mathfrak{P}(\Sigma\cap\nu_0^{-1}(]1.5,3.5[)) \Rrightarrow 0.$$

  The first and second rows are exact since linear subdivisions preserve (liftable) singular simplices. The third row is exact since the covering $\{\Sigma\backslash\nu_0^{-1}([0,1.5]), \nu_0^{-1}(]0,3.5[)\cap\Sigma\}$ possesses a subordinated partition of unity made up of invariant controlled functions (for example, $\{f\circ\nu_0, 1-(f\circ\nu_0)\}$ cf. 2.1). The Five's Lemma gives:

$$\mathfrak{P}(\Sigma\backslash\nu_0^{-1}([0,1.5])), \mathfrak{P}(\nu_0^{-1}(]0,3.5[)\cap\Sigma), D_0\cap(\Sigma\backslash F)), \mathfrak{P}(\Sigma\cap\nu_0^{-1}(]1.5,3.5[)) \Longrightarrow \mathfrak{P}(\Sigma\backslash F, D_0\cap(\Sigma\backslash F)).$$

From First Step we get $\mathfrak{P}(\Sigma\backslash\nu_0^{-1}([0,1.5]))$ and $\mathfrak{P}(\Sigma\cap\nu_0^{-1}(]1.5,3.5[))$. Finally, since $D_0\cap\Sigma$ is equivariantly diffeomorphic to $(S_0\cap\Sigma)\times]0,\infty[$ then $\mathfrak{P}(\nu_0^{-1}(]0,3.5[)\cap\Sigma), D_0\cap(\Sigma\backslash F)) \Longleftrightarrow \mathfrak{P}((S_0\cap\Sigma)\times]0,3.5[), (S_0\cap\Sigma)\times]0,1[))$. Retracting the second factor, we get $\mathfrak{P}(\nu_0^{-1}(]0,3.5[)\cap\Sigma), D_0\cap(\Sigma\backslash F)) \Longleftrightarrow \mathfrak{P}(S_0\cap\Sigma, S_0\cap\Sigma)$. But $\mathfrak{P}(S_0\cap\Sigma, S_0\cap\Sigma)$ is clearly true! ♣

**5.9. Remark.** Keeping track of the inclusion $I\colon \Omega^*_{v}\left(M/\mathbb{S}^3\right) \hookrightarrow \underline{\Omega}^*_{v}(M)$ one shows that it induces the morphism $\pi^*\colon H^*\left(M/\mathbb{S}^3\right) \to H^*(M)$.

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