

A Gysin sequence for semifree actions of \mathbf{S}^3 *

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Abstract

In this work we shall consider smooth semifree (i.e., free outside the fixed point set) actions of \mathbf{S}^3 on a manifold M . We exhibit a Gysin sequence relating the cohomology of M with the intersection cohomology of the orbit space M/\mathbf{S}^3 . This generalizes the usual Gysin sequence associated with a free action of \mathbf{S}^3 .

Given a free action of the group of unit quaternions \mathbf{S}^3 on a differentiable manifold M , there exists a long exact sequence relating the deRham cohomology of the manifold M with the deRham cohomology of the orbit space M/\mathbf{S}^3 . This is the Gysin sequence (see for example [1, pag.179]):^{1 2}

$$(1) \quad \dots \rightarrow H^i(M) \xrightarrow{f^*} H^{i-3}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \rightarrow \dots$$

where f is the integration along the fibers of the natural projection $\pi: M \rightarrow M/\mathbf{S}^3$ and $[e] \in H^4(M/\mathbf{S}^3)$ is the Euler class of Φ . This paper is devoted to generalizing this relationship to the case where Φ is allowed to have fixed points (semifree action).

In this context the orbit space M/\mathbf{S}^3 is no longer a manifold but a stratified pseudomanifold, a notion introduced by Goresky and MacPherson in [7]. The Gysin sequence we get in this case is

$$(2) \quad \dots \rightarrow H^i(M) \xrightarrow{f^*} IH_{\bar{r}}^{i-3}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} IH_{\bar{r}+4}^{i+1}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \rightarrow \dots$$

where \bar{r} and $\overline{r+4}$ are two perversities and $[e] \in IH_{\bar{r}}^4(M/\mathbf{S}^3)$ is the Euler class of Φ . The exact statement is given in Theorem 4.7. A similar sequence has been already found for circle actions [8]. Finally, we show a relationship between the existence of a section of π and the vanishing of the Euler class $[e]$. This result generalizes the situation of the free case

The work is organized as follows. In Section 1, we introduce simple stratified spaces, which are singular spaces including the orbit space M/\mathbf{S}^3 as a special case. Section 2 is devoted to recalling the notion of intersection cohomology with the perversity introduced by MacPherson in [9]. The main tool we use to construct the Gysin Sequence is the complex of invariant forms, which is studied in Section 3. Finally, we construct the Gysin sequence (2) in the last Section.

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In this paper, a manifold is supposed to be without boundary and smooth (of class C^∞). From now on, we fix a manifold M with dimension m and $\Phi: \mathbf{S}^3 \times M \rightarrow M$ a smooth semifree action, that is, Φ is free out of the set $M^{\mathbf{S}^3}$ of fixed points (which will be different from M).

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1 Simple stratified sets

We prove that the action Φ induces on M and M/\mathbf{S}^3 a particular structure of stratified set.

1.1 Let E be a stratified set [10], we shall say that E is **simple** if there exists a stratum R with $E = \overline{R}$ (such R is said to be **regular**), and any other stratum S is closed (S is said to be **singular**). The second condition implies that the singular strata are disjoint. The dimension of E is, by definition, $\dim R$. We shall write \mathcal{S} to represent the family of singular strata.

1.2 We know (cf. [10]) that for each stratum $S \in \mathcal{S}$ there exist a neighborhood T_S of S , a compact manifold L_S and a fiber bundle $\tau_S: T_S \rightarrow S$ satisfying:

- a) the fiber of τ_S is the cone $cL_S = L_S \times [0, 1[/ L_S \times \{0\}$,
- b) the restriction map $\tau_S|_S$ is the identity,
- c) the restriction $\tau_S: (T_S - S) \rightarrow S$ is a smooth fiber bundle with fiber $L_S \times]0, 1[$, whose structural group is $\text{Diff}(L_S)$, the group of diffeomorphisms of L_S , and
- d) $T_S \cap T_{S'} = \emptyset$ if $S \neq S'$.

The family $\{T_S / S \in \mathcal{S}\}$ is said to be a **family of tubes**. Notice that, according to c), there exists a smooth map $\lambda_S: (T_S - S) \rightarrow]0, 1[$ such that the restriction $\tau_S: \lambda_S^{-1}(]0, \varepsilon[) \rightarrow S$, for $\varepsilon \in [0, 1]$, is a fiber bundle with fiber $L_S \times]0, \varepsilon[$. We shall write $D_S = \lambda_S^{-1}(]0, 1/2[)$; in fact, D_S is the half of T_S .

1.3 The manifold M inherits from the action³ Φ a natural structure of stratified set where the singular strata are the connected components of $M^{\mathbf{S}^3}$ and the regular stratum R is $M - M^{\mathbf{S}^3}$. This stratified set is simple because the open set $M - M^{\mathbf{S}^3}$ is dense.

Since each singular stratum S of M is an invariant submanifold of M , we construct a tubular neighborhood $(T_S, \tau_S, S, \mathbf{S}^{\ell_S})$ satisfying:

- i) T_S is an open neighborhood of S ,
- ii) $\tau_S: T_S \rightarrow S$ is a smooth fiber bundle with fiber the open disk D^{ℓ_S+1} and $O(\ell_S + 1)$ as a structural group,
- iii) the restriction of τ_S to S is the identity,
- iv) τ_S is equivariant, that is, $\tau_S(g \cdot y) = g \cdot \tau_S(y)$,
- v) there exists an orthogonal action $\Psi^S: \mathbf{S}^3 \times \mathbf{S}^{\ell_S} \rightarrow \mathbf{S}^{\ell_S}$ and an atlas $\mathcal{A}_S = \{(U, \varphi)\}$ such that $\varphi: \tau_S^{-1}(U) \rightarrow U \times D^{\ell_S+1}$ is equivariant, that is, $\varphi(\Phi(g, x)) = (\tau_S(x), [\Phi^S(g, \theta), r])$ for each $g \in \mathbf{S}^3$ and $x = \varphi^{-1}(\tau_S(x), [\theta, r]) \in \tau_S^{-1}(U)$. Here we have identified D^{ℓ_S+1} with the cone $c\mathbf{S}^{\ell_S} = \mathbf{S}^{\ell_S} \times [0, 1[/ \mathbf{S}^{\ell_S} \times \{0\}$ and written $[\theta, r]$ an element of $c\mathbf{S}^{\ell_S}$.

Notice that the action Φ^S is free and therefore the codimension of S is a multiple of 4. Consider, for each singular stratum S , a tubular neighborhood T_S verifying $T_S \cap T_{S'} = \emptyset$ if $S \neq S'$. Thus, the family $\{T_S\}$ is a family of tubes.

Let $\pi: M \rightarrow M/\mathbf{S}^3$ denote the canonical projection. The orbit space M/\mathbf{S}^3 inherits naturally from M a structure of stratified set, the strata are $\pi(R) = \pi(M - M^{\mathbf{S}^3})$, the regular stratum, with dimension $m - 3$, and $\{\pi(S) / S \in \mathcal{S}\}$, the singular strata. The local description given by v) shows that M/\mathbf{S}^3 is a simple stratified set.

For each $S \in \mathcal{S}$ the image $\pi(T_S)$ is a neighborhood of $\pi(S)$. The map $\rho_S: \pi(T_S) \rightarrow \pi(S)$ given by $\rho_S(\pi(x)) = \pi(\tau_S(x))$ is well defined. It is easy to show that $(\pi(\tau_S), \rho_S, \pi(S), \mathbf{S}^{\ell_S}/\mathbf{S}^3)$ verify §1.2 a)-d). Then, the family $\{\pi(T_S) / S \in \mathcal{S}\}$ is a family of tubes.

1.4 Consider the commutative diagram

³For the notions related with actions we refer to [3].

$$\begin{array}{ccc}
D_S - S & \xrightarrow{\tau_S} & S \\
\pi \downarrow & & \downarrow \pi \\
\pi(D_S - S) & \xrightarrow{\rho_S} & \pi(S)
\end{array}$$

Since the restriction of π to the fibers of τ_S is a submersion ($(\mathbf{S}^{\ell_S} \times]0, 1/2[) \mapsto (\mathbf{S}^{\ell_S} / \mathbf{S}^3 \times]0, 1/2[)$), we get the relation $\pi_*\{Ker(\tau_S)_*\} = Ker(\rho_S)_*$. This will be used in §3.3.

2 Intersection cohomology

We recall the notion of intersection cohomology [4] using the notion of perversity introduced by MacPherson in [9].

2.1 Cartan's filtration. Let $\kappa: N \rightarrow C$ be a smooth submersion between two manifolds N and C . For each differential form $\omega \neq 0$ on N , we define the **perverse degree** of ω , written $\|\omega\|_C$, as the smallest integer k verifying:

$$\text{If } \xi_0, \dots, \xi_k \text{ are vectorfields on } N \text{ tangents to the fibers of } \kappa, \text{ then } i_{\xi_0} \cdots i_{\xi_k} \omega \equiv 0.$$

Here, i_{ξ_j} denotes the interior product by ξ_j . We shall write $\|0\|_C = -\infty$. For each $k \geq 0$ we put $F_k \Omega_N^* = \{\omega \in \Omega^*(N) / \|\omega\|_C \leq k \text{ and } \|d\omega\|_C \leq k\}$. This is the **Cartan's filtration** of κ [4]. Notice that, for $\alpha, \beta \in \Omega^*(N)$, we get the relations

$$(3) \quad \|\alpha + \beta\|_C \leq \max(\|\alpha\|_C, \|\beta\|_C) \quad \text{and} \quad \|\alpha \wedge \beta\|_C \leq \|\alpha\|_C + \|\beta\|_C.$$

2.2 Let E be a simple stratified set. A **perversity** is a map $\bar{q}: \mathcal{S} \rightarrow \mathbb{Z}$ (see [9]). A differential form ω on R is a \bar{q} -**intersection differential form** if for each $S \in \mathcal{S}$ the restriction $\omega|_{D_S}$ belongs to $F_{\bar{q}(S)} \Omega_{D_S}^*$. We shall denote by $\Omega_{\bar{q}}^*(E)$ the complex of \bar{q} -intersection differential forms of E . Remark: for the case $\mathcal{S} = \emptyset$, the complex $\Omega_{\bar{q}}^*(E)$ is exactly the deRham complex $\Omega^*(E)$ of E . The cohomology of the complex $\Omega_{\bar{q}}^*(E)$ is the **intersection cohomology** of E , written $IH_{\bar{q}}^*(E)$. This denomination is justified by §2.5.

Locally, the stratified set E looks like $\mathbb{R}^k \times cL_S$, where L_S is a compact manifold. Here, we have the following computational result:

Proposition 2.3 *For any perversity \bar{q} we obtain $IH_{\bar{q}}^i(\mathbb{R}^k \times cL_S) \cong \begin{cases} H^i(L_S) & \text{if } i \leq \bar{q}(S) \\ 0 & \text{if } i > \bar{q}(S). \end{cases}$*

Proof. Since the maps $pr: \mathbb{R}^k \times cL_S \rightarrow \mathbb{R}^{k-1} \times cL_S$ and $J: \mathbb{R}^{k-1} \times cL_S \rightarrow \mathbb{R}^k \times cL_S$ defined by $pr(x_1, \dots, x_k, [y, r]) = (x_2, \dots, x_k, [y, r])$ and $J(x_2, \dots, x_k, [y, r]) = (0, x_2, \dots, x_k, [y, r])$, verify $\|pr^* \omega\|_S \leq \|\omega\|_S$ and $\|J^* \eta\|_S \leq \|\eta\|_S$, for each $\omega \in \Omega^*(\mathbb{R}^{k-1} \times L_S \times]0, 1[)$ and $\eta \in \Omega^*(\mathbb{R}^k \times L_S \times]0, 1[)$, then the induced operators

$$pr^*: \Omega_{\bar{q}}^*(\mathbb{R}^{k-1} \times cL_S) \rightarrow \Omega_{\bar{q}}^*(\mathbb{R}^k \times cL_S) \quad \text{and} \quad J^*: \Omega_{\bar{q}}^*(\mathbb{R}^k \times cL_S) \rightarrow \Omega_{\bar{q}}^*(\mathbb{R}^{k-1} \times cL_S)$$

are well defined. Notice that the composition $J^* pr^*$ is the identity.

Consider the homotopy operator $h: \Omega^*(\mathbb{R}^k \times L_S \times]0, 1[) \rightarrow \Omega^{*-1}(\mathbb{R}^k \times L_S \times]0, 1[)$ given by $h(\omega = \alpha + dx_1 \wedge \beta) = \int_0^1 \beta \wedge dx_1$, where $\alpha, \beta \in \Omega^*(\mathbb{R}^k \times L_S \times]0, 1[)$ do not involve dx_1 . It verifies

$$(4) \quad dh\omega + hd\omega = \omega - pr^* J^* \omega.$$

Now, the relation $\|h\omega\|_S \leq \|\omega\|_S$ implies that h is a homotopy between pr^*J^* and the identity on $\Omega_{\bar{q}}^*(\mathbb{R}^k \times cL_S)$. We have proved $IH_{\bar{q}}^*(\mathbb{R}^k \times cL_S) \cong IH_{\bar{q}}^*(cL_S)$. Moreover, by the equalities $\Omega_{\bar{q}}^i(cL_S) = \Omega^i(L_S \times]0, 1[)$, if $i < \bar{q}(S)$, $\Omega_{\bar{q}}^{\bar{q}(S)}(cL_S) \cap d^{-1}(0) = \Omega^{\bar{q}(S)}(L_S \times]0, 1[) \cap d^{-1}(0)$ and by previous calculation we get $IH_{\bar{q}}^i(\mathbb{R}^k \times cL_S) \cong H^i(L_S)$ for $i \leq \bar{q}(S)$.

It remains to prove that, given a cycle $\omega \in \Omega_{\bar{q}}^i(cL_S)$ with $i > \bar{q}(S)$, there exists $\eta \in \Omega_{\bar{q}}^{i-1}(cL_S)$ with $d\eta = \omega$. Write $\omega = \alpha + dr \wedge \beta$, where α, β do not involve dr (r variable of $]0, 1[$); observe that $\alpha \equiv \beta \equiv 0$ on $L_S \times]0, 1/2[$. Then, since $\omega = d \int_0^{\cdot} \beta \wedge dr$, it suffices to take $\eta = \int_0^{\cdot} \beta \wedge dr$. ♣

The intersection cohomology satisfies the Mayer-Vietoris property as it is stated in [1, pag.94].

Proposition 2.4 *Given an open covering $\mathcal{U} = \{U\}$ of E , there exists a subordinated partition of the unity $\{f_U\}$ verifying: $\omega \in \Omega_{\bar{q}}^*(E) \implies f_U \omega \in \Omega_{\bar{q}}^*(U)$.*

Proof. A **controlled map** $f: E \rightarrow \mathbb{R}$ is defined to be a continuous map, differentiable on each stratum, such that the restriction to the fibers of each $\tau_S: D_S \rightarrow S$ is a constant map [11]. Notice that we have the equality $\max(\|f\|_S, \|df\|_S) = 0$. Then, the result follows from the fact that \mathcal{U} possesses a subordinated partition of unity made up of controlled functions [11, pag.8]. ♣

Two perversities \bar{p} and \bar{q} are **dual** if $\bar{p}(S) + \bar{q}(S) = \dim L_S - 1$ for each $S \in \mathcal{S}$. For example, the **zero perversity** $\bar{0}$, defined by $\bar{0}(S) = 0$, and the **top perversity** \bar{t} , defined by $\bar{t}(S) = \dim L_S - 1$, are dual. The relationship between the intersection homology $IH_{\bar{q}}^*(E)$ of [6] and the intersection cohomology is given by:

Proposition 2.5
$$IH_{\bar{q}}^*(E) \cong IH_{\bar{q}}^{\bar{p}}(E)$$

Proof. Consider first the case $E = \mathbb{R}^k \times cL_S$ as in §2.3. Following [6] and [9] we get: $IH_{\bar{q}}^{\bar{p}}(\mathbb{R}^k \times cL_S) \cong \begin{cases} H_i(L_S) & \text{if } i \leq \dim L_S - 1 - \bar{p}(S) \\ 0 & \text{if } i \geq \dim L_S - \bar{p}(S) \end{cases}$, which is isomorphic to $IH_{\bar{q}}^i(E)$ (see §2.3).

This shows that the intersection cohomology and the intersection homology are locally isomorphic. The passage from the local case to the global case cannot be made as in [7] because the axiomatic presentation of the intersection homology has not yet been extended to the new perversities. But we can proceed as in [2] by showing that the usual integration of differential forms over simplices induces a morphism between $IH_{\bar{q}}^*(E)$ and $\text{Hom}(IH_{\bar{q}}^{\bar{p}}(E), \mathbb{R})$; such morphism turns out to be an isomorphism because of Mayer-Vietoris and previous local calculation. Since the proof is similar to that of [2], we send the reader to this work. ♣

The following result has also been proved in [9].

Corollary 2.6 *Suppose that each link L_S is connected (that is, E is **normal**). Then $IH_{\bar{0}}^*(E) \cong H^*(E)$.*

Proof. It suffices to consider the isomorphism $IH_{\bar{q}}^{\bar{t}}(E) \cong H_*(E)$ proved in [9]. ♣

Corollary 2.7 *If E is a manifold then $IH_{\bar{q}}^*(E) \cong H^*(E)$, for each perversity $\bar{0} \leq \bar{q} \leq \bar{t}$.*

Proof. Since E is normal, Corollary 2.6 reduces the problem to prove that the inclusion $\Omega_{\bar{0}}^*(E) \hookrightarrow \Omega_{\bar{q}}^*(E)$ induces an isomorphism in cohomology. Applying §2.4 and §2.3 and taking into account the inequalities $0 \leq \bar{q}(S) \leq \dim L_S - 1$ we transform the problem to showing $H^i(L_S) = 0$ for $0 < i \leq \bar{q}(S)$. But this is exactly the same as showing that L_S is a cohomology sphere, and this arises from the fact that M is a manifold. ♣

3 Invariant forms

A good simplification in the construction of the Gysin sequence is the use of invariant forms.

3.1 The **fundamental vectorfields** X_1, X_2, X_3 of Φ are the vectorfields of M defined by $X_i(x) = T_e\Phi_x(\ell_i)$, $i = 1, 2, 3$, where $\{\ell_1, \ell_2, \ell_3\}$ is a basis of the Lie algebra of \mathbf{S}^3 . These vectorfields can be chosen to verify $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$ and $[X_3, X_1] = X_2$. The zeroset for each of them is exactly $M^{\mathbf{S}^3}$.

It is well known that the subcomplex of invariant forms

$$I\Omega^*(M) = \{\omega \in \Omega^*(M) / g^*\omega = \omega \text{ for each } g \in \mathbf{S}^3\} = \{\omega \in \Omega^*(M) / L_{X_i}\omega = 0, \quad i = 1, 2, 3\}$$

computes the cohomology of M (see for example [5]). We prove now a similar result for

$$I\Omega_{\bar{q}}^*(M) = \{\omega \in \Omega_{\bar{q}}^*(M) / L_{X_i}\omega = 0, \quad i = 1, 2, 3\}.$$

Proposition 3.2 *For each perversity $\bar{0} \leq \bar{q} \leq \bar{t}$ we have $H^*(I\Omega_{\bar{q}}(M)) \cong H^*(M)$.*

Proof. We first apply §2.4 (with \mathcal{U} made up of invariant sets and $\{f_U\}$ to be invariant controlled maps) and we reduce the problem to $M = \mathbb{R}^k \times c\mathbf{S}^{\ell_S}$. Here, the action of \mathbf{S}^3 is given by

$$(5) \quad (g, (x_1, \dots, x_k, [y, r])) \mapsto (x_1, \dots, x_k, [\Phi^S(g, y), r]).$$

Consider $\mathbb{R}^k \times c\mathbf{S}^{\ell_S}$ as the product $\mathbb{R} \times (\mathbb{R}^{k-1} \times c\mathbf{S}^{\ell_S})$. Notice that the fundamental vectorfields of $\mathbb{R}^k \times c\mathbf{S}^{\ell_S}$ (resp. $\mathbb{R}^{k-1} \times c\mathbf{S}^{\ell_S}$) are $X_i = (\underbrace{0, \dots, 0}_k, Y_i, 0)$ (resp. $Z_i = (\underbrace{0, \dots, 0}_{k-1}, Y_i, 0)$) where Y_i are the

fundamental vectorfields of \mathbf{S}^{ℓ_S} , $i = 1, 2, 3$. Write pr , J and h the operators given by §2.3 for this decomposition. The equalities $pr_*X_i = Z_i$, $J_*Z_i = X_i$ and $i_{X_i}h = hi_{X_i}$ show that these operators are equivariant. Proceeding as in §2.3, we first reduce the problem to the case $M = \mathbb{R}^{k-1} \times c\mathbf{S}^{\ell_S}$ and finally to the case $M = c\mathbf{S}^{\ell_S}$. Again, the operators used in §2.3 to reduce the problem to \mathbf{S}^{ℓ_S} , are equivariant. Here, the inclusion $I\Omega^*(\mathbf{S}^{\ell_S}) \hookrightarrow \Omega^*(\mathbf{S}^{\ell_S})$ induces an isomorphism in cohomology because Φ^S is free. ♣

3.3 For any differential form $\alpha \in \Omega^*(\pi(M - M^{\mathbf{S}^3}))$ the pull-back $\pi^*\alpha$ is an invariant form. According §1.4, it satisfies

$$(6) \quad \|\pi^*\alpha\|_S = \|\alpha\|_{\pi(S)},$$

for each $S \in \mathcal{S}$.

Let μ be a riemannian metric on $R = M - M^{\mathbf{S}^3}$ invariant by the action of Φ and satisfying $\chi_i(X_j) = \delta_{i,j}$ for $i, j \in \{1, 2, 3\}$. The **fundamental forms** of Φ are the differential forms on $M - M^{\mathbf{S}^3}$ defined by $\chi_i = \mu(X_i, -)$, $i = 1, 2, 3$. They satisfy

$$(7) \quad \|\chi_i\|_S = 1.$$

Let $e \in \Omega^4(\pi(R))$ be a closed form representing the Euler class of the action $\Phi: \mathbf{S}^3 \times R \rightarrow R$. Then we can choose $\eta \in \Omega^3(R)$ so that $i_{X_3}i_{X_2}i_{X_1}\eta = 0$ and $d\eta = d(\chi_1 \wedge \chi_2 \wedge \chi_3) - \pi^*e$ (cf. [5, pag. 322]). Notice that the relation $\|e\|_{\pi(S)} \leq 4$ holds for each $S \in \mathcal{S}$. The class $[e] \in IH_4^4(M/\mathbf{S}^3)$ is called the **Euler class** of Φ . It coincides with the usual one when the action Φ is free.

4 Gysin sequence

The Gysin sequence is constructed by using the integration along the fibers of π ; this operator is very simple when we are dealing with invariant differential forms.

4.1 Consider ω an invariant differential form. The differential form $i_{X_3}i_{X_2}i_{X_1}\omega$ is also invariant ($L_{X_i}i_{X_j} = i_{X_j}L_{X_i} + i_{[X_i, X_j]}$); moreover, $i_{X_i}i_{X_3}i_{X_2}i_{X_1}\omega = 0$ for $i = 1, 2, 3$ and therefore $i_{X_3}i_{X_2}i_{X_1}\omega$ is a basic form.

That is, there exists $\eta \in \Omega^*(\pi(R))$ with $i_{X_3}i_{X_2}i_{X_1}\omega = (-1)^{|\omega|}\pi^*\eta$ where $|\omega| = \text{degree of } \omega$. Notice that $i_{X_3}i_{X_2}i_{X_1}d\omega = -di_{X_3}i_{X_2}i_{X_1}\omega$.

The **integration along the fibers** of π is defined to be the operator $\int : I\Omega^*(R) \longrightarrow \Omega^{*-1}(\pi(R))$, where $\int \omega = \eta$. It is a differential operator. Notice that $\int \pi^*\alpha = 0$ and $\int (-1)^{|\alpha|}\chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^*\alpha = \alpha$, for any $\alpha \in \Omega^*(\pi(R))$.

4.2 If the action is free, the short exact sequence

$$0 \longrightarrow \Omega^*(M/\mathbf{S}^3) \xrightarrow{\pi^*} I\Omega^*(M) \xrightarrow{\int^*} \Omega^{*-3}(M/\mathbf{S}^3) \longrightarrow 0,$$

where ι is the inclusion, induces the following long exact sequence

$$\dots \rightarrow H^i(M) \xrightarrow{\int^*} H^{i-3}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \rightarrow \dots,$$

where $[e] \in H^4(M/\mathbf{S}^3)$ is the Euler class of Φ .

If the action Φ is not free, the previous sequence is not longer an exact one (see §4.9). But, we are going to show that by considering the intersection differential forms of M instead of the differential forms, we also get a Gysin sequence relating in this case the intersection cohomology of M/\mathbf{S}^3 with the cohomology of M . This sequence arises from the study of the short exact sequence $0 \longrightarrow Ker \int \xrightarrow{\iota} I\Omega_{\bar{q}}^*(M) \xrightarrow{\int^*} Im \int \longrightarrow 0$, and more precisely, from the comparison of $Ker \int$ and $Im \int$ with $\Omega_{\bar{q}}^*(M/\mathbf{S}^3)$.

There will come out a shift on the perversities involved, due to the perverse degree of e . For this reason we fix three perversities \bar{q} (of M), \bar{r} and $\bar{r} + 4$ (of M/\mathbf{S}^3) satisfying: $\bar{r}(\pi(S)) = \bar{q}(S) - 4$, $\bar{r} + 4(\pi(S)) = \bar{q}(S)$, and $\bar{0} \leq \bar{q} \leq \bar{t}$.

4.3 Kernel of \int . By construction we have $Ker \int = \{\omega \in I\Omega_{\bar{q}}^*(M) / i_{X_3}i_{X_2}i_{X_1}\omega = 0\}$. For each $\alpha \in \Omega^*(\pi(R))$ we have $\|\pi^*\alpha\|_S = \|\alpha\|_{\pi(S)}$ (cf. §3.3) and $\int \pi^*\alpha = 0$. Thus, the operator $\pi^*: \Omega_{\bar{r}+4}^*(M/\mathbf{S}^3) \rightarrow Ker \int$ is well defined. In fact, we have:

Proposition 4.4 *The operator $\pi^*: \Omega_{\bar{r}+4}^*(M/\mathbf{S}^3) \longrightarrow Ker \int$ induces an isomorphism in cohomology.*

Proof. We first apply §2.4 (with \mathcal{U} made up of invariant sets and $\{f_U\}$ to be invariant controlled maps) and we reduce the problem to $M = \mathbb{R}^k \times c\mathbf{S}^{\ell_S}$. Consider $pr': \mathbb{R}^k \times c\mathbf{S}^{\ell_S}/\mathbf{S}^3 \rightarrow \mathbb{R}^{k-1} \times c\mathbf{S}^{\ell_S}/\mathbf{S}^3$ the natural projection as in §2.3. Set $\pi: \mathbb{R}^k \times c\mathbf{S}^{\ell_S} \rightarrow \mathbb{R}^k \times c\mathbf{S}^{\ell_S}/\mathbf{S}^3$ and $\pi': \mathbb{R}^{k-1} \times c\mathbf{S}^{\ell_S} \rightarrow \mathbb{R}^{k-1} \times c\mathbf{S}^{\ell_S}/\mathbf{S}^3$ the natural projections. With the notations of §3.2, we have $pr'\pi = \pi'pr$. The relations $pr_*X_i = Z_i$, $J_*Z_i = X_i$ and $i_{X_i}h = hi_{X_i}$ imply

$$(8) \quad \int pr^* = pr^* \int', \quad \int' J^* = J^* \int, \quad \text{and} \quad \int h = h \int,$$

where \int (resp. \int') is the integration along the fibers of π (resp. π'). We conclude that the diagram

$$\begin{array}{ccc} \Omega_{\bar{r}+4}^*(\mathbb{R}^k \times c\mathbf{S}^{\ell_S}/\mathbf{S}^3) & \xrightarrow{\pi^*} & Ker \left\{ \int : I\Omega_{\bar{r}+4}^*(\mathbb{R}^k \times c\mathbf{S}^{\ell_S}) \longrightarrow \Omega^{*-3}(\mathbb{R}^k \times \mathbf{S}^{\ell_S}/\mathbf{S}^3 \times]0, 1]) \right\} \\ \uparrow (pr')^* & & \uparrow pr^* \\ \Omega_{\bar{r}+4}^*(\mathbb{R}^{k-1} \times c\mathbf{S}^{\ell_S}/\mathbf{S}^3) & \xrightarrow{(\pi')^*} & Ker \left\{ \int' : I\Omega_{\bar{r}+4}^*(\mathbb{R}^{k-1} \times c\mathbf{S}^{\ell_S}) \longrightarrow \Omega^{*-3}(\mathbb{R}^{k-1} \times \mathbf{S}^{\ell_S}/\mathbf{S}^3 \times]0, 1]) \right\} \end{array}$$

is well defined and commutative. The vertical rows are quasi-isomorphisms (same procedure as §2.3). This firstly reduces the problem to $M = \mathbb{R}^{k-1} \times c\mathbf{S}^{\ell_S}$, and finally to $M = c\mathbf{S}^{\ell_S}$.

In order to prove that

$$\pi^*: IH_{\frac{i}{r+4}}^i(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) \longrightarrow H^i(Ker \left\{ \int: I\Omega_{\frac{i}{r+4}}^*(c\mathbf{S}^{\ell_S}) \longrightarrow \Omega^*(\mathbf{S}^{\ell_S}/\mathbf{S}^3 \times]0, 1[) \right\})$$

is an isomorphism in cohomology, we distinguish three cases.

- $i < \bar{r}(\pi(S)) + 4$. Here, we have $\Omega_{\frac{i}{r+4}}^i(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) = \Omega^i(\mathbf{S}^{\ell_S}/\mathbf{S}^3 \times]0, 1[)$, and $(Ker \int)^i = \{\omega \in I\Omega^i(\mathbf{S}^{\ell_S} \times]0, 1[) / i_{(Y_3,0)}i_{(Y_2,0)}i_{(Y_1,0)}\omega = 0\}$. Contracting the second factor to a point, and proceeding as before, we reduce the problem to prove that

$$\pi^*: H^i(\mathbf{S}^{\ell_S}/\mathbf{S}^3) \longrightarrow H^i(\{\omega \in I\Omega^*(\mathbf{S}^{\ell_S}) / i_{Y_3}i_{Y_2}i_{Y_1}\omega = 0\})$$

is an isomorphism. But, since the action Φ^S is free, we already know that the map $\pi^*: \Omega^*(\mathbf{S}^{\ell_S}/\mathbf{S}^3) \rightarrow Ker \left\{ \int: I\Omega^*(\mathbf{S}^{\ell_S}) \longrightarrow \Omega^{*-3}(\mathbf{S}^{\ell_S}/\mathbf{S}^3) \right\}$ induces an isomorphism in cohomology.

- $i = \bar{r}(\pi(S)) + 4$. We can proceed in the same way because $\Omega_{\frac{i}{r+4}}^i(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) \cap d^{-1}(0) = \Omega^i(\mathbf{S}^{\ell_S}/\mathbf{S}^3 \times]0, 1[) \cap d^{-1}(0)$, and $(Ker \int)^i \cap d^{-1}(0) = \{\omega \in I\Omega^i(\mathbf{S}^{\ell_S} \times]0, 1[) / i_{(Y_3,0)}i_{(Y_2,0)}i_{(Y_1,0)}\omega = 0\} \cap d^{-1}(0)$.
- $i > \bar{r}(\pi(S)) + 4$. Since $IH_{\frac{i}{r+4}}^i(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) = 0$ it suffices to prove that for any $\omega \in I\Omega^i(\mathbf{S}^{\ell_S} \times]0, 1[)$ satisfying

$$1) \ \omega = 0 \text{ on } \mathbf{S}^{\ell_S} \times]0, 1/2[, \quad 2) \ i_{(Y_3,0)}i_{(Y_2,0)}i_{(Y_1,0)}\omega = 0, \quad \text{and} \quad 3) \ d\omega = 0,$$

there exists $\eta \in I\Omega^{i-1}(\mathbf{S}^{\ell_S} \times]0, 1[)$ verifying 1) and 2) with $d\eta = \omega$. Write $\omega = \alpha + dr \wedge \beta$ where $\alpha, \beta \in I\Omega^*(\mathbf{S}^{\ell_S} \times]0, 1[)$ do not involve dr . We define $\eta = \int_0^{\cdot} \beta \wedge dr$, which clearly satisfies 1) and $d\eta = \omega$. Since Y_1, Y_2, Y_3 do not involve $\partial/\partial r$, then we also have 2). ♣

4.5 Image of \int . For each differential form $\alpha \in \Omega^*(\pi(R))$ we get $\max(\|\chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^*\alpha\|_S, \|d(\chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^*\alpha)\|_S) \leq 4 + \|\alpha\|_{\pi(S)}$ (cf. (3) and (7)). Since $\int(-1)^{|\alpha|}\chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^*\alpha = \alpha$, we conclude that $\Omega_{\bar{r}}^*(M/\mathbf{S}^3)$ is a subcomplex of $Im \int$.

Proposition 4.6 *The inclusion $\Omega_{\bar{r}}^*(M/\mathbf{S}^3) \hookrightarrow Im \int$ induces an isomorphism in cohomology.*

Proof. Given an invariant function $f = \pi^*f_0: M \rightarrow \mathbb{R}$ and an invariant differential form $\omega \in I\Omega^*(M)$, we get $\int f\omega = f_0 \int \omega$. We can therefore apply §2.4 and reduce the problem to the case $M = \mathbb{R}^k \times c\mathbf{S}^{\ell_S}$, where the action is given by (5).

Proceeding as §4.4 we arrive to the case $M = c\mathbf{S}^{\ell_S}$. Here, in order to prove that the induced map

$$IH_{\frac{i}{r}}^i(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) \longrightarrow H^i(Im \left\{ \int: I\Omega_{\frac{i}{r}}^*(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) \longrightarrow \Omega^{*-3}(\mathbf{S}^{\ell_S}/\mathbf{S}^3 \times]0, 1[) \right\})$$

is an isomorphism for $i \geq 0$, we distinguish four cases.

- $i < \bar{r}(\pi(S))$. In this case we have $I\Omega_{\bar{r}}^i(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) = \Omega^i(\mathbf{S}^{\ell_S}/\mathbf{S}^3 \times]0, 1[)$ and $(Im f)^i = \{f\omega / \omega \in I\Omega^{i+3}(\mathbf{S}^{\ell_S} \times]0, 1[)\}$, which is exactly $\Omega^i(\mathbf{S}^{\ell_S}/\mathbf{S}^3 \times]0, 1[)$.
- $i = \bar{r}(\pi(S))$. We can proceed in the same way because $I\Omega_{\bar{r}}^i(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) \cap d^{-1}(0) = \Omega^i(\mathbf{S}^{\ell_S}/\mathbf{S}^3 \times]0, 1[) \cap d^{-1}(0)$, and $(Im f)^i \cap d^{-1}(0) = \{f\omega/\omega \in I\Omega^{i+3}(\mathbf{S}^{\ell_S} \times]0, 1[)\} \cap d^{-1}(0)$.
- $i = \bar{r}(\pi(S)) + 1$. Since $IH_{\bar{r}}^i(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) = 0$ and $i + 3 = \bar{q}(S)$ we need to prove that for any $\omega \in I\Omega^{i+3}(\mathbf{S}^{\ell_S} \times]0, 1[)$ verifying

$$1) \ d\omega = 0 \text{ on } \mathbf{S}^{\ell_S} \times]0, 1/2[, \quad \text{and} \quad 2) \ d\int \omega = 0,$$

there exists $\eta \in I\Omega^{i+2}(\mathbf{S}^{\ell_S} \times]0, 1[)$ with $d\int \eta = \int \omega$. We project $\mathbf{S}^{\ell_S} \times]0, 1[$ onto $\mathbf{S}^{\ell_S} \times \{1/4\} \equiv \mathbf{S}^{\ell_S}$.

Relations (4) and (8) give $\int \omega = \int pr^* J^* \omega + d\int h\omega - hd\int \omega = \int pr^* J^* \omega + d\int h\omega$, where $pr^* J^* \omega, h\omega \in I\Omega^{i+3}(\mathbf{S}^{\ell_S} \times]0, 1[)$. By construction, the differential form $J^* \omega$ is a cycle of $I\Omega^{i+3}(\mathbf{S}^{\ell_S})$. Since $0 < i+3 = \bar{q}(S) \leq \ell_S - 1$, we find $\gamma \in I\Omega^{i+2}(\mathbf{S}^{\ell_S})$ with $d\gamma = J^* \omega$. Now, we can choose $\eta = pr^* \gamma + h\omega$.

- $i > \bar{r}(\pi(S)) + 1$. Since $IH_{\bar{r}}^i(c\mathbf{S}^{\ell_S}/\mathbf{S}^3) = 0$ and $i + 3 > \bar{q}(S)$ we need to prove that for any $\omega \in I\Omega^{i+3}(\mathbf{S}^{\ell_S} \times]0, 1[)$ verifying

$$1) \ \omega = 0 \text{ on } \mathbf{S}^{\ell_S} \times]0, 1/2[, \quad \text{and} \quad 2) \ d\int \omega = 0,$$

there exists $\eta \in I\Omega^{i+2}(\mathbf{S}^{\ell_S} \times]0, 1[)$ satisfying 1) with $d\int \eta = \int \omega$. It suffices to choose $\eta = (-1)^i \chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \pi^* \int_0^- \beta \wedge dr$, where $\int \omega = \alpha + dr \wedge \beta$ as in §4.4. ♣

We arrive to the main result of this work.

Theorem 4.7 *Let $\Phi: \mathbf{S}^3 \times M \rightarrow M$ be a semifree action. Then, there exists a long exact sequence*

$$(9) \quad \dots \rightarrow H^i(M) \xrightarrow{f^*} IH_{\bar{r}}^{i-3}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} IH_{\bar{r}+4}^{i+1}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \rightarrow \dots,$$

where

- \int is the integration along the fibers of the natural projection $\pi: M \rightarrow M/\mathbf{S}^3$,
- \bar{r} is a perversity of M/\mathbf{S}^3 verifying $-4 \leq \bar{r}(\pi(S)) \leq \ell_S - 5$,
- $\bar{r} + 4$ is the perversity of M/\mathbf{S}^3 defined by $\overline{\bar{r} + 4}(\pi(S)) = \bar{r}(\pi(S)) + 4$, and
- $[e] \in IH_{\bar{r}+4}^4(M/\mathbf{S}^3)$ is the Euler class of Φ .

Proof. Consider \bar{q} the perversity of M defined by $\bar{q}(S) = \bar{r}(\pi(S)) + 4$. The short exact sequence $0 \rightarrow Ker f \xrightarrow{\iota} I\Omega^*(M) \xrightarrow{f^*} Im f \rightarrow 0$ induces the long exact sequence (cf. §2.7)

$$\dots \rightarrow H^i(M) \xrightarrow{f^*} H^{i-3}(Im f) \xrightarrow{\delta} H^{i+1}(Ker f) \xrightarrow{\pi^*} H^{i+1}(M) \rightarrow \dots$$

The connecting homomorphism is defined by $\delta[\alpha] = [(-1)^{|\alpha|} d(\chi_1 \wedge \chi_2 \wedge \chi_3) \wedge \pi^* \alpha]$, which is $[(-1)^{|\alpha|} \pi^*(e \wedge \alpha)]$ on $H^*(Ker f)$ (cf. §3.3). It suffices now to apply §4.4 and §4.6. ♣

Corollary 4.8 *Let $\Phi: \mathbf{S}^3 \times M \rightarrow M$ be a semifree action. The long exact sequences*

$$\dots \rightarrow H^i(M) \xrightarrow{f^*} H^{i-3}(M/\mathbf{S}^3, M^{\mathbf{S}^3}/\mathbf{S}^3) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \rightarrow \dots,$$

and

$$\dots \rightarrow H^i(M) \xrightarrow{f^*} H^{i-3}(M/\mathbf{S}^3) \xrightarrow{\wedge[e]} IH_{\frac{4}{4}}^{i+1}(M/\mathbf{S}^3) \xrightarrow{\pi^*} H^{i+1}(M) \rightarrow \dots,$$

are exact, where for the second one we have assumed M/\mathbf{S}^3 to be without boundary.

Proof. In both cases we apply the previous Theorem taking into account Corollary 2.6. For the first one we consider the perversity \bar{r} defined by $\bar{r}(\pi(S)) = -4$. By definition, $IH_{\bar{r}}^*(M/\mathbf{S}^3)$ is the cohomology of the complex made up of differential forms on $M - M^{\mathbf{S}^3}/\mathbf{S}^3$ vanishing on a neighborhood of $M^{\mathbf{S}^3}/\mathbf{S}^3$; therefore $IH_{\bar{r}}^*(M/\mathbf{S}^3) \cong H^*(M/\mathbf{S}^3, M^{\mathbf{S}^3}/\mathbf{S}^3)$.

For the second case, we consider the perversity $\bar{r} = \bar{0}$. This perversity satisfies the condition c) of the previous Theorem because if M/\mathbf{S}^3 has no boundary then $\ell_S > 5$ for each $S \in \mathcal{S}$. ♣

4.9 The sequence (1) does not become necessarily (9). Let us give an example. Consider the unit sphere $\mathbf{S}^{4\ell+3}$ of $\mathbf{HP}^{\ell+1}$, where \mathbf{HP} are the quaternions. The product by quaternions induce the action $\Psi: \mathbf{S}^3 \times \mathbf{S}^{4\ell+3} \rightarrow \mathbf{S}^{4\ell+3}$. Identify $\mathbf{S}^{4\ell+4}$ with the suspension $\Sigma\mathbf{S}^{4\ell+3} = \mathbf{S}^{4\ell+3} \times [-1, 1]/\{\mathbf{S}^{4\ell+3} \times \{1\}, \mathbf{S}^{4\ell+3} \times \{-1\}\}$. Consider the action $\Phi: \mathbf{S}^3 \times \mathbf{S}^{4\ell+4} \rightarrow \mathbf{S}^{4\ell+4}$ defined by $\Phi(\theta, [x, t]) = [\Psi(\theta, x), t]$. The sequence (1) becomes

$$\dots \rightarrow H^i(\mathbf{S}^{4\ell+4}) \rightarrow H^{i-3}(\Sigma\mathbf{HP}^{\ell}) \rightarrow H^{i+1}(\Sigma\mathbf{HP}^{\ell}) \rightarrow H^{i+1}(\mathbf{S}^{4\ell+4}) \rightarrow \dots,$$

which cannot be exact because $\chi(\mathbf{S}^{4\ell+4}) \neq 0$.

We finish the work with a geometrical interpretation of the vanishing of the Euler class, generalizing [5, pag.321].

Proposition 4.9 *If the principal fibration $\pi: (M - M^{\mathbf{S}^3}) \rightarrow (M - M^{\mathbf{S}^3})/\mathbf{S}^3$ has a section, then $[e] = 0$.*

The existence of a section of $\pi: (M - M^{\mathbf{S}^3}) \rightarrow (M - M^{\mathbf{S}^3})/\mathbf{S}^3$ implies the vanishing of the Euler class $[e']$ of the action $\Phi': \mathbf{S}^3 \times (M - M^{\mathbf{S}^3}) \rightarrow (M - M^{\mathbf{S}^3})$. Thus, the singular strata must have at most codimension four and therefore $F_4\Omega_{D_S}^* = \Omega^*(D_S - S)$, for each $S \in \mathcal{S}$. This implies $IH_{\frac{4}{4}}^*(M/\mathbf{S}^3) = H^*((M - M^{\mathbf{S}^3})/\mathbf{S}^3)$. We have finished the proof because $[e] = [e']$. ♣

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