MORPHISMS FROM P2 TO Gr(2,C4)

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MORPHISMS FROM $\mathbb{P}^2$ TO $Gr(2, \mathbb{C}^4)$.

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Abstract. In this note we study morphisms from $\mathbb{P}^2$ to $Gr(2, \mathbb{C}^4)$ from the point of view of the cohomology class they represent in the Grassmannian. This leads to some new result about projection of d-uple imbedding of $\mathbb{P}^2$ to $\mathbb{P}^5$.

1. Introduction

We denote by $\mathbb{P}^2$ the projective plane over the field of complex numbers and $Gr(2, \mathbb{C}^4)$ be the Grassmannian of two-dimentional quotient spaces of $\mathbb{C}^4$. In this paper we investigate the possible types of non-constant morphisms $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$. Any non-constant morphism $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ determines a cohomology class in $H^4(Gr(2, \mathbb{C}^4), \mathbb{Z})$. We consider the following problem:
Determine the necessary and sufficient conditions that a cohomology class of $H^4(Gr(2, \mathbb{C}^4), \mathbb{Z})$ has to satisfy in order to be represented by a morphism from $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$?

It is easy to see that if a cohomology class is represented by a morphism from $\mathbb{P}^2$ then it has to satisfy an obvious necessary condition (see Lemma (3.3)). We show that in general this condition is not sufficient. The following result shows that there are classes satisfying the condition but are not represented by morphisms:

Theorem 1.1. *Let c and a be two integers. Assume $c \geq 4$ and $1 \leq a \leq c - 2$ or $c^2 - c + 2 \leq a \leq c^2 - 1$. Then the cohomology class $(a, c^2 - a)$ is not represented by a morphism $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$.*

The following theorem shows that in some cases morphisms exist:

Theorem 1.2. *0) For every $c \geq 1$ the cohomology classes $(0, c^2)$ and $(c^2, 0)$ are represented by morphisms $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$.*

1) Let $1 \leq c \leq 3$ and $0 \leq \ell \leq c^2$ be integers. Then the cohomology class $(\ell, c^2 - \ell)$ is represented by a morphism $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$.

2) If $c = 4$ and $3 \leq \ell \leq 13$ then there are morphisms $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ representing the cohomology class $(\ell, 16 - \ell)$.

1991 *Mathematics Subject Classification.* 14F17.

3) Let $c \geq 5$ be an integer. Let k be the largest integer such that $k.c \leq (c^2 - 3c + 2)/2$. Then for every integer ℓ in one of the following intervals $[t(c-3)+2, t.c]$ for $1 \leq t \leq k$ or $[(c^2 - 3c + 2)/2 + 1, c^2/2]$ there are morphisms $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ representing the cohomology class $(\ell, c^2 - \ell)$. Also, for every such ℓ there is a morphism whose cohomology class is $(c^2 - \ell, \ell)$.

It is also shown here (see, Remark (3.12)) that for every integer $n \geq 1$ there are morphisms

$$f_n : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$$

such that f_n is one to one onto its image and $f_n^*(\mathcal{O}_{Gr(2,\mathbb{C}^4)}(1)) = \mathcal{O}_{\mathbb{P}^2}(n)$.

Acknowledgments: The third author would like to thank, IFIM, Université Lille, and Université d' Artois, France.

2. Morphisms to Grassmannian

Here we recall some results about morphisms from a variety to a Grassmannian variety.

Definition 2.1. Let X be a variety over the field $\mathbb{C}$ of complex numbers and s be a positive integer. A vector bundle E over X is said to be generated by s sections if there a surjection of vector bundles

$$\mathbb{C}^s \otimes \mathcal{O}_X \to E.$$

Definition 2.2. Let X be a projective variety. Let r and k be two positive integers. Let E_1 and E_2 be two vector bundles of rank r over X. Two vector bundle surjections

$$\phi_1 : \mathbb{C}^{r+k} \otimes \mathcal{O}_X \to E_1,$$

and

$$\phi_2 : \mathbb{C}^{r+k} \otimes \mathcal{O}_X \to E_2,$$

are said to be equivalent if there exists an isomorphsim of vector bundles $\psi : E_1 \to E_2$ over X such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^{r+k} \otimes \mathcal{O}_X & \xrightarrow{\phi_1} & E_1 \\ \| & & \downarrow \psi \\ \mathbb{C}^{r+k} \otimes \mathcal{O}_X & \xrightarrow{\phi_2} & E_2 \end{array}$$

The following two lemmas are well known(see, for example [4]). We include the proofs of these lemmas for the sake of completeness.

Lemma 2.3. *Let X be a projective variety. Let r and k be two positive integers. There is a natural bijection between the following two sets:*

1) The set of equivalence classes of surjection of vector bundles

$$\phi : \mathbb{C}^{r+k} \otimes \mathcal{O}_X \to E,$$

where E is a vector bundle of rank r on X.

2) The set of morphisms $f : X \to Gr(r, \mathbb{C}^{r+k})$, where $Gr(r, \mathbb{C}^{r+k})$ is the Grassmannian of r dimensional quotient of $\mathbb{C}^{r+k}$.

Proof: Given a surjection $\phi : \mathbb{C}^{r+k} \otimes \mathcal{O}_X \to E$ by sending

$$x \mapsto \{\mathbb{C}^{r+k} \to E_x\}$$

defines a morphism $f : X \longrightarrow Gr(r, \mathbb{C}^{r+k})$. This defines a map from the set in 1) to the set in 2).

To prove the existence of the map in the other direction, we first note the following: on $Gr(r, \mathbb{C}^{r+k})$ there is a canonical vector bundle surjection $\mathbb{C}^{r+k} \otimes \mathcal{O}_{Gr(r,\mathbb{C}^{r+k})} \longrightarrow Q$, where Q is the rank r bundle on $Gr(r, \mathbb{C}^{r+k})$ whose fiber at $x \in Gr(r, \mathbb{C}^{r+k})$ is the quotient vector space corresponding to x.

Now given a morphism $f : X \longrightarrow Gr(r, \mathbb{C}^{r+k})$ by pulling back by f the canonical surjection of vector bundles on the Grassmannian, we get a surjection of vector bundles

$$\mathbb{C}^{r+k} \otimes \mathcal{O}_X = f^*(\mathbb{C}^{r+k} \otimes \mathcal{O}_{Gr(r,\mathbb{C}^{r+k})}) \longrightarrow f^*(Q).$$

This gives a map from the set in 2) to the set in 1). These two maps are clearly inverse to each other hence we get the required bijection. $\square$

Theorem 2.4. *Let X be a projective variety. There is a natural bijection between the following two sets:*

1) The set of equivalence classes of surjections of vector bundles

$$\mathbb{C}^{r+k} \otimes \mathcal{O}_X \longrightarrow E$$

with $rank(E) = r$ and $\det(E)$ is ample.

2) The set of morphisms $f : X \longrightarrow Gr(r, \mathbb{C}^{r+k})$ with f finite (onto its image).

Proof: Given an element of the set in 1), i.e., a surjection, $\mathbb{C}^{r+k} \otimes \mathcal{O}_X \longrightarrow E$ with $\mathrm{rk(E) = r}$ and $\det(E)$ is ample, then as in Lemma(2.3) it determines a morphism $f : X \longrightarrow Gr(r, \mathbb{C}^{r+k})$, also, it follows from Lemma(2.3), that $f^*(Q) \simeq E$. Hence $f^*\det(Q) \simeq \mathrm{det(E)}$. Since $\det(E)|_{f^{-1}(f(x))} \simeq f^*\mathrm{det(Q)}|_{\mathrm{f^{-1}(f(x))}}$ is trivial, the ampleness assumption on $\det(E)$ implies $\mathrm{dim(f^{-1}(f(x)))} \leq 0$. Thus, if $\det(E)$ is ample, then f is finite onto its image.

In the other direction, if $f : X \longrightarrow Gr(r, \mathbb{C}^{r+k})$ is finite morphism onto its image, then

$$\mathrm{det(f^*Q) \simeq f^*(detQ)}$$

is ample, as pull-back of an ample bundle remains ample under finite morphism. $\square$

Let L be an vector subspace of dimension one in $\mathbb{C}^4$. Then the projective 2 plane

$$\mathbb{P}^2_L = Gr(2, \mathbb{C}^4/L)$$

is naturally a subvariety of $Gr(2, \mathbb{C}^4)$. Similarly, if H is an one dimensional quotient vector space of $\mathbb{C}^4$, then the projective 2 plane

$$\mathbb{P}^2_H = Gr(2, (\mathbb{C}^4)^*/H^*)$$

is also naturally a subvariety of $Gr(2, \mathbb{C}^4)$, where $(\mathbb{C}^4)^*$ (resp. H^*) denotes the dual vector space of $\mathbb{C}^4$ (resp. of H). Note that $\mathbb{P}^2_H$ is the subset of $Gr(2, \mathbb{C}^4)$ consists of all those quotient vector spaces of dimension two of $\mathbb{C}^4$ which has H as their quotient.

Definition 2.5. A morphism $f : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ is said to be trivial if it is one of the following:

- a) f is a constant morphism.
- b) Image of f is $\mathbb{P}^2_L$ for some one dimensional subspace L of $\mathbb{C}^4$.
- c) Image of f is $\mathbb{P}^2_H$ for some one dimensional quotient H of $\mathbb{C}^4$.

Example 2.6. It is known that (see, for example [3]) the Veronese surface (2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$) is contained in a smooth quadric in $\mathbb{P}^5$. As any two smooth quadrics are isomorphic via a projective automorphism we see that there is an embedding of $\mathbb{P}^2$ in $Gr(2, \mathbb{C}^4)$. In fact, the global sections

$$s_1 = (X, Z), \quad s_2 = (Y, X), \quad s_3 = (Z, Y) \text{ and } s_4 = (X, 0)$$

of $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ gives a vector bundle surjection

$$\mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1).$$

Then by Lemma (2.3) we get morphism $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$. It is easy to see that ϕ is an embedding and ϕ composed with the natural embedding of $Gr(2, \mathbb{C}^4)$ into $\mathbb{P}^5$ is given by the quadrics

$$X^2 - YZ, \quad XY - Z^2, \quad Y^2 - XZ, \quad XZ, \quad X^2, \quad XY.$$

As these quadrics form a basis for the space of quadrics on $\mathbb{P}^2$ we get that the veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ factors through $Gr(2, \mathbb{C}^4)$.

Example 2.7. More generally, for any positive integers a and b the global sections

$$s_1 = (X^a, Z^b), \quad s_2 = (Y^a, X^b), \quad s_3 = (Z^a, Y^b) \text{ and } s_4 = (X^a, 0)$$

of $\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$ gives a vector bundle surjection

$$\mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b).$$

Then by Lemma (2.3) we get a morphism $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$.

Example 2.8. Let $T_{\mathbb{P}^2}$ denotes the tangent bundle of $\mathbb{P}^2$. Then one has an exact sequence (See, page 409 of [2])

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3 \to T_{\mathbb{P}^2} \to 0.$$

¿From this we see that the space of all global sections $\mathrm{H}^0(T_{\mathbb{P}^2})$ of the tangent bundle is a vector space of dimension eight. Now it is easy to see that if we choose four linearly independent general sections of the tangent bundle we get a surjective morphism

$$\mathcal{O}_{\mathbb{P}^2}^4 \to T_{\mathbb{P}^2}.$$

This surjection give rise to a morphism from $\mathbb{P}^2$ to $Gr(2, \mathbb{C}^4)$. Since the tangent bundle is not direct sum of line bundles, this morphism is different from the morphisms given by example above.

3. Non-trivial morphisms from $\mathbb{P}^2$ to $Gr(2, \mathbb{C}^4)$

On the Grassmaian $Gr(2, \mathbb{C}^4)$ one has universal exact sequence:

$$0 \to S \to \mathbb{C}^4 \otimes \mathcal{O}_{Gr(2,\mathbb{C}^4)} \to Q \to 0, \tag{1}$$

where S and Q are respectively the universal sub bundle and quotient bundle of rank two on $Gr(2, \mathbb{C}^4)$. The fiber of Q (resp. of S) at a point $p \in Gr(2, \mathbb{C}^4)$ is the two dimensional quotient space (resp. subspace, which is the kernal of this quotient map) of $\mathbb{C}^4$ corresponding to the point p. It is known (see, page 197 and 411 of [2]) that the cohomology group $H^4(Gr(2, \mathbb{C}^4), \mathbb{Z})$ is equal to

$$\mathbb{Z}[c_2(Q)] \oplus \mathbb{Z}[c_2(S)], \tag{2}$$

where $c_2(Q)$ (resp. $c_2(S)$) is the second Chern class of Q (resp. of S). If $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ is a non constant morphism then the cohomology class of $\phi_*([\mathbb{P}^2])$ is an element of $H^4(Gr(2, \mathbb{C}^4), \mathbb{Z})$. It is easy to see that the morphism b) (resp. c)) of Example(2.5) gives the cohomology class $(0, 1)$ (resp. $(1, 0)$) of the decomposition in (2) of the cohomology group. The exact sequence corresponding to cohomology class $(0, 1)$ is

$$0 \to \Omega^1(1) \to \mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \to 0. \tag{3}$$

The dual of the exact sequence (3) correspondence to the cohomology class $(1, 0)$.

Question 1) a) Given the cohomology class (a, b) of the decomposition in (2) of the cohomology group does there exists a morphism $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ such that the cohomology class of this morphism is (a, b)?

b) For which cohomology class (a, b) does there exists a generically injective morphism $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ such the cohomology class of this morphism is (a, b)?

Question 2) Let Q be the vector bundle on $Gr(2,\mathbb{C}^4)$ as in the equation (1). For which cohomology classes (a,b) does there exists a morphism $\phi : \mathbb{P}^2 \to Gr(2,\mathbb{C}^4)$ the bundle $\phi^*(Q)$ is indecomposable?

Remark 3.1. *Let $\phi : \mathbb{P}^2 \to Gr(2,\mathbb{C}^4)$ be a morphism. By pulling back the universal exact sequence* (1) *on $Gr(2,\mathbb{C}^4)$ we get an exact sequence*

$$0 \to \phi^*(S) \to \mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \phi^*(Q) \to 0, \tag{4}$$

of vector bundles on $\mathbb{P}^2$. By dualizing the exact sequence (4) *we get another exact sequence*

$$0 \to \phi^*(Q)^\vee \to \mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \phi^*(S)^\vee \to 0, \tag{5}$$

of vector bundles on $\mathbb{P}^2$. By Lemma (2.3) the surjection

$$\mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \phi^*(S)^\vee \to 0, \tag{6}$$

gives a morphism from $\mathbb{P}^2$ to $Gr(2,\mathbb{C}^4)$ which we denote by $\phi^\vee$ and call the dual morphism. If (a,b) is the cohomology class of a morphism ϕ then it is clear that the cohomology class of the dual morphism $\phi^\vee$ is (b,a).

Remark 3.2. *Let $\phi : \mathbb{P}^2 \to Gr(2,\mathbb{C}^4)$ be a closed immersion. Assume that ϕ is a non trivial imbedding as in Definition(2.5). It follows from* [10] *the morphism ϕ or $\phi^\vee$, is given by a surjection*

$$\mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1). \tag{7}$$

If the morphism ϕ is given by (7) *then $\phi_*([\mathbb{P}^2]) = (1,3)$ and the class of $(\phi^\vee)_*([\mathbb{P}^2]) = (3,1)$. Thus the only classes (a,b) represented by regularly imbedded $\mathbb{P}^2$ in $Gr(2,\mathbb{C}^4)$ are $(1,0),(0,1),(1,3)$ and $(3,1)$.*

Lemma 3.3. *Let $\phi : \mathbb{P}^2 \to Gr(2,\mathbb{C}^4)$ be a non constant morphism. Then the cohomolgy class given by the morphism ϕ is of the form $(a, c^2 - a)$, for some integers $c > 0$ and $0 \leq a \leq c^2$.*

Proof: By pulling back the universal exact sequence (1) by ϕ we get the following exact sequence of vector bundles on $\mathbb{P}^2$:

$$0 \to \phi^*(S) \to \mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \phi^*(Q) \to 0. \tag{8}$$

Since, the bundle $\phi^*(Q)$ is non-trivial and is a quotient of trivial bundle, we see that the Chern classes $c_1(\phi^*(Q))$ and $c_2(\phi^*(Q))$ are both non-negative integers (we identify $H^4(\mathbb{P}^2,\mathbb{Z})$ with $\mathbb{Z}$ by sending the class $[H].[H]$ to 1 in $\mathbb{Z}$, where $[H]$ is the class of a hyper plane). Set $a = c_2(\phi^*(Q))$ and $c = c_1(\phi^*(Q))$. Note that $c^2 - a = c_2(\phi^*(S)) = c_2((\phi^*(S))^\vee)$, where $(\phi^*(S))^\vee$ is the dual of the bundle $\phi^*(S)$. Since $(\phi^*(S))^\vee$ is generated by sections we must have $c^2 - a \geq 0$. This completes the proof of the lemma. $\square$

Remark 3.4. *The following question arises naturally. Given (a, c^2-a), with $c > 0$ and $0 \leq a \leq c^2$ does there exists a morphism $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ such that the cohomolgy class given by the morphism ϕ is of the form $(a, c^2 - a)$? In general this question has negative answer (see Lemma (3.6) below). Also, we give below partial answer to the above question (see Theorem(3.7) and Theorem(3.8)). We believe that there are no morphisms from $\mathbb{P}^2$ which represent the remaining cohomology classes.*

We need the following theorem:

Theorem 3.5. *(*Cayley-Bacharach theorem*) Let S be a non-singular surface and Z be a sub scheme of S of dimension zero. Then there exists a rank two vector bundle E on S with a section s such that the zero sub scheme of s is Z if and only if for every point $p \in Z$ the linear system*

$$|\mathcal{I}_Z K_S \otimes \det(E)| = |\mathcal{I}_{Z-\{p\}} K_S \otimes \det(E)|,$$

where $\mathcal{I}_Z$ (respectively, $\mathcal{I}_{Z-\{p\}}$) denotes the ideal sheaf of Z (respectively, of $Z - \{p\}$).

Proof: This is a consequence of [Theorem 7.[11]]. (See, [2] page 731, for the case Z is reduced).

Lemma 3.6. *There is no morphism $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ such that the cohomolgy class given by the morphism ϕ is of the form $(1, 15)$.*

Proof: Assume there exists a morphism $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ such that the cohomolgy class given by the morphism ϕ is of the form $(1, 15)$. By pulling back the universal exact sequence (1) by ϕ gives the following exact sequence of vector bundles on $\mathbb{P}^2$:

$$0 \to \phi^*(S) \to \mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \phi^*(Q) \to 0. \tag{9}$$

with $c_2(\phi^*(Q)) = [H]^2$ and $c_2(\phi^*(S)) = 15[H]^2$, where $[H]$ is the hyper plane class. More over it can be easily seen that $c_1(\phi^*(Q)) = 4[H]$. Since $E = \phi^*(Q)$ is generated by sections there exists a section of E which vanishes at exactly one point p of $\mathbb{P}^2$ with multiplicity one. But then by Cayley-Bacharach theorem(3.5) the point must be a base point for the complete linear system of the line bundle $\det(E) \otimes K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$. But this is a contradiction. This proves the required result. $\square$

More generally we have the following:

Theorem 3.7. *Let c and a be two integers. Assume $c \geq 4$ and $1 \leq a \leq c - 2$ or $c^2 - c + 2 \leq a \leq c^2 - 1$. Then the cohomology class $(a, c^2 - a)$ is not represented by a morphism $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$.*

Proof: If $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ is a morphism such that the cohomology class of the morphism is ϕ is $(a, c^2 - a)$ then the cohomology class of the dual morphism (see 3.6) $\phi^\vee : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ is $(c^2 - a, a)$. So it is enough to show that there are no morphisms from $\mathbb{P}^2$ to $Gr(2, \mathbb{C}^4)$ such that the cohomolgy class is given by $(a, c^2 - a)$ for $1 \leq a \leq c - 2$. But note that for any zero dimensional sub scheme Z of $\mathbb{P}^2$ the natural morphism

$$H^0(\mathcal{O}_{\mathbb{P}^2}(c-3)) \to H^0(\mathcal{O}_Z(c-3)$$

is surjective, if length of $\mathcal{O}_Z$ is less than or equal to $c - 2$. This will imply the following: if Z is a zero dimensional sub scheme of $\mathbb{P}^2$ such that the length of $\mathcal{O}_Z$ is less than or equal to $c - 2$, then for any $p \in Z$

$$H^0(\mathcal{I}_Z(c-3)) \neq H^0(\mathcal{I}_{Z-p}(c-3)), \tag{10}$$

where $\mathcal{I}_T$ denotes the ideal sheaf of the sub scheme T. On the other hand if there exists a morphism $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ such that the cohomology class of the morphism is ϕ is $(a, c^2 - a)$ then the pull back E of the universal quotient bundle on $Gr(2, \mathbb{C}^4)$ has $c_1(E) = c, c_2(E) = a$. Since E is generated by sections we can find section s such that the scheme Z of zeros of this section is zero dimensional and length of $\mathcal{O}_Z$ is equal to $a \leq c - 2$. Now by Cayley-Bacharach theorem (3.5) we must have

$$H^0(\mathcal{I}_Z(c-3)) = H^0(\mathcal{I}_{Z-\{p\}}(c-3)),$$

for every $p \in Z$. But this leads to a contradiction to (10). This proves the reqired result. $\square$

Theorem 3.8. *0) For every $c \geq 1$ the cohomology classes $(0, c^2)$ and $(c^2, 0)$ are represented by morphisms $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$.*

1) Let $1 \leq c \leq 3$ and $0 \leq \ell \leq c^2$ be integers. Then the cohomology class $(\ell, c^2 - \ell)$ is represented by a morphism $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$.

2) If $c = 4$ and $3 \leq \ell \leq 13$ then there are morphisms $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ representing the cohomology class $(\ell, 16 - \ell)$.

3) Let $c \geq 5$ be an integer. Let k be the largest integer such that $k.c \leq (c^2 - 3c + 2)/2$. Then for every integer ℓ in one of the following intervals $[t(c-3) + 2, t.c]$ for $1 \leq t \leq k$ or $[(c^2 - 3c + 2)/2 + 1, c^2/2]$ there are morphisms $\mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ representing the cohomology class $(\ell, c^2 - \ell)$. Also, for every such ℓ there is a morphism whose cohomology class is $(c^2 - \ell, \ell)$.

Proof: 0) If $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a finite morphism then $f^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_{\mathbb{P}^2}(n)$ for some integer $n > 0$. Then the degree of the morphism is equal to n^2. Thus we see that if a cohomology class of the form $(0, a)$ or $(a, 0)$ is represented by a non constant morphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ if and only if $a = n^2$ for a positive integer n.

1) If $1 \leq c \leq 3$ then for any zero dimensional sub scheme Z the vector space $H^0(I_Z(c-3))$ is 0. Hence by Cayley-Bacharach theorem(3.5) there exists a vector bundle E and a section s such that $c_1(E) = c$ and zero scheme of s is Z. It is enough to consider the case $\ell \leq c^2/2$, for if $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ such that the cohomology class of the morphism given by ϕ is $(\ell, c^2 - \ell)$ then the cohomology class of dual morphism (see 3.1) $\phi^\vee$ is $(c^2 - \ell, \ell)$. For $\ell \leq c^2/2$, by considering any reduced sub scheme Z consisting of ℓ points we get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to E \to I_Z(c) \to 0,$$

where I_Z denotes the ideal sheaf of Z, where E is the vector bundle obtained by Cayley Bacharach theorem. Then we see that E is generated by sections and hence by 4 sections. This gives the required morphism ϕ.

2) For $\ell = 3$ choose Z a reduced scheme consisting of three points lying on a line. Then we see that Z satisfies Cayley - Bacharach conditions with respect to line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$. Thus there exists vector bundle E with $c_1(det(E)) = 4[H]$ and a section s such that $(s)_0 = Z$. Also, We get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to E \to I_Z(4) \to 0,$$

where I_Z denotes the ideal sheaf of Z. Then we see that E is generated by sections and hence by 4 sections. This gives the required morphism ϕ.

For $4 \leq \ell \leq 8$ Let $Z = \{P_1, \ldots, P_\ell\}$ be a reduced closed scheme consisting of ℓ points such that no three points lie on a line. Then we see that Z satisfies Cayley - Bacharach conditions with respect to line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$. Thus there exists vector bundle E with $c_1(det(E)) = 4[H]$ and a section s such that $(s)_0 = Z$. Also, We get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to E \to I_Z(4) \to 0,$$

where I_Z denotes the ideal sheaf of Z. Then we see that E is generated by sections and hence by 4 sections. This gives the required morphism ϕ. For $8 \leq \ell \leq 13$ the dual morphism $\phi^\vee$ of appropriate ϕ above gives the reqired morphism.

3) Let t and ℓ be as in the Theorem. Let $Z = \{P_1, \ldots, P_\ell\}$ be a reduced closed scheme consisting of ℓ points such that no $r.c+1$ points lie on a curve of degree r for $1 \leq r \leq t$. Note that Z satisfies Cayley - Bacharach conditions with respect to line bundle $\mathcal{O}_{\mathbb{P}^2}(c-3)$. Thus there exists vector bundle E with $c_1(det(E)) = d[H]$ and a section s such that $(s)_0 = Z$. Also, We get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to E \to I_Z(d) \to 0,$$

where I_Z denotes the ideal sheaf of Z. Then we see that E is generated by sections and hence by 4 sections. This morphism ϕ corresponds to the cohomology class $(\ell, c^2-\ell)$. The dual morphism $\phi^\vee$ corresponds to the cohomology class $(c^2-\ell, \ell)$. For ℓ in the interval $[(c^2-3c+2)/2+1, c^2/2]$ let $Z = \{P_1, \ldots, P_\ell\}$ be a reduced closed scheme consisting of ℓ points such that no $r.c+1$ points lie on a curve of degree r for $1 \le r \le c-3$. Then the rest of the proof is as before. □

Lemma 3.9. *Let $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ be a non constant morphism. Assume that $\phi^*(Q)$ is decomposable. Then the cohomolgy class given by the morphism ϕ is of the form $(a.b, (a+b)^2 - a.b)$ for some non negative integers a, b with $a+b > 0$. Moreover, for any such tuple $(a.b, (a+b)^2 - a.b)$, there are morphisms $\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ whose corresponding cohomology class is $(a.b, (a+b)^2 - a.b)$.*

Proof: By pulling back the universal exact sequence ((1)) by ϕ gives the following exact sequence of vector bundles on $\mathbb{P}^2$:

$$0 \to \phi^*(S) \to \mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \phi^*(Q) \to 0. \tag{11}$$

By assumption $\phi^*(Q) \simeq \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$. Since the bundle $\phi^*(Q)$ is quotient of trivial bundle implies we must have $a, b \ge 0$. On the other hand ϕ is non constant morphism implies $a+b > 0$. Now, it is easy to see that $c_2(\phi^*(Q)) = a.b$ and $c_2(\phi^*(S)) = (a+b)^2 - a.b$ Last assertion of the Lemma follows from Example(2.7). □

Proposition 3.10. *Let X and Y be two irreducible projective varieties. Let S be any irreducible quasi-projective variety and $s_0 \in S$ be a point. Let*

$$F : X \times S \to Y$$

be a morphism. Assume that $F_s := F|_{X \times s} : X \to Y$ is finite for all $s \in S$ and F_{s_0} is a birational onto its image. Then there is an open subvariety U of S such that $s_0 \in U$ and for $s \in U$ the morphism F_s is a birational onto its image.

Proof: Consider the morphism $G = F \times Id_S : X \times S \to Y \times S$. Then the assumption F_s is finite implies the morphism G is finite and proper. Hence $\mathcal{G} = G_*(\mathcal{O}_{X \times S})$ is coherent sheaf of $\mathcal{O}_{Y \times S}$ modules. Let $Z \subset Y \times S$ be the sub variety on which the sheaf $G_*(\mathcal{O}_{X \times S})$ is supported. Then clearly the map $p : Z \to S$, restriction of the natural projection, is surjective. The section $1 \in \mathcal{O}_{X \times S}$ gives an inclusion of $\mathcal{O}_Z$ in $\mathcal{G}$. Let $\mathcal{F} = \mathcal{G}/\mathcal{O}_Z$. Let $Z_1 \subset Y \times S$ be the sub variety on which the sheaf $\mathcal{F}$ supported. Let $q : Z_1 \to S$ be the natural projection and let $U = \{s \in S | \mathrm{dim} q^{-1}(s) < \dim(X)\}$ then we see that by semi continuity (see, page 95, Exercise (3.22) [6]), U is an open subset and is non-empty as $s_0 \in U$. For $s \in U$ the morphism F_s is an isomorphism on $X \times s - G^{-1}(q^{-1}(s)$. Since G is finite $G^{-1}(q^{-1}(s)$ is proper closed sub set of $X \times s$ and hence the morphism F_s is birational onto its image. This proves the Proposition. $\square$

Proposition 3.11. *Let a, b be two coprime positive integers. Let*

$$f_\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$$

be the morphism given by a surjection

$$\phi : \mathbb{C}^4 \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(a) \otimes \mathcal{O}_{\mathbb{P}^2}(b).$$

Then for a generic choice of ϕ the morphism is birational onto its image.

Proof: Since the set of surjections ϕ is an open subset of

$$Hom(\mathbb{C}^4, H^0(\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)))$$

The result follows from Proposition(3.10), if we show the existence of one such f_ϕ. If ϕ is given by the matrix

$$\begin{bmatrix} X^a & Z^b \\ Y^a & X^b \\ Z^a & Y^b \\ X^a & 0 \end{bmatrix}$$

Since a, b are coprime, it is easy to see that the morphism $f_\phi : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$ corresponding to ϕ is birational onto its image. $\square$

Remark 3.12. *By choosing $a = 1$ and $b = n$ and using the fact that $Gr(2, \mathbb{C}^4)$ is imbedded in $\mathbb{P}^5$ as a smooth quadric we get the following: for every integer $n \geq 1$ there are morphisms*

$$f_n : \mathbb{P}^2 \to Gr(2, \mathbb{C}^4)$$

such that f_n is one to one onto its image and $f_n^*(\mathcal{O}_{Gr(2,\mathbb{C}^4)}(1)) = \mathcal{O}_{\mathbb{P}^2}(n)$.

References

[1] J. D'Almeida: *Une involution sur un espace de modules de fibrés instantons* Bull. Soc. Math. Fr. **128** (2000), 577–584.

[2] P. Griffiths and J. Harris: Principles of Algebraic Geometry. Wiley Interscience Publication. 1978.

[3] S. P. Inamdar and D. S. Nagaraj: *Cycle class map and restriction of subvarieties* J. Ramanujan Math. Soc. **17**, No.2(2002) 85–91.

[4] F. Laytimi and D. S. Nagaraj: *Vector bundles generated by sections and morphism to Grassmannian* (preprint) https://www.imsc.res.in/eprints/45/

[5] W. Fulton and R. Lazarsfeld: *On the connectedness of degeneracy loci and special divisors* Acta Math. 146 (1981), no. 3-4, 271–283.

[6] R. Hartshorne: Algebraic Geometry. Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York, 1977.

[7] I. Kausz: *A Gieseker type degeneration of moduli stack of vector bundles on curves* Trans. Amer. Math. Soc. 357(2005) 4897-4955.

[8] R. Lazarsfeld: *Some applications of the theory of Positive vector bundles* In: Complete Intersections, C.I.M.E., Acireale 1983. Lect. Notes. Math. vol. 1092, Berlin, Heidelberg, New York: Springer, 1984.

[9] C. Okonek, M. Schneider and H. Spindler: Vector bundles on complex projective spaces. Progress in Mathematics, No. 3, Birkhäuser, Boston, Mass., 1980.

[10] Hiroshi Tango: *On (n-1)-dimensional projective spaces contained in the Grassmann variety* $Gr(n, 1)$. J. Math. Kyoto Univ. 14-3 (1974) 415-460.

[11] Sheng-Li Tan and Eckart Viehweg: *A note on Caylay-Bacharach property for vector bundles* Complex analysis and Algebraic geometry. A volume in memory of Michael Schneider. Editors: Thomes Peternell and Frank-Olaf Schreyer. Walter de Gruyter & Co., Berlin, (2000) 361-373.

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